Solution

Week 44 (7/14/03)

Relatively prime numbers

The probability that two random numbers both have a given prime p as a factor is $1/p^2$. The probability that they do not have p as a common factor is thus $1 - 1/p^2$. Therefore, the probability that two numbers have no common prime factors is

$$P = (1 - 1/2^2)(1 - 1/3^2)(1 - 1/5^2)(1 - 1/7^2)(1 - 1/11^2)\cdots. \qquad (1)$$

Using

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad (2)$$

this can be rewritten as

$$P = \Big((1 + 1/2^2 + 1/2^4 + \cdots)(1 + 1/3^2 + 1/3^4 + \cdots)\cdots\Big)^{-1}. \qquad (3)$$

By the Unique Factorization Theorem (every positive integer is expressible as the product of primes in exactly one way), we see that the previous expression is equivalent to

$$P = (1 + 1/2^2 + 1/3^2 + 1/4^2 + 1/5^2 + 1/6^2 + \cdots)^{-1}. \qquad (4)$$

And since the sum of the squares of the reciprocals of all of the positive integers is known to be $\pi^2/6$, the desired probability is $P = 6/\pi^2 \approx 61\%$.

Remarks:

1. The probability that n random numbers all have a given prime p as a factor is $1/p^n$. So the probability that they do not all have p as a common factor is $1 - 1/p^n$. In exactly the same manner as above, we find that the probability, P_n, that n numbers have no common factor among all of them is

$$P_n = (1 + 1/2^n + 1/3^n + 1/4^n + 1/5^n + 1/6^n + \cdots)^{-1}. \qquad (5)$$

This is, by definition, the Riemann zeta function, $\zeta(n)$. It can be calculated exactly for even values of n, but only numerically for odd values. For the case of $n = 4$, we can use the known value $\zeta(4) = \pi^4/90$ to see that the probability that four random numbers do not all have a common factor is $P_4 = 90/\pi^4 \approx 92\%$.

2. We can also perform the somewhat silly exercise of applying this result to the case of $n = 1$. The question then becomes: What is the probability, P_1, that a randomly chosen positive integer does not have a factor? Well, 1 is the only positive integer without any factors, so the probability is $1/\infty = 0$. And indeed,

$$\begin{aligned} P_1 &= (1 - 1/2)(1 - 1/3)(1 - 1/5)(1 - 1/7)\ldots \\ &= (1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6\cdots)^{-1} \\ &= 1/\infty, \end{aligned} \qquad (6)$$

because the sum of the reciprocals of all of the positive integers diverges.

3. Let $\phi(n)$ equal the number of integers less than n that are relatively prime to n. Then $\phi(n)/n$ equals the probability that a randomly chosen integer is relatively prime to n. (This is true because any integer is relatively prime to n if and only if its remainder, when divided by n, is relatively prime to n.) The result of our original problem therefore tells us that the average value of $\phi(n)/n$ is $6/\pi^2$.

4. To be precise about what we mean by probabilities in this problem, we really should word the question as: Let N be a very large integer. Pick two random integers less than or equal to N. What is the probability that these numbers are relatively prime, in the limit where N goes to infinity?

   The solution would then be slightly modified, in that the relevant primes p would be cut off at N, and "edge effects" due to the finite size of N would have to be considered. It is fairly easy to see that the answer obtained in this limit is the same as the answer obtained above.