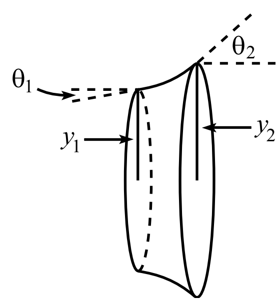
Solution

Week 63 (11/24/03)

Minimal surface

First Solution: By symmetry, the surface is obtained by rotating a certain function $y(x)$ around the x-axis. Our goal is to find $y(x)$. Consider a thin vertical cross-sectional ring of the surface, as shown below. The ratio of the circumferences of the circular boundaries of the ring is y_2/y_1.



The condition that the bubble be in equilibrium is that the tension (force per unit length) throughout the surface is constant, because otherwise there would be a net force on some little patch. Therefore, the requirement that the horizontal forces on the ring cancel is $y_1 \cos\theta_1 = y_2 \cos\theta_2$, where the θ's are the angles of the surface, as shown. In other words, $y\cos\theta$ is constant throughout the surface. But $\cos\theta = 1/\sqrt{1+y'^2}$, so we have

$$\frac{y}{\sqrt{1+y'^2}} = \text{Constant} \qquad \Longrightarrow \qquad 1+y'^2 = By^2, \tag{1}$$

where B is some constant. At this point, motivated by the facts that $1+\sinh^2 z = \cosh^2 z$ and $d(\cosh z)/dz = \sinh z$, we can guess that the solution to this differential equation is

$$y(x) = \frac{1}{b}\cosh b(x+d), \tag{2}$$

where $b = \sqrt{B}$, and d is a constant of integration. Or, we can do things from scratch by solving for $y' \equiv dy/dx$ and then separating variables to obtain (again with $b = \sqrt{B}$)

$$dx = \frac{dy}{\sqrt{(by)^2 - 1}}\,. \tag{3}$$

We can then use the fact that the integral of $1/\sqrt{z^2-1}$ is $\cosh^{-1} z$, to obtain the same result as in eq. (2).

Eq. (2) gives the general solution in the case where the rings may have unequal radii. The constants b and d are determined by the boundary conditions (the facts that the y values equal the radii of the rings at the x-values of the rings). In the special case at hand where the radii are equal, the two boundary conditions give $r = (1/b)\cosh b(\pm\ell + d)$, where $x=0$ has been chosen to be midway between the rings. Therefore, $d=0$, and so the constant b is determined from

$$r = \frac{1}{b}\cosh b\ell. \tag{4}$$

Our solution for $y(x)$ then

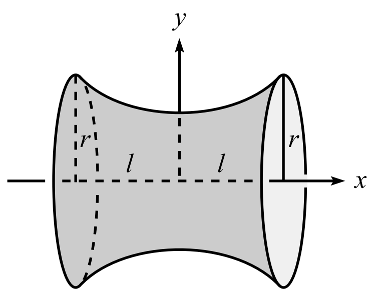
$$y(x) = \frac{1}{b}\cosh bx. \tag{5}$$

There is, however, an ambiguity in this solution, in that there may be two solutions for b in eq. (4). We'll comment on this in the first "Remark" at the end of the problem.

Second Solution: We can also solve this problem by using a "principle of least action" type of argument, which takes advantage of the fact that the surface is the one with the minimum area. There are two ways of going about this. One is quick, and the other is lengthy. A sketch of the lengthy way is the following. The area of the surface in the following figure is

$$\int_{-\ell}^{\ell} 2\pi y\sqrt{1+y'^2}\,dx, \tag{6}$$

where $y' \equiv dy/dx$.



In analogy with the principle of least action, our "Lagrangian" (from a physicist's point of view) is $L = 2\pi y\sqrt{1+y'^2}$, and in order for the area to be minimized, L must satisfy the Euler-Lagrange equation,

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y}\,. \tag{7}$$

It is, alas, rather tedious to work through all of the necessary differentiations here. If you so desire, you can show that this equation does in fact lead to eq. (1). But let's instead just do things the quick way. If we consider x to be a function of y (there's no need to worry about any double-value issues, because the Euler-Lagrange formalism deals with local variations), we may write the area as

$$\int_{-\ell}^{\ell} 2\pi y\sqrt{1+x'^2}\,dy, \tag{8}$$

where $x' \equiv dx/dy$. Our "Lagrangian" is now $L = 2\pi y\sqrt{1+x'^2}$, and the Euler-Lagrange equation gives

$$\frac{d}{dy}\left(\frac{\partial L}{\partial x'}\right) = \frac{\partial L}{\partial x} \qquad \Longrightarrow \qquad \frac{d}{dy}\left(\frac{yx'}{\sqrt{1+x'^2}}\right) = 0. \tag{9}$$

The zero on the right-hand side makes everything nice and easy, because it tells us that $yx'/\sqrt{1+x'^2}$ is constant. Defining this constant to be $1/b$, and then solving for x' and separating variables, gives

$$dx = \frac{dy}{\sqrt{(by)^2 - 1}}, \tag{10}$$

which is identical to eq. (3). The solution then concludes as above.

Let us now determine the maximum value of ℓ/r for which the minimal surface exists. If ℓ/r is too large, then we will see that there is no solution for b in eq. (4). In short, the minimal "surface" turns out to be the two given circles, attached by a line, which isn't a nice two-dimensional surface. If you perform the experiment with soap bubbles (which want to minimize their area), and if you pull the rings too far apart, then the surface will break and disappear, as it tries to form the two circles.

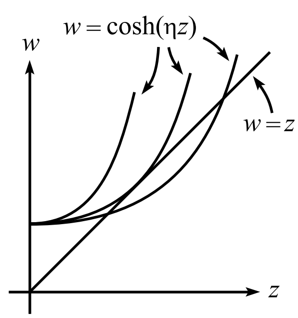
Define the dimensionless quantities,

$$\eta \equiv \frac{\ell}{r}, \qquad \text{and} \qquad z \equiv br. \tag{11}$$

Then eq. (4) becomes

$$z = \cosh \eta z. \tag{12}$$

If we make a rough plot of the graphs of $w = z$ and $w = \cosh \eta z$ for a few values of η, as shown below, we see that there is no solution for z if η is too big. The limiting value of η for which there exists a solution occurs when the curves $w = z$ and $w = \cosh \eta z$ are tangent; that is, when the slopes are equal in addition to the functions being equal.



Let η_0 be the limiting value of η, and let z_0 be the place where the tangency occurs. Then equality of the values and the slopes gives

$$z_0 = \cosh(\eta_0 z_0), \qquad \text{and} \qquad 1 = \eta_0 \sinh(\eta_0 z_0). \tag{13}$$

Dividing the second of these equations by the first gives

$$1 = (\eta_0 z_0) \tanh(\eta_0 z_0). \tag{14}$$

This must be solved numerically. The solution is

$$\eta_0 z_0 \approx 1.200. \tag{15}$$

Plugging this into the second of eqs. (13) gives

$$\left(\frac{\ell}{r}\right)_{\max} \equiv \eta_0 \approx 0.663. \tag{16}$$

Note also that $z_0 = 1.200/\eta_0 = 1.810$. We see that if ℓ/r is larger than 0.663, then there is no solution for $y(x)$ that is consistent with the boundary conditions. Above this value of ℓ/r, the soap bubble minimizes its area by heading toward the shape of just two disks, but it will pop well before it reaches that configuration.

REMARKS:

1. As mentioned at the end of the first solution above, there may be more than one solution for the constant b in eq. (5). In fact, the preceding graph shows that for any $\eta < 0.663$, there are two solutions for z in eq. (12), and hence two solutions for b in eq. (4). This means that there are two possible surfaces that might solve our problem. Which one do we want? It turns out that the surface corresponding to the smaller value of b is the one that minimizes the area, while the surface corresponding to the larger value of b is the one that (in some sense) maximizes the area.

   We say "in some sense" because the surface is actually a saddle point for the area. It can't be a maximum, after all, because we can always make the area larger by adding little wiggles to it. It's a saddle point because there does exist a class of variations for which is has the maximum area, namely ones where the "dip" in the curve is continuously made larger (just imagine lowering the midpoint in a smooth manner). The reason why this curve arises in the first solution above is that we simply demanded that the surface be in equilibrium; it just happens to be an *unstable* equilibrium in this case. The reason why it arises in the second solution above is that the Euler-Lagrange technique simple sets the "derivative" equal to zero and thus does not differentiate between maxima, minima, and saddle points.

2. How does the area of the limiting surface (with $\eta_0 = 0.663$) compare with the area of the two circles? The area of the two circles is

$$A_{\rm c} = 2\pi r^2. \tag{17}$$

   The area of the limiting surface is

$$A_{\rm s} = \int_{-\ell}^{\ell} 2\pi y\sqrt{1+y'^2}\,dx. \tag{18}$$

   Using eq. (5), this becomes

$$\begin{aligned} A_{\rm s} &= \int_{-\ell}^{\ell} \frac{2\pi}{b}\cosh^2 bx\,dx \\ &= \int_{-\ell}^{\ell} \frac{\pi}{b}(1+\cosh 2bx)\,dx \\ &= \frac{2\pi\ell}{b} + \frac{\pi\sinh 2b\ell}{b^2}\,. \end{aligned} \tag{19}$$

   But from the definitions of η and z, we have $\ell = \eta_0 r$ and $b = z_0/r$ for the limiting surface. Therefore, $A_{\rm s}$ can be written as

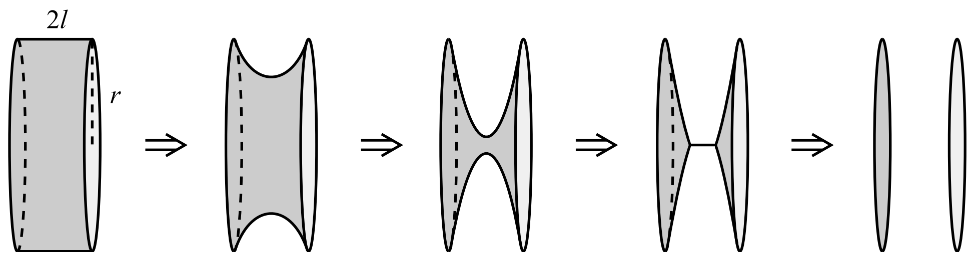
$$A_{\rm s} = \pi r^2\left(\frac{2\eta_0}{z_0} + \frac{\sinh 2\eta_0 z_0}{z_0^2}\right). \tag{20}$$

   Plugging in the numerical values ($\eta_0 \approx 0.663$ and $z_0 \approx 1.810$) gives

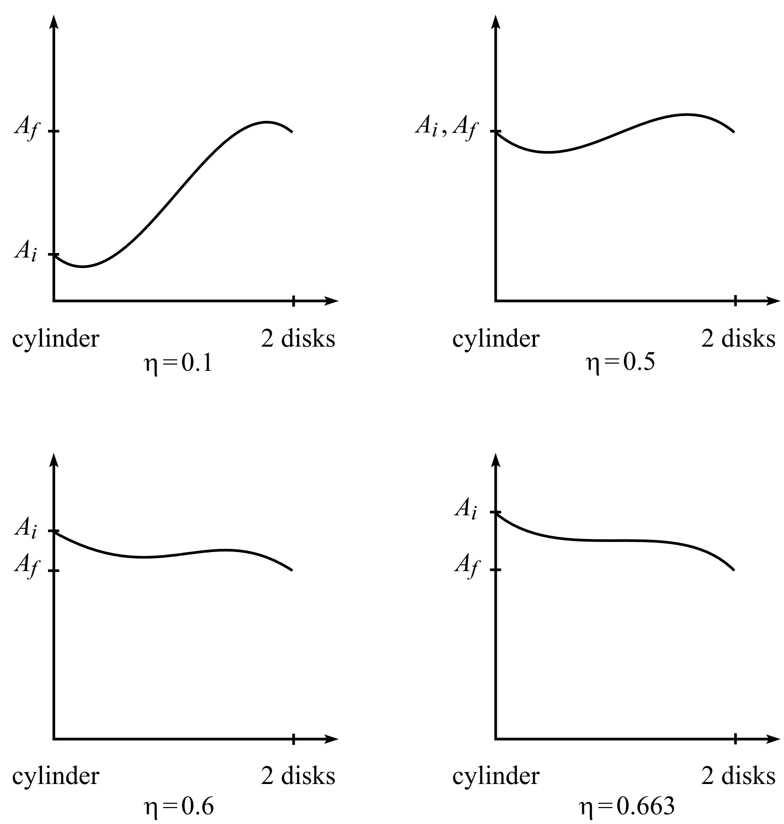
$$A_{\rm c} \approx (6.28)r^2, \qquad \text{and} \qquad A_{\rm s} \approx (7.54)r^2. \tag{21}$$

   The ratio of $A_{\rm s}$ to $A_{\rm c}$ is approximately 1.2 (it's actually $\eta_0 z_0$, as you can show). The limiting surface therefore has a larger area. This is expected, of course, because for $\ell/r > \eta_0$ the surface tries to run off to one with a smaller area, and there are no other stable configurations besides the cosh solution we found.

3. How does the area of the surface change if we gradually transform it from a cylinder to the two disks? There are many ways to go about doing this transformation, but let's just be vague and say that we pick a nice smooth method that passes through the two cosh solutions (if $\eta < 0.663$) that we found above. The transformation might look something like:



The area of the starting cylinder is $A_i = (2\ell)(2\pi r) = 4\pi r\ell$, and the area of the ending two disks is $A_f = 2\pi r^2$. Note that the ratio of these is $A_i/A_f = 2\ell/r \equiv 2\eta$. For a given $\eta \equiv \ell/r$, what does the plot of the changing area look like? Below are four qualitative plots, for four values of η. We've imagined changing η by keeping r (and hence A_f) fixed and changing ℓ. Ignore the actual measure along the axes; just look at the general shape.



We see that $\eta = 0.663$ is the value for which the maximum and minimum merge into one point with zero slope. For higher values of η, there are no points on the curve that have zero slope.

The actual values of the area along these curves is nebulous, because we haven't been quantitative about exactly how we're varying the surface. But the area at the maximum and minimum (at which points we have one of our cosh surfaces) can be found from eq. (20), which says that

$$A_{\rm s} = \pi r^2 \left(\frac{2\eta}{z} + \frac{\sinh 2\eta z}{z^2} \right), \tag{22}$$

for general $\eta \equiv \ell/r$ and $z \equiv br$. For a given η, the two solutions for z are found from eq. (12). Plugging each of these into eq. (22) gives the areas at the maximum and minimum.