Solution

Week 56   (10/6/03)

Stirling's formula

Let's first prove the result,

$$N! = \int_0^\infty x^N e^{-x} dx \equiv I_N. \tag{1}$$

The proof by induction proceeds as follows. Integrating by parts gives

$$\int_0^\infty x^N e^{-x} dx = -x^N e^{-x} dx \Big|_0^\infty + N \int_0^\infty x^{N-1} e^{-x} dx. \tag{2}$$

The first term on the right-hand side is zero, so we have $I_N = N I_{N-1}$. Therefore, if $I_{N-1} = (N-1)!$, then $I_N = N!$. Since it is indeed true that $I_0 = 0! = 1$, we see that $I_N = N!$ for all N.

Let us now write $x^N e^{-x}$ as $\exp(N \ln x - x) \equiv \exp(f(x))$, and then expand $f(x)$ in a Taylor series about its maximum, which occurs at $x = N$. Computing the first two derivatives of $f(x)$, evaluated at N, gives the following Taylor series in the exponent.

$$
\begin{aligned}
N! &= \int_0^\infty \exp\left(-N + N\ln N - \frac{(x-N)^2}{2N} + \cdots \right) dx \\
&\approx N^N e^{-N} \int_0^\infty \exp\left(-\frac{(x-N)^2}{2N} \right) dx.
\end{aligned}
\tag{3}
$$

If N is very large, we can let the integral run from $-\infty$ to ∞, with negligible error. Letting $y \equiv x - N$, we have

$$
\begin{aligned}
N! &\approx N^N e^{-N} \int_{-\infty}^\infty e^{-y^2/2N} dy \\
&= N^N e^{-N} \sqrt{2\pi N},
\end{aligned}
\tag{4}
$$

as desired.

REMARK: We have used the fact that $\int_{-\infty}^\infty e^{-y^2} dy = \sqrt{\pi}$. This can be proved in the following way, where we make use of a change of variables from cartesian to polar coordinates. Let I be the desired integral. Then

$$
\begin{aligned}
I^2 &= \int_{-\infty}^\infty e^{-x^2} dx \int_{-\infty}^\infty e^{-y^2} dy \\
&= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)} dx\, dy \\
&= \int_0^{2\pi} \int_0^\infty e^{-r^2} r\, dr\, d\theta \\
&= 2\pi \left(-\frac{e^{-r^2}}{2} \right) \Big|_0^\infty \\
&= \pi.
\end{aligned}
\tag{5}
$$

A change of variables then gives $\int_{-\infty}^{\infty} e^{-y^2/n}\,dy = \sqrt{n\pi}$.

The calculation of the higher-order corrections is a bit messier, because we have to keep track of more terms in the Taylor expansion of $f(x)$. Let's find the order-$1/N$ correction first. Our strategy will be to write the integrand in eq. (1) as a gaussian plus small corrections. Computing the first four derivatives of $f(x)$, evaluated at N, gives us the following Taylor series in the exponent (letting $y \equiv x - N$, and letting the limits of integration run from $-\infty$ to ∞).

$$
\begin{aligned}
N! &= \int_{\infty}^{\infty} \exp\left(-N + N \ln N - \frac{y^2}{2N} + \frac{y^3}{3N^2} - \frac{y^4}{4N^3} + \cdots\right) dy \\
&\approx N^N e^{-N} \int_{\infty}^{\infty} \exp\left(-\frac{y^2}{2N}\right) \exp\left(\frac{y^3}{3N^2} - \frac{y^4}{4N^3}\right) dy \\
&\approx N^N e^{-N} \int_{\infty}^{\infty} \exp\left(-\frac{y^2}{2N}\right) \left(1 + \left[\frac{y^3}{3N^2} - \frac{y^4}{4N^3}\right]\right. \\
&\qquad\qquad\qquad\qquad\qquad\left. + \frac{1}{2}\left[\frac{y^3}{3N^2} - \frac{y^4}{4N^3}\right]^2 + \cdots\right) dy.
\end{aligned}
$$
(6)

Since terms with odd powers of y integrate to zero, we obtain (to leading orders in $1/N$),

$$
N! \approx N^N e^{-N} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2N}\right)\left(1 - \frac{y^4}{4N^3} + \frac{1}{2}\left[\frac{y^3}{3N^2}\right]^2 + \cdots\right) dy. \tag{7}
$$

At this point, we need to know how to calculate integrals of the form $\int_{-\infty}^{\infty} x^{2n} e^{-ax^2}\,dx$. Using $\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\pi}a^{-1/2}$, and successively differentiating with respect to a, we obtain

