Solution

Week 42 (6/30/03)

How much change?

If the item costs between $N/2$ and N dollars, then you can buy only one item. These extremes will produce remainders of $N/2$ and 0, respectively. The average amount of money left over in this region, which has length $N(1-1/2)=N/2$, is therefore $N/4$.

Likewise, if the item costs between $N/3$ and $N/2$ dollars, then you can buy only two items. These extremes will produce remainders of $N/3$ and 0, respectively. The average amount of money left over in this region, which has length $N(1/2-1/3)=N/6$, is therefore $N/6$.

Continuing in this manner, we see that if the item costs between $N/(n+1)$ and N/n, then you can buy only n items. These extremes will produce remainders of $N/(n+1)$ and 0, respectively. The average amount of money left over in this region, which has length $N(1/n-1/(n+1))=N/n(n+1)$, is therefore $N/2(n+1)$.

The expected amount, M, of money left over is therefore

$$\begin{aligned}
M &= \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)\frac{N}{2(n+1)} \\
&= \frac{N}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)}-\frac{1}{(n+1)^2}\right) \\
&= \frac{N}{2}\sum_{n=1}^{\infty}\left(\left[\frac{1}{n}-\frac{1}{n+1}\right]-\frac{1}{(n+1)^2}\right) \\
&= \frac{N}{2}\left(1-\left(\frac{\pi^2}{6}-1\right)\right) \\
&= N\left(1-\frac{\pi^2}{12}\right).
\end{aligned}$$

In the third line, we have used the fact that the sum in brackets telescopes to 1, and also that $\sum_{n=1}^{\infty}1/n^2=\pi^2/6$. Since $\pi^2/12\approx 0.82$, the amount of money left over is roughly $(0.18)N$ dollars. Note that what we have essentially done in this problem is find the area under the graph in the following figure.