Solution

Week 79 (3/15/04)

Propelling a car

Let the speed of the car be $v(t)$. Consider the time interval when a mass dm enters the car. Conservation of momentum gives

$$\begin{aligned} (dm)u + mv &= (m+dm)(v+dv) \\ \Longrightarrow \quad dm(u-v) &= m\,dv, \end{aligned} \tag{1}$$

where we have dropped the second-order $dm\,dv$ term. Separating variables and integrating gives

$$\begin{aligned} \int_M^m \frac{dm}{m} = \int_0^v \frac{dv}{u-v} \qquad &\Longrightarrow \qquad \ln\left(\frac{m}{M}\right) = -\ln\left(\frac{u-v}{u}\right) \\ &\Longrightarrow \qquad m = \frac{Mu}{u-v}\,. \end{aligned} \tag{2}$$

Note that $m \to \infty$ as $v \to u$, as it should.

How does m depend on time? Mass enters the car at a rate $\sigma(u-v)/u$, because although you throw the balls at speed u, the relative speed of the balls and the car is only $(u-v)$. Therefore,

$$\frac{dm}{dt} = \frac{(u-v)\sigma}{u}\,. \tag{3}$$

Substituting the m from eq. (2) into this equation gives

$$\begin{aligned} \int_0^v \frac{dv}{(u-v)^3} &= \int_0^t \frac{\sigma\,dt}{Mu^2} \\ \Longrightarrow \quad \frac{1}{2(u-v)^2} - \frac{1}{2u^2} &= \frac{\sigma t}{Mu^2} \\ \Longrightarrow \quad v(t) &= u\left(1 - \frac{1}{\sqrt{1+\frac{2\sigma t}{M}}}\right). \end{aligned} \tag{4}$$

Note that $v \to u$ as $t \to \infty$, as it should. Integrating this speed to obtain the position gives

$$x(t) = ut - \frac{Mu}{\sigma}\sqrt{1+\frac{2\sigma t}{M}} + \frac{Mu}{\sigma}\,. \tag{5}$$

We see that even though the speed approaches u, the car will eventually be an arbitrarily large distance behind a ball with constant speed u. For example, pretend that the first ball missed the car and continued to travel forward at speed u.