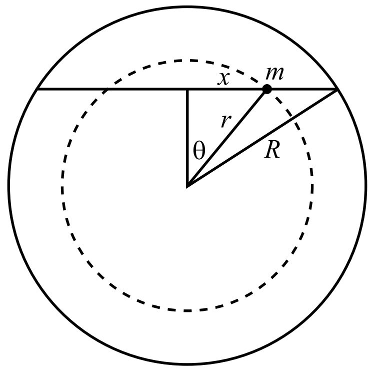
Solution

Week 41 (6/23/03)

Speedy travel

Let the earth's mass be M, and let its radius be R. Consider the object when it is a distance x from the center of the tube, at radius r, as shown.



The gravitational force on the object is due to the mass of the earth that is inside the radius r. Since mass is proportional to volume, the mass inside the radius r is $M(r/R)^3$. The force on the object is therefore

$$F = \frac{G\Big(M(r/R)^3\Big)m}{r^2} = \frac{GMmr}{R^3}\,. \tag{1}$$

We are interested in the component of the force along the tube, which is

$$F_x = -F\sin\theta = -\frac{GMmr}{R^3}\left(\frac{x}{r}\right) = -\left(\frac{GMm}{R^3}\right)x. \tag{2}$$

Therefore, $F = ma$ along the tube gives

$$-\left(\frac{GMm}{R^3}\right)x = m\ddot{x} \qquad \Longrightarrow \qquad \ddot{x} = -\left(\frac{GM}{R^3}\right)x. \tag{3}$$

This equation describes simple harmonic motion, with frequency

$$\omega = \sqrt{\frac{GM}{R^3}}\,. \tag{4}$$

Therefore, the roundtrip time is

$$\begin{aligned} T = \frac{2\pi}{\omega} &= 2\pi\sqrt{\frac{R^3}{GM}} \\ &= 2\pi\sqrt{\frac{(6.38\cdot 10^6\,\mathrm{m})^3}{(6.67\cdot 10^{-11}\,\mathrm{m}^3/\mathrm{kg}\cdot\mathrm{s}^2)(5.98\cdot 10^{24}\,\mathrm{kg})}} \\ &\approx 5060\,\mathrm{s} \\ &\approx 84 \text{ minutes}. \end{aligned} \tag{5}$$

The one-way time to the other end of the tube is therefore 42 minutes. That's quick! Note that this result is independent of where chord is. The chord can be a diameter of the earth, or it can be a straight tube spanning the 20-foot width of a room. Neglecting friction, an object will take 42 minutes to go from one side of the room to the other. The very slight component of gravity along the tube is enough to do the trick.