$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{-ax^2}\,dx &= \sqrt{\pi}a^{-1/2}, \\
\int_{-\infty}^{\infty} x^2 e^{-ax^2}\,dx &= \frac{1}{2}\sqrt{\pi}a^{-3/2}, \\
\int_{-\infty}^{\infty} x^4 e^{-ax^2}\,dx &= \frac{3}{4}\sqrt{\pi}a^{-5/2}, \\
\int_{-\infty}^{\infty} x^6 e^{-ax^2}\,dx &= \frac{15}{8}\sqrt{\pi}a^{-7/2}.
\end{aligned}
$$
(8)

Letting $a \equiv 1/2N$ here, eq. (7) gives

$$
\begin{aligned}
N! &\approx N^N e^{-N} \sqrt{\pi}\left((2N)^{1/2} - \frac{1}{4N^3}\frac{3}{4}(2N)^{5/2} + \frac{1}{18N^4}\frac{15}{8}(2N)^{7/2}\right) \\
&= N^N e^{-N} \sqrt{2\pi N}\left(1 + \frac{1}{12N}\right).
\end{aligned}
$$
(9)

Note that to obtain all the terms of order $1/N$, it is necessary to include the $(y^3/3N^2)^2$ term in eq. (7). This is an easy term to forget.

If you like these sorts of calculations, you can go a step further and find the order-$1/N^2$ correction. It turns out that you have to keep terms out to the $-y^6/6N^5$ term

in the expansion in the first line of eq. (6). Furthermore, you must keep terms out to the $[\cdots]^4/4!$ term in the expansion in the last line of eq. (6). The relevant extra terms that take the place of the "$\cdots$" in eq. (7) then turn out to be

$$\left[-\frac{y^6}{6N^5}\right] + \frac{1}{2}\left[\left(-\frac{y^4}{4N^3}\right)^2 + 2\left(\frac{y^3}{3N^2}\right)\left(\frac{y^5}{5N^4}\right)\right]$$
$$+\frac{1}{3!}\left[3\left(\frac{y^3}{3N^2}\right)^2\left(-\frac{y^4}{4N^3}\right)\right] + \frac{1}{4!}\left[\left(\frac{y^3}{3N^2}\right)^4\right], \tag{10}$$

where we have grouped these terms via square brackets according to which term in the e^z series expansion in the last line of eq. (6) they come from. To do all of the necessary integrals in the modified eq. (7), we'll need the next three integrals in the list in eq. (8). They are

$$
\begin{aligned}
\int_{-\infty}^{\infty} x^8 e^{-ax^2}\,dx &= \frac{3\cdot5\cdot7}{2^4}\sqrt{\pi}a^{-9/2}, \\
\int_{-\infty}^{\infty} x^{10} e^{-ax^2}\,dx &= \frac{3\cdot5\cdot7\cdot9}{2^5}\sqrt{\pi}a^{-11/2}, \\
\int_{-\infty}^{\infty} x^{12} e^{-ax^2}\,dx &= \frac{3\cdot5\cdot7\cdot9\cdot11}{2^6}\sqrt{\pi}a^{-13/2}.
\end{aligned}
\tag{11}
$$

Putting the terms of eq. (10) in place of the "$\cdots$" in eq. (7), we find that the coefficient of $N^N e^{-N}\sqrt{2\pi N}$ equals $1/N^2$ times

$$-\frac{1}{6}\frac{3\cdot5}{2^3}2^3 + \frac{1}{2}\left(\frac{1}{16}+\frac{2}{15}\right)\frac{3\cdot5\cdot7}{2^4}2^4 - \frac{1}{3!}\frac{3}{36}\frac{3\cdot5\cdot7\cdot9}{2^5}2^5 + \frac{1}{4!}\frac{1}{81}\frac{3\cdot5\cdot7\cdot9\cdot11}{2^6}2^6$$
$$= \frac{1}{288}. \tag{12}$$

Therefore, we may write Stirling's formula as

$$N! \approx N^N e^{-N}\sqrt{2\pi N}\left(1 + \frac{1}{12N} + \frac{1}{288N^2}\right). \tag{13}$$

This result of $1/288$ is rather fortuitous, because it is the third term in the Taylor series for $e^{1/12}$. This means that we can write $N!$ as

$$
\begin{aligned}
N! &= N^N e^{-N}\sqrt{2\pi N}\left(e^{1/12N} + \mathcal{O}(1/N^3)\right) \\
&\approx N^N e^{-N+1/12N}\sqrt{2\pi N}.
\end{aligned}
\tag{14}
$$

It turns out that the order-$1/N^3$ correction is *not* equal to $1/(3!\cdot12^3)$, which is the next term in the expansion for $e^{1/12}$.