Solution

Week 82 (4/5/04)

Standing in a line

First solution: Let T_N be the expected number of people who are able to make the given statement. If we consider everyone except the last person in line, then this group of $N-1$ people has by definition T_{N-1} people who are able to make the statement. Let us now add on the last person. There is a $1/N$ chance that she is the tallest, in which case she is able to make the statement. Otherwise she cannot. Therefore, we have

$$T_N = T_{N-1} + \frac{1}{N}\,. \tag{1}$$

Starting with $T_1 = 1$, we therefore inductively find

$$T_N = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}\,. \tag{2}$$

For large N, this goes like $\ln N$, which grows very slowly with N.

Second solution: Let T_N be the desired average. Consider the location of the tallest person. If he is the last person in the line (which occurs with probability $1/N$), then the problem reduces to that for the $N-1$ people in front of him. So in this case, we can expect $1+T_{N-1}$ people who are able to make the given statement.

If the tallest person is the second to last person in the line (which occurs with probability $1/N$), then the problem reduces to that for the $N-2$ people in front of him (because the person behind him is not able to make the statement). So in this case, we can expect $1+T_{N-2}$ people who are able to make the given statement.

Continuing along these lines, and adding up all N possibilities for the location of the tallest person, we find

$$\begin{aligned} T_N &= \frac{1}{N}\Big((1+T_{N-1}) + (1+T_{N-2}) + \cdots + (1+T_1) + (1+T_0)\Big) \\ \Longrightarrow \quad NT_N &= N + T_{N-1} + T_{N-2} + \cdots + T_1. \end{aligned} \tag{3}$$

Writing down the analogous equation for $N-1$,

$$(N-1)T_{N-1} = (N-1) + T_{N-2} + T_{N-2} + \cdots + T_1, \tag{4}$$

and then subtracting this from eq. (3), yields

$$T_N = T_{N-1} + \frac{1}{N}\,, \tag{5}$$

which agrees with the first solution.