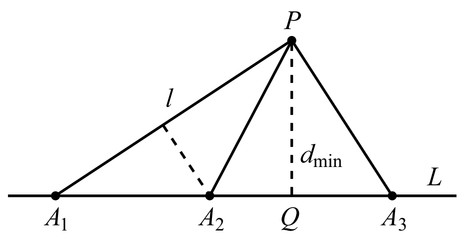
Solution

Week 20 (1/27/03)

Collinear points

Draw all the lines determined by the points. From the assumption of the problem, there are at least three points on each of these lines. Consider all of the distances between any of the points and any of the lines. (Many of these distances are zero, of course, for points lying on a given line.) Since there is a finite number of points and lines, there is a finite number of these distances.

Assume that all the points do not lie on one line, so that some of these distances are nonzero. Then there is a smallest nonzero distance, d_{min} (it may occur more than once). Consider a point P, and line L, associated with d_{min}, as shown.



Let Q be the projection of point P onto L. Since L contains at least three points, at least two of them must lie on the same side of Q (or one may coincide with Q). Call these points A_1 and A_2. Let ℓ be the line through P and A_1. Then the distance from A_2 to ℓ is less than d_{min} (because this distance is less than or equal to the distance from Q to ℓ, which is strictly less than the distance from Q to P). But this contradicts our assumption that d_{min} was the smallest nonzero distance. Hence, there can be no smallest nonzero distance. Therefore, all the distances are zero, and all the points must lie on one line.