Solution

Week 31 (4/14/03)

Simultaneous claps

First Solution: The relative speed of A and B is obtained from the velocity-addition formula, which yields

$$\beta_{\text{rel}} = \frac{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}}{2}\,. \tag{1}$$

In B's frame, A's clock runs slow by a factor $\gamma = 1/\sqrt{1-\beta_{\text{rel}}^2} = 2$. Hence, if A claps her hands when her clock reads T (which happens to be $L/\sqrt{2}c$, but we won't need the actual value), then B claps her hands when her clock reads $2T$. But likewise, in A's frame, B's clock runs slow by a factor $\gamma = 2$. Hence, A will make her second clap at $4T$, and so on. A and B therefore increase their separation by a factor of 4 between successive claps of A. Their separation after A's nth clap therefore equals

$$d_n = 4^{n-1}L. \tag{2}$$

In the case where A and B move at a general speed v, their relative speed equals

$$\beta_{\text{rel}} = \frac{2\beta}{1+\beta^2}\,. \tag{3}$$

The associated γ factor is

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{2\beta}{1+\beta^2}\right)^2}} = \frac{1+\beta^2}{1-\beta^2}\,. \tag{4}$$

From the reasoning above, the separation after A's nth clap equals

$$d_n = L\left(\frac{1+\beta^2}{1-\beta^2}\right)^{2(n-1)}\,. \tag{5}$$

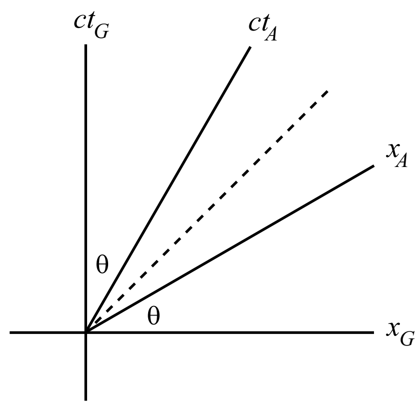
This agrees with eq. (2) when $\beta = 1/\sqrt{3}$.

Second Solution: We can also solve this problem by using Minkowski diagrams. Such diagrams show what the x and ct axes of one frame look like with respect to the x and ct axes of another frame. As we will see below, these diagrams make things very easy to visualize, and provide an easy geometrical way of determining various quantities.

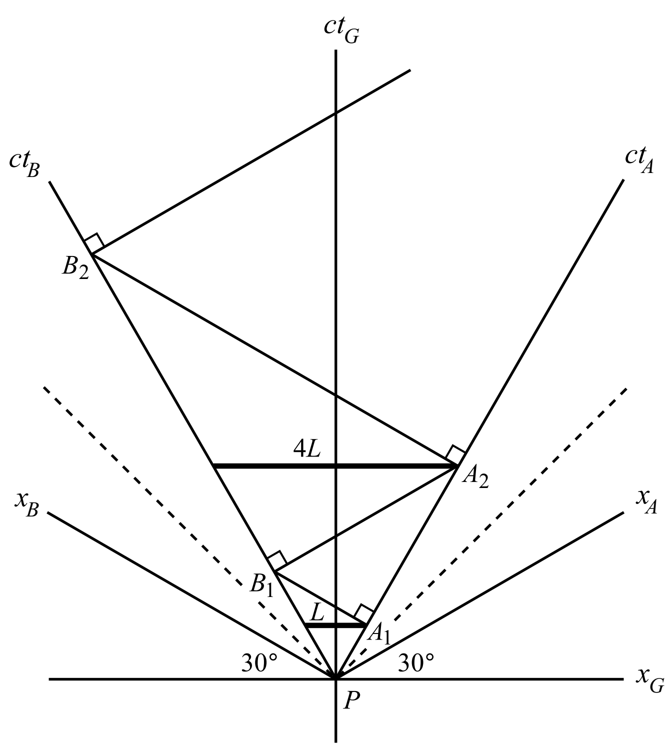
A basic Minkowski diagram is shown below. This diagram is what arises when A travels to the right at speed v. Her x_A and ct_A axes are tilted inward at an angle θ with respect to the x_G and ct_G axes of the ground frame. The angle θ satisfies $\tan\theta = v/c$. These facts (and many others) of a Minkowski diagram can be derived from the Lorentz transformations,

$$\begin{aligned} x_A &= \gamma(x_G - \beta c t_G) \\ ct_A &= \gamma(ct_G - \beta x_G), \end{aligned} \tag{6}$$

where $\beta \equiv v/c$.



