

Solution

Week 11 (11/25/02)

Break or not break?

There are two possible reasonings, which seem to create a paradox:

- To an observer in the original rest frame, the spaceships stay the same distance, d , apart. Therefore, in the frame of the spaceships, the distance between them, d' , must equal γd . This is true because d' is the distance that gets length-contracted down to d . After a long enough time, γ will differ appreciably from 1, so the string will be stretched by a large factor. Therefore, it *will* break.
- Let A be the rear spaceship, and let B be the front spaceship. From A 's point of view, it looks like B is doing exactly what he is doing (and vice versa). A says that B has the same acceleration that he has. So B should stay the same distance ahead of him. Therefore, the string should *not* break.

The first reasoning is correct. The string *will* break. So that's the answer to our problem. But as with any good relativity paradox, we shouldn't feel at ease until we've explained what's wrong with the wrong reasoning.

The problem with the second reasoning is that A does *not* see B doing exactly what he is doing. Rather, A sees B 's clock running fast. Perhaps the easiest way to show this is via the gravitational time-dilation effect. Since A and B are accelerating, they may be considered (by the Equivalence Principle) to be in a gravitational field, with B "higher" in the field. But high clocks run fast in a gravitational field. Hence, A sees B 's clock running fast (and B sees A 's clock running slow). A therefore sees B 's engine running faster, and so B pulls away from A . Therefore, the string eventually breaks.

REMARKS:

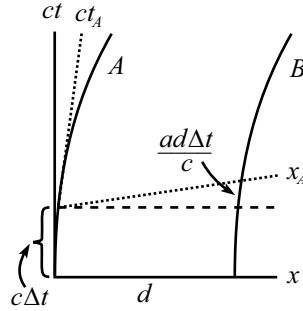
1. There is one slight (inconsequential) flaw in the first reasoning above. There is not one "frame of the spaceships". Their frames differ, since they measure a relative speed between themselves. It is therefore not clear exactly what is meant by the "length" of the string, because it is not clear what frame the measurement should take place in. This ambiguity, however, does not change the fact that A and B observe their separation to be (essentially) γd .

If we want there to eventually be a well-defined "frame of the spaceships", we can simply modify the problem by stating that after a while, the spaceships stop accelerating simultaneously, as measured by someone in the original inertial frame. (Equivalently, A and B turn off their engines after equal proper times.) What A will see is the following. B pulls away from A . B then turns off his engine. The gap continues to widen. But A continues to fire his engine until he reaches B 's speed. They then sail onward, in a common frame, keeping a constant separation (which is greater than the original separation, by a factor γ .)

2. The main issue in this problem is that it depends exactly on how we choose to accelerate an extended object. If we accelerate a stick by pushing on the back end (or by pulling on the front end), its length will remain essentially the same in its own frame, and it will become shorter in the original frame. But if we arrange for each end (or perhaps a number of points on the stick) to speed up in such a way that they always

move at the same speed with respect to the original frame, then the stick will be torn apart.

3. There is no need to invoke the Equivalence Principle to show that A sees B 's clock run fast. We can demonstrate this effect completely within the realm of special relativity. Consider the following Minkowski diagram (about which we'll invoke a few standard facts).



After a small time Δt , the x_A -axis (which consists of simultaneous events, as observed by A) tilts upward with slope $v/c = a \Delta t/c$. It therefore intersects B 's worldline at a ct value of

$$c \Delta t + \frac{ad \Delta t}{c} = c \Delta t \left(1 + \frac{ad}{c^2} \right). \quad (1)$$

A therefore sees B 's clock run fast by a factor $(1 + ad/c^2)$, which is the standard gravitational time-dilation result.

Another derivation of the ad/c^2 result is the following. Consider the situation a short time after the start. An outside observer sees A 's and B 's clocks showing the same time. Therefore, by the usual vd/c^2 loss-of-simultaneity result in special relativity, B 's clock must read vd/c^2 more than A 's, in the moving frame. The increase per unit time, as viewed by A , must therefore be $(vd/c^2)/t = ad/c^2$, as above.

At any later time, we can repeat (roughly) either of the above two derivations in the instantaneous rest frame of A . Note that any special-relativistic time-dilation or length-contraction effects will be second order in (v/c) , and hence negligible for small v .