Solution

Week 57 (10/13/03)

Throwing a beach ball

On both the way up and the way down, the total force on the ball is

$$F = -mg - m\alpha v. \tag{1}$$

On the way up, v is positive, so the drag force points downward, as it should. And on the way down, v is negative, so the drag force points upward.

Our strategy for finding v_f will be to produce two different expressions for the maximum height, h, and then equate them. We'll find these two expressions by considering the upward and then the downward motion of the ball. In doing so, we will need to write the acceleration of the ball as

$$a = \frac{dv}{dt} = \frac{dy}{dt}\frac{dv}{dy} = v\frac{dv}{dy}\,. \tag{2}$$

For the upward motion, $F = ma$ gives

$$\begin{aligned} -mg - m\alpha v &= mv\frac{dv}{dy} \\ \Longrightarrow \qquad \int_0^h dy &= -\int_{v_0}^0 \frac{v\,dv}{g+\alpha v}\,. \end{aligned} \tag{3}$$

where we have taken advantage of the fact that we know that the speed of the ball at the top is zero. Writing $v/(g+\alpha v)$ as $[1 - g/(g+\alpha v)]/\alpha$, we may evaluate the integral to obtain

$$h = \frac{v_0}{\alpha} - \frac{g}{\alpha^2}\ln\left(1 + \frac{\alpha v_0}{g}\right). \tag{4}$$

Now let us consider the downward motion. Let v_f be the final speed, which is a positive quantity. The final velocity is then the negative quantity, $-v_f$. Using $F = ma$, we similarly obtain

$$\int_h^0 dy = -\int_0^{-v_f} \frac{v\,dv}{g+\alpha v}\,. \tag{5}$$

Performing the integration (or just replacing the v_0 in eq. (4) with $-v_f$) gives

$$h = -\frac{v_f}{\alpha} - \frac{g}{\alpha^2}\ln\left(1 - \frac{\alpha v_f}{g}\right). \tag{6}$$

Equating the expressions for h in eqs. (4) and (6) gives an implicit equation for v_f in terms of v_0,

$$v_0 + v_f = \frac{g}{\alpha}\ln\left(\frac{g+\alpha v_0}{g-\alpha v_f}\right). \tag{7}$$

Remarks: In the limit of small α (more precisely, in the limit $\alpha v_0/g \ll 1$), we can use $\ln(1+x) = x - x^2/2 + \cdots$ to obtain approximate values for h in eqs. (4) and (6). The results are, as expected,

$$h \approx \frac{v_0^2}{2g}\,, \qquad \text{and} \qquad h \approx \frac{v_f^2}{2g}\,. \tag{8}$$

We can also make approximations for large α (or large $\alpha v_0/g$). In this limit, the log term in eq. (4) is negligible, so we obtain $h \approx v_0/\alpha$. And eq. (6) gives $v_f \approx g/\alpha$, because the argument of the log must be very small in order to give a very large negative number, which is needed to produce a positive h on the left-hand side. There is no way to relate v_f and h is this limit, because the ball quickly reaches the terminal velocity of $-g/\alpha$ (which is the velocity that makes the net force equal to zero), independent of h.

Let's now find the times it takes for the ball to go up and to go down. We'll present two methods for doing this.

First method: Let T_1 be the time for the upward path. If we write the acceleration of the ball as $a = dv/dt$, then $F = ma$ gives

$$\begin{aligned} -mg - m\alpha v &= m\frac{dv}{dt} \\ \Longrightarrow \qquad \int_0^{T_1} dt &= -\int_{v_0}^{0} \frac{dv}{g+\alpha v}\,. \end{aligned} \tag{9}$$

$$T_1 = \frac{1}{\alpha}\ln\left(1+\frac{\alpha v_0}{g}\right). \tag{10}$$

In a similar manner, we find that the time T_2 for the downward path is

$$T_2 = -\frac{1}{\alpha}\ln\left(1-\frac{\alpha v_f}{g}\right). \tag{11}$$

Therefore,

$$T_1 + T_2 = \frac{1}{\alpha}\ln\left(\frac{g+\alpha v_0}{g-\alpha v_f}\right). \tag{12}$$

Using eq. (7), we have

$$T_1 + T_2 = \frac{v_0+v_f}{g}\,. \tag{13}$$

This is shorter than the time in vacuum (namely $2v_0/g$) because $v_f < v_0$.

Second method: The very simple form of eq. (13) suggests that there is a cleaner way to calculate the total time of flight. And indeed, if we integrate $m\,dv/dt = -mg - m\alpha v$ with respect to time on the way up, we obtain $-v_0 = -gT_1 - \alpha h$ (because $\int v\,dt = h$). Likewise, if we integrate $m\,dv/dt = -mg - m\alpha v$ with respect to time on the way down, we obtain $-v_f = -gT_2 + \alpha h$ (because $\int v\,dt = -h$). Adding these two results gives eq. (13). This procedure only works, of course, because the drag force is proportional to v.

Remarks: The fact that the time here is shorter than the time in vacuum isn't obvious. On one hand, the ball doesn't travel as high in air as it would in vacuum (so you might think that $T_1 + T_2 < 2v_0/g$). But on the other hand, the ball moves slower in air (so you might think that $T_1 + T_2 > 2v_0/g$). It isn't obvious which effect wins, without doing a calculation.

For any α, you can use eq. (10) to show that $T_1 < v_0/g$. But T_2 is harder to get a handle on, because it is given in terms of v_f. But in the limit of large α, the ball quickly reaches terminal velocity, so we have $T_2 \approx h/v_f \approx (v_0/\alpha)/(g/\alpha) = v_0/g$. Interestingly, this is the same as the downward (and upward) time for a ball thrown in vacuum.