An important point to note in this diagram is that all events on the x_A-axis have $t_A = 0$. Therefore, all events on the x_A-axis are simultaneous in A's frame. Likewise, all events on any line parallel to the x_A-axis are simultaneous with each other in A's frame. Armed with this fact, we can easily solve the given problem. The following diagram shows what the axes of A's and B's frames look like with respect to the ground frame.



A makes her first clap at event A_1. At this event, she is a distance L from B, as measured in the ground frame. B's first clap (which occurs simultaneously with A_1, as measured by B) is obtained by drawing a line through A_1 parallel to the x_B-axis. Since the x_B-axis is perpendicular to the ct_A-axis (due to the $30° = \tan^{-1}(1/\sqrt{3})$ angles in the diagram), we obtain the right angle shown.

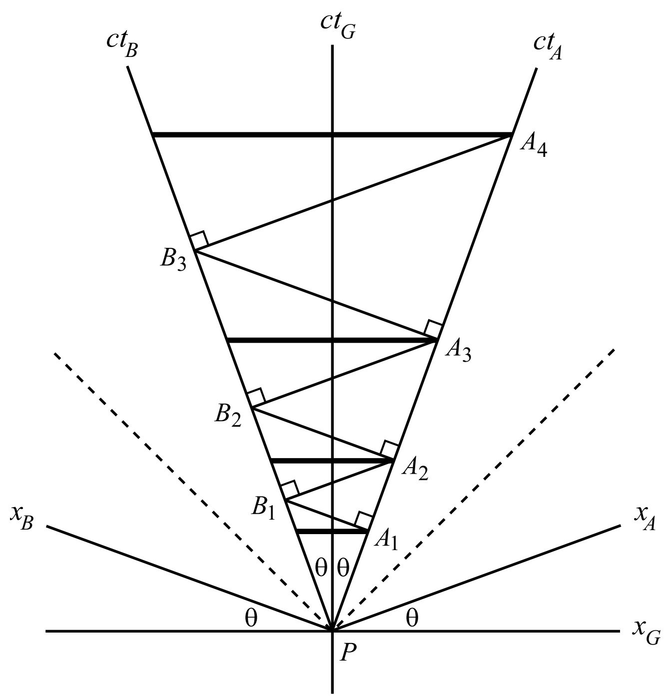
Similarly, A's second clap (which occurs simultaneously with B_1, as measured by A) is obtained by drawing a line through B_1 parallel to the x_A-axis. Since the x_A-axis is perpendicular to the ct_B-axis, we obtain the right angle shown.

Continuing in this manner, we can locate all subsequent claps. Making use of the plethora of 30°-60°-90° triangles in the figure, we see that $PB_1 = 2PA_1$, and $PA_2 = 2PB_1$, and so on. Therefore, $PA_2 = 4PA_1$, and $PA_3 = 4PA_2$, and so on. Looking now at the equilateral triangles whose top sides (the bold lines in the figure) give the distances between A and B (as measured in the ground frame), we see that these distances increase by a factor of 4 during each interval between A's claps. The separation after A's nth clap therefore equals

$$d_n = 4^{n-1}L, \tag{7}$$

in agreement with eq. (2).

In the case where A and B move at a general speed v, we obtain the following figure.



Using the same reasoning above, but now with a plethora of right triangles with an angle of 2θ, we see that $PA_2 = PA_1/\cos^2 2\theta$, and so on. Looking now at the isosceles triangles with vertex angle 2θ, whose top sides (the bold lines in the figure) give the distances between A and B (as measured in the ground frame), we see that these distances increase by a factor of $1/\cos^2 2\theta$ during each interval between A's claps. The separation after A's nth clap therefore equals

$$d_n = \frac{L}{(\cos 2\theta)^{2(n-1)}} \,. \tag{8}$$

Using $\tan\theta = \beta$, you can show that $\cos 2\theta = (1-\beta^2)/(1+\beta^2)$. Therefore,

$$d_n = L\left(\frac{1+\beta^2}{1-\beta^2}\right)^{2(n-1)} , \tag{9}$$

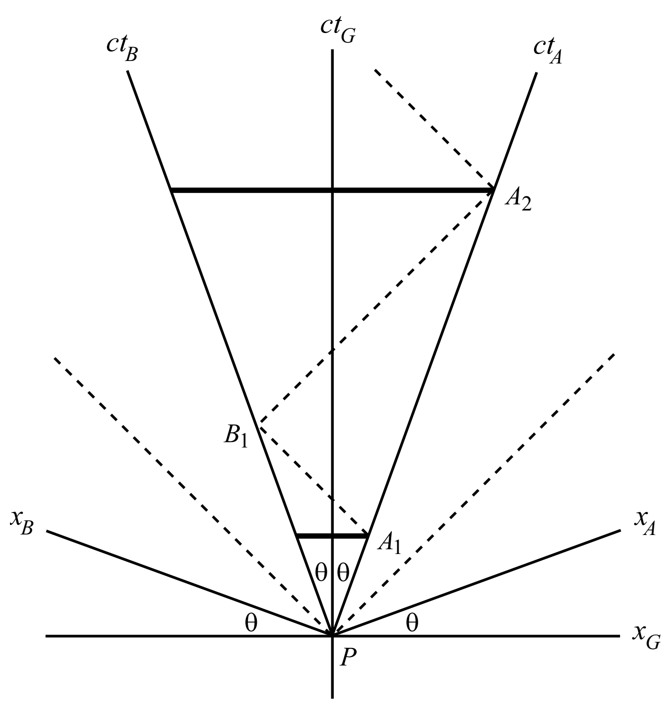
in agreement with eq. (5).

REMARK: In the above problem, each person clapped her hands simultaneously (as measured by her) with the other person's clap. We may also consider the (entirely different) setup where each person claps her hands when she *sees* the other person's clap. In this new setup, we are concerned with the actual travel of photons from one person to the other, whereas in the original problem, the travel of any photons was irrelevant.

The ratio of the B's clock reading when she makes her first clap, to A's clock reading when she makes her first clap, is simply the (inverse of the) longitudinal doppler factor $\sqrt{(1-\beta_{\rm rel})/(1+\beta_{\rm rel})}$. (This is clear if you imagine A continually sending out a series of light flashes to B.) Using the $\beta_{\rm rel}$ from eq. (3), we find this ratio to be $(1+\beta)/(1-\beta)$. We can use this same reasoning on successive claps, and so the separation between A and B after A's nth clap equals

$$d_n = L\left(\frac{1+\beta}{1-\beta}\right)^{2(n-1)}. \qquad (10)$$

We can also solve this new setup by using a Minkowski diagram. The solution proceeds just as it did with the Minkowski diagrams above, except that now we must draw the 45° lines which describe a photon's travel, instead of the lines of simultaneity (which were parallel to the x_A and x_B axes) that we drew above. These 45° lines appear as follows.



We must find the ratio of PB_1 to PA_1. You can show that the law of signs in triangle PB_1A_1 gives

$$\frac{PB_1}{PA_1} = \frac{\sin(135^\circ - \theta)}{\sin(45^\circ - \theta)}. \qquad (11)$$

Using the trig sum formula for sine, we find

$$\frac{PB_1}{PA_1} = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \frac{1+\tan\theta}{1-\tan\theta} = \frac{1+\beta}{1-\beta}. \qquad (12)$$

Likewise, this is the ratio of PA_2 to PB_1. Therefore, $PA_2/PA_1 = (1+\beta)^2/(1-\beta)^2$, and so the separation between A and B after A's nth clap equals

$$d_n = L\left(\frac{1+\beta}{1-\beta}\right)^{2(n-1)}, \tag{13}$$

in agreement with eq. (10). As you can check, this is larger than the answer to the original problem, given in eq. (9). This must be the case, because the photons take some nonzero time to travel between the A and B.