Solution

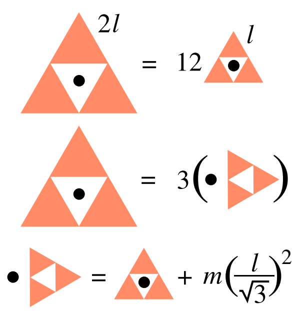
Week 9 (11/11/02)

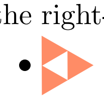
Fractal moment

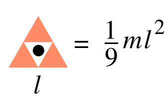
The strategy here will be to compute the moment of inertia by using scaling arguments, along with the parallel-axis theorem. To do this, we will need to compare the I for our triangle of side ℓ to that of a triangle of side 2ℓ. So let us scale up our triangle by a factor of 2 and examine what happens to the integral $I = \int r^2\, dm$. We get a simple factor of 2^2 from the r^2, but happens to the dm?

A solid triangle would yield a factor of $2^2 = 4$ in the dm (since area is proportional to length squared), but our fractal object is a bit different. The mass scales in a strange way. Doubling the size of our triangle increases its mass by a factor of only 3. This is true because the doubled triangle is simply made up of *three* of the smaller ones, plus an empty triangle in the middle. Thus, the dm picks up a factor of 3, and so the I for a fractal triangle of side 2ℓ is $4 \cdot 3 = 12$ times that of a fractal triangle of side ℓ (where the axes pass through any two corresponding points).

In what follows, we'll use pictures to denote the I's of the fractal objects around the dots shown. In terms of these pictures, we have,



The first line comes from the scaling argument, the second is obvious (moments of inertia simply add), and the third comes from the parallel-axis theorem (you can show that the distance between the dots is $\ell/\sqrt{3}$). Equating the right-hand sides of the first two equations, and then using the third to eliminate  , gives



Remarks: This result is larger than the I for a uniform triangle (which happens to be $m\ell^2/12$), because the mass is generally further away from the center in the fractal case.

When we increase the side length of our fractal triangle by a factor of 2, the factor of 3 in the dm is between the factor of $2^1 = 2$ relevant to a one-dimensional object, and the factor of $2^2 = 4$ relevant to a two-dimensional object. So in some sense our object has a dimension between 1 and 2. It is reasonable to define the dimension, d, of an object as the number for which f^d is the increase in "volume" when the dimensions are increased by a factor f. For our fractal triangle, we have $2^d = 3$, and so $d = \log_2 3 \approx 1.58$.