Field Theory Methods in Two-Dimensional and Heterotic String Theories

A dissertation presented
by

Morten Ernebjerg

to
The Department of Physics
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
in the subject of

Physics

Harvard University
Cambridge, Massachusetts
January 2007

©2007 - Morten Ernebjerg

All rights reserved.

Thesis advisor Author

Professor Andrew Strominger **Morten Ernebjerg**

Field Theory Methods in Two-Dimensional and Heterotic String Theories

Abstract

This thesis has three parts. In the first, we study the Das-Jevicki collective field description of arbitrary classical solutions in the $c = 1$ matrix model, which are believed to describe nontrivial spacetime backgrounds in 2D string theory. Our analysis naturally includes the case of a Fermi droplet cosmology. We cast the droplet collective field theory in standard coordinates and comment on the form of the interactions.

In the second part, we prove the existence of topological rings in $(0, 2)$ theories containing non-anomalous left-moving $U(1)$ currents by which they may be twisted. While the twisted models are not topological, their ground operators form a ring under non-singular OPE which reduces to the (a,c) or (c,c) ring at $(2, 2)$ points and define a quantum sheaf cohomology. In the special case of Calabi-Yau compactifications, these rings are shown to exist globally on the moduli space in many cases.

In the third part, we construct worldsheet descriptions of heterotic flux vacua as the IR limits of $\mathcal{N} = 2$ gauge theories. Spacetime torsion is incorporated via a 2D Green-Schwarz mechanism in which a doublet of axions cancels a one-loop gauge anomaly. Manifest $(0, 2)$ supersymmetry and the compactness of the gauge theory instanton moduli space suggest that these models, which include Fu-Yau models, are stable against worldsheet instantons.

Contents

Citations to Previously Published Work

The original material presented in this thesis is almost entirely derived from three articles co-authored with fellow researchers—any additions or changes of emphasis relative to these publications reflect only the opinion of the current author. The material in chapter 2 was first published in

> "Collective Field Description of Matrix Cosmologies", M. Ernebjerg, J.L. Karczmarek, and J. M. Lapan, JHEP **0409**, 065 (2004) [hep-th/0405187].

Chapter 4 is based on

> "Topological Heterotic Rings", A. Adams, J. Distler, and M. Ernebjerg, Adv. Theor. Math. Phys. **10**, 657-682 (2006) [hep-th/0506263],

while chapter 5 contains material published in

> "Linear Models for Flux Vacua", A. Adams, M. Ernebjerg, and J. M. Lapan [hep-th/0611084] (submitted to Adv. Theor. Math. Phys.).

Electronic preprints are available in the online `hep-th` archive at `http://arXiv.org`

Acknowledgments

This thesis owes its existence to the generous intellectual gifts of many people. My advisor Andy Strominger combined a great calmness with deep insight and supported me in all my endeavors. Allan Adams has been a most extraordinary and brilliant string guide, and Josh Lapan was an excellent companion, with dry humor and endless mental energy. Joanna Karczmarek and Jacques Distler helped shape my thinking, and together with Michelle Cyrier I learned the value of things that do not work. It has been a great privilege to work with these people. My office-mates, Devin Walker, Kyriakos Papadodimas, and Aaron Simons, provided intellectual input, and welcome distractions. Itay Yavin was a great virtual office-mate. The members of the Harvard high energy theory group generated an unrivalled intellectual environment and I am grateful to all its members, in particular Nima Arkani-Hamed, Chris Beasley, Melanie Becker, Sergei Gukov, Lubos Motl, Andy Neitzke, Suvrat Raju, Shiraz Minwalla, Li-Sheng Tseng, Toby Wiseman, and Xi Yin. Shiraz helped on my first foray into string theory. Lubos agreed to serve on my thesis committee on very short notice, as did Masahiro Morii. Sheila Ferguson always had an open door and sound advice, and Nancy Partridge kept the String Palace humming. I am also deeply indebted to my former teachers Flemming Clausen and John Wheater, who showed me the beauty and joy of hard thinking. Stanislas Leibler, Roy Kishony, and the members of the Kishony Lab complemented this by opening a whole new world for me. I thank the Center for Mathematical Sciences at Zheijang University for hospitality. This work was partly funded by DOE under contract No. DE-FG02-91ER40654.

Finally, I can only fall short in expressing my deep gratitude to my parents for their tireless support and encouragement. And Silke leaves me speechless.

Jeg tilegner denne afhandling
mindet om min morfar,
Peter Bach Jensen.

Chapter 1

Introduction

In its ambition and reach, string theory surpasses every previous attempt at a unified description of the physical universe. So vast is the theoretical structure that central underlying principles still remain to be uncovered. Faced with such complexity, researchers have used a cornucopia of techniques to trace the contours of the theory, drawing on insights from both mathematics and other branches of physics. This thesis continues that work, focusing on applying methods from field theory.

1.1 Themes of this Thesis

Field theory is itself a very rich topic and in the context of string theory it sometimes appears as just a low-energy approximation. To set the stage for the following chapters, it will therefore be useful to recall how and in what forms it appears *within* string theory proper.

As a string moves through a spacetime of arbitrary dimension, it traces out a two-dimensional surface known as the *worldsheet*—we shall use the symbol Σ to refer to it. The dynamics and interaction of the string can now be captured by a theory defined on the worldsheet and governed by an action

$$S_\Sigma[g,\Phi] = \frac{1}{2\pi\alpha'}\int_\Sigma d^2z\sqrt{-g}\,\mathcal{L}[g,\Phi] \tag{1.1}$$

where g is the metric on Σ and Φ signifies all other fields. The constant α' has dimensions of length squared and sets the characteristic scale of the strings; it is typically assumed to be of order of the Plank length squared, $\sqrt{\alpha'} \sim 10^{-35}$m. The spacetime coordinates of the string appear as bosonic fields in the action, so that motion through spacetime maps to a change in the vacuum expectation values of these fields . In the simplest case, S_Σ simply evaluates the area of the worldsheet traced out in spacetime (the Nambu-Goto action), but we shall be interested in more complex theories. In particular, in chapters 3-5 we will work entirely with supersymmetric theories.

In the worldsheet approach, the partition function of a string propagating through space can be written as the Polyakov path integral:

$$Z = \sum_\Sigma \int \mathcal{D}g\mathcal{D}\Phi\; e^{iS_\Sigma[g,\Phi]/\hbar}\;. \tag{1.2}$$

The sum runs over all possible worldsheet topologies and is the string equivalent of the expansion of a field theory amplitude in terms of Feynman diagrams. In the case of oriented, closed strings, the allowed topologies are specified simply by their genus (fig. 1.1).

The freedom in choosing coordinates on the worldsheet Σ appears as a gauge symmetry of S_Σ under the corresponding transformations of the metric. Using this

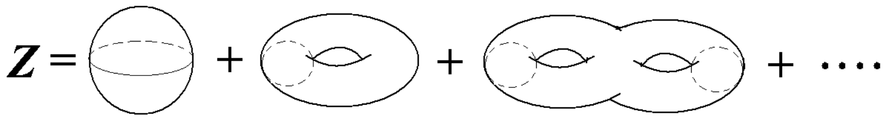

Figure 1.1: The topological expansion of the string partition function in terms of worldsheets of different genus.

freedom, the metric can always be brought into the standard form (here in Euclidean signature)

$$g_{ab} = e^{2\phi}\delta_{ab} \tag{1.3}$$

where the field ϕ parameterizes rescalings. Metrics that differ only by such a rescaling—*Weyl equivalent metrics*—define the same embedding of the worldsheet in spacetime and hence we typically take it to be a further gauge-symmetry and demand that ϕ decouple. This gives a scale invariant and hence conformally invariant world-sheet theory.[1] However, as we shall see in chapter 2, it may occasionally be advantageous to keep the Weyl mode ϕ explicit.

Imposing conformal invariance on the worldsheet leads to the constraint that a supersymmetric string must move in a ten-dimensional spacetime.[2] But it does much more than that. As discussed in chapter 3, a string moving in a spacetime with a given metric and gauge-field backgrounds is described by a non-linear sigma model that depends on these background fields. Conversely, the demand that the sigma model be conformal imposes constraints on the spacetime fields that translate into (string-corrected) equations of motions. It is no mean feat to capture ten-dimensional

[1]Scale invariance is generally thought to imply conformal invariance and in two dimensions this can be rigourously proved [44]

[2]Strictly speaking, we are not required to interpret all sectors of the theory, geometrically and theories with non-geometric sectors can be constructed.

physics with a two-dimensional theory and even by itself this should provide a strong motivation for considering a theory based on one-dimensional entities rather than point particles.

The worldsheet approach to string theory is thus intimately connected with the dynamics of the spacetime in which the string moves. One can also make the spacetime perspective the starting point and work directly with the low-energy field theories that follow from the string dynamics. Throughout the development of string theory, these two points of view have interacted extensively and fruitfully. Indeed, the worldsheet-spacetime connection will be the central common theme of all the work presented here. The emphasis will be mainly on the worldsheet perspective and how it can reflect and enhance our understanding of the spacetime physics that may ultimately govern nature. One point of contact that we shall discuss extensively is the relation between supersymmetry in the two contexts.

The worldsheet approach derives much of its power from the extraordinary range of analytical tools available for treating two-dimensional field theories. Working in only one (spacetime[3]) dimension imposes such strong constraints that non-trivial behavior is hard to come by. In two dimensions, the constraints are still powerful enough to allow explicit analysis in many cases (though exact solutions are still the exception rather than the rule for generic theories). However, field theories in 2D display a wide variety of intricate features, some of which have no equivalents in higher dimensions. A few examples, all of which will explicitly or implicitly play a role in this thesis, are: the topological nature of gravity, integrability, infinite conformal group, and the existence of twisted chiral superfields. This makes two-dimensional field theory an

[3]In what follows, 'dimension' will always refer to *spacetime* dimension, unless otherwise stated.

extraordinarily rich topic and though we shall not delve into the volumes of technical work on it, the power of two-dimensional field theory will be another common theme of this thesis.

We hasten to add that this is not meant to imply that the worldsheet approach is intrinsically superior—in fact, it also has a number of distinct drawbacks. In particular, much of the physics of non-perturbative objects like D-branes and black holes is more transparent from the spacetime point of view. Related to this is the fact that one cannot incorporate Ramond-Ramond background fields into the most common formulation of worldsheet theory (the Ramond-Neveu-Schwarz string).[4] In chapter 5 we avoid this particular obstacle by working in heterotic string theory in which Ramond-Ramond fields are absent. In turn, much of the inspiration for the material in that chapter comes from previous work from the spacetime perspective.

A final overarching theme, which chapters 3-5 have in common, is heterotic theories, i.e. theories in which the holomorphic and antiholomorphic fields are not governed by the same dynamics. These theories allow a natural incorporation of gauge fields, a fact that played an important role in bringing string theory into the mainstream of high-energy theory in the early 1980s. Since then, other ways of generating realistic gauge groups have been found,[5] and the emerging understanding that all ten-dimensional string theories are only limits of an overarching theory (M-theory) has meant that the heterotic string has moved somewhat out of the limelight. The new results in heterotic string theory discussed in this thesis can serve as a reminder that the lower amount of symmetry, compared to Type II strings, also makes for

[4] These difficulties can to some extent be overcome with Berkovits' modified pure-spinor formulation [21, 22] which, however, is technically quite difficult.

[5] A prime example is D-brane constructions, see e.g. [24, 119].

very rich dynamics, and as a hint that the heterotic string may be analytically more tractable than was previously thought.

The results presented here deal primarily with developing new tools for studying string theory rather than with attempts at constructing realistic or semi-realistic phenomenological models. Nonetheless, one may hope that the insights gained will help move us forward towards an understanding of how string theory could give rise to the physics we observe on experimentally accessible length-scales. In particularly, the tools developed in chapter 5 provide a way of probing the low-energy dynamics of a much broader range of possible string theories than hitherto.

1.2 An Overview

The work described in this thesis is derived from three separate articles and correspondingly falls into three main parts. The first is devoted to strings in two spacetime dimensions and the construction of time-dependent backgrounds—*cosmologies*— in this theory using matrix models. Together with a brief introduction, it forms chapter 2. The main result is an explicit construction of a class of Das-Jevicki effective field theories for closed universes undergoing time evolution.

The last two main parts are both concerned with results from heterotic string theory and also share a focus on worldsheet supersymmetry and its consequences. The introductory material is therefore presented in a separate chapter, ch. 3, which also discusses the connection between worldsheet and spacetime physics.

Chapter 4 extends the concept of chiral rings to the context of theories with only $(0,2)$ supersymmetry. The central result is a proof of the existence of chiral rings in a wide range of these heterotic theories. The rings are shown to exists in an open neighborhood of any $(2, 2)$ point in moduli space and globally for conformal models with sufficiently few left-moving fermions.

Finally, chapter 5 is devoted to the construction of a gauged linear sigma model for spacetimes with torsion, i.e. non-vanishing H-flux. This model provides an analytically tractable implementation of a recently discovered rigorous solution for torsionfull supersymmetric compactifications. It is also the first example of a new class of linear models and casts new light on the connection between spacetime torsion and worldsheet theories.

Further background material on string theory can be found in the range of textbooks that are now available. The classic volumes by Green, Schwarz, and Witten [63, 62], though dated in a number of respects, are still notable for their excellent introduction to the role of geometry. The current author learned mainly from the textbooks by Polchinski [102, 103], which treat many topics in great depth and cover progress up until the mid-1990s, including D-branes and dualities. The emphasis of string theory has changed in several ways since then and the textbook by Becker, Becker, and Schwarz [18], which provides an excellent, up-to-date overview of the later developments, is poised to become the new standard work. It is particularly well suited as background for chapter 5 of this thesis.

Needless to say, the research presented here is, by non-physicist standards, highly technical. But as public research, it seems appropriate to make at least the general ideas accessible to non-specialist. In this spirit, I have attempted to convey a flavor of my work in a brief non-technical summary, intended for non-scientists, in appendix A.

Chapter 2

Cosmological Matrix Models

String theory in 1+1 dimensions has the exquisite and rare property that it is exactly solvable. This makes it a perfect laboratory for investigating many phenomena that are difficult to describe explicitly in higher dimensions. One such phenomenon—and arguably one of the most crucial missing elements in a full understanding of string theory—is time-evolution of backgrounds. On the grandest scale, this is the question of the evolution of the entire universe. It goes without saying that even a full understanding of cosmology in 2D string theory can, at best, be a caricature of what we might find in our own universe. However, given the importance of these questions and their intractability in even moderately realistic scenarios, such models are more than mere toys. Because 2D string theory can be solved explicitly, it offers the opportunity to thoroughly investigate the robust features that may have counterparts in more realistic models of time-dependent string theory.

In this chapter, we shall describe some first steps towards a general understanding of cosmological evolution in 2D string theory using matrix models. This topic was first investigated by Karczmarek and Strominger [85, 84] and by Alexandrov *et al.* [6, 7, 8]. Here, we extend their work by constructing models of finite closed universes evolving with time. Along the way, we prove the general existence of a certain convenient choice of coordinates (which we dub *Alexandrov coordinates*) and discuss the spacetime interpretation of such models [52].

General introductions to 2D string theory and matrix models can be found in e.g. [59, 87, 104, 7]. In the review of these topics in the following section, references for specific details of the derivations have been omitted to reduce clutter, but the interested reader can find them in the cited reviews.

2.1 Strings in Two Dimensions and Matrix Models

Doing field theory path integrals is hard, but doing string theory path integrals is much worse. As we have seen, computing the amplitude of some particular process involving closed strings moving in spacetime requires us to sum over all possible string worldsheets. Doing such an integral over all possible surfaces is extremely complicated in all but the simplest cases and may seem an impossible task. However, it turns out that in particular cases there is in fact a very elegant (albeit rather indirect) way to calculate it.

Consider how one might go about computing such an integral numerically. To parameterize the surfaces in an approximate way, we could tile them with regular polygons. The end result is in fact rather insensitive to the exact choice of polygons, so for convenience we will only consider triangulations, tilings by