

Solution

Week 8 (11/4/02)

Sub-rectangles

Put the rectangle in the x - y plane, with its sides parallel to the x and y axes, and consider the function,

$$f(x, y) = e^{2\pi ix} e^{2\pi iy}. \quad (1)$$

Claim: *The integral of $f(x, y)$ over a rectangle, whose sides are parallel to the axes, is zero if and only if at least one pair of sides has integer length.*

Proof:

$$\begin{aligned} \int_a^b \int_c^d e^{2\pi ix} e^{2\pi iy} dx dy &= \int_a^b e^{2\pi ix} dx \int_c^d e^{2\pi iy} dy \\ &= -\frac{1}{4\pi^2} (e^{2\pi ib} - e^{2\pi ia}) (e^{2\pi id} - e^{2\pi ic}) \\ &= -\frac{e^{2\pi ia} e^{2\pi ic}}{4\pi^2} (e^{2\pi i(b-a)} - 1) (e^{2\pi i(d-c)} - 1). \end{aligned} \quad (2)$$

This equals zero if and only if at least one of the factors in parentheses is zero. In other words, it equals zero if and only if at least one of $(b - a)$ and $(d - c)$ is an integer. ■

Since we are told that each of the smaller rectangles has at least one integer pair of sides, the integral of f over each of these smaller rectangles is zero. Therefore, the integral of f over the whole rectangle is also zero. Hence, from the Claim, we see that the whole rectangle has at least one integer pair of sides.

REMARKS:

1. This same procedure works for the analogous problem in higher dimensions. Given an N -dimensional rectangular parallelepiped which is divided into smaller ones, each of which has the property that at least one edge has integer length, then the original parallelepiped must also have this property. In three dimensions, for example, this follows from considering integrals of the function $f(x, y, z) = e^{2\pi ix} e^{2\pi iy} e^{2\pi iz}$, in the same manner as above.
2. We can also be a bit more general and make the following statement. Given an N -dimensional rectangular parallelepiped which is divided into smaller ones, each of which has the property that at least n “non-equivalent” edges (that is, ones that aren’t parallel) have integer lengths, then the original parallelepiped must also have this property.

This follows from considering the $\binom{N}{m}$ functions (where $m \equiv N + 1 - n$) of the form,

$$f(x_{j_1}, x_{j_2}, \dots, x_{j_m}) = e^{2\pi ix_{j_1}} e^{2\pi ix_{j_2}} \dots e^{2\pi ix_{j_m}}, \quad (3)$$

where the indices j_1, j_2, \dots, j_m are a subset of the indices $1, 2, \dots, N$. (For example, the previous remark deals with $n = 1$, $m = N$, and only $\binom{N}{N} = 1$ function.) We’ll leave it to you to show that the integrals of these $\binom{N}{m}$ functions, over an N -dimensional rectangular parallelepiped, are *all* equal to zero if and only if at least n “non-equivalent” edges of the parallelepiped have integer length.