Solution

Week 73 (2/2/04)

Chain on a scale

First solution: Let y be the height of the top of the chain, and let F be the desired force applied by the scale. The net force on the whole chain is $F - (\sigma L)g$ (with upward taken to be positive). The momentum of the chain is $(\sigma y)\dot{y}$. Note that this is negative, since $\dot{y}$ is negative. Equating the net force with the change in momentum gives

$$\begin{aligned} F - \sigma L g &= \frac{d(\sigma y \dot{y})}{dt} \\ &= \sigma y \ddot{y} + \sigma \dot{y}^2. \end{aligned} \tag{1}$$

The part of the chain that is still above the scale is in free-fall. Therefore, $\ddot{y} = -g$. And $\dot{y} = \sqrt{2g(L-y)}$, which is the usual result for a falling object. Putting these into eq. (1) gives

$$\begin{aligned} F &= \sigma L g - \sigma y g + 2\sigma(L-y)g \\ &= 3\sigma(L-y)g. \end{aligned} \tag{2}$$

This answer has the expected property of equaling zero when $y = L$, and also the interesting property of equaling $3(\sigma L)g$ right before the last bit touches the scale. Once the chain is completely on the scale, the reading will suddenly drop down to the weight of the chain, namely $(\sigma L)g$.

Second solution: The normal force from the scale is responsible for doing two things. It holds up the part of the chain that already lies on the scale, and it also changes the momentum of the atoms that are suddenly brought to rest when they hit the scale. The first of these two parts of the force is simply the weight of the chain already on the scale, which is $F_{\text{weight}} = \sigma(L-y)g$.

To find the second part of the force, we need to find the change in momentum, dp, of the part of the chain that hits the scale during a given time, dt. The amount of mass that hits the scale in a time dt is $dm = \sigma|dy| = \sigma|\dot{y}|\,dt = -\sigma\dot{y}\,dt$. This mass initially has velocity $\dot{y}$, and then it is abruptly brought to rest. Therefore, the change in momentum is $dp = 0 - (dm)\dot{y} = \sigma\dot{y}^2\,dt$. The force required to cause this change in momentum is thus

$$F_{dp/dt} = \frac{dp}{dt} = \sigma\dot{y}^2. \tag{3}$$

But as in the first solution, we have $\dot{y} = \sqrt{2g(L-y)}$. Therefore, the total force from the scale is

$$\begin{aligned} F &= F_{\text{weight}} + F_{dp/dt} \\ &= \sigma(L-y)g + 2\sigma(L-y)g \\ &= 3\sigma(L-y)g. \end{aligned} \tag{4}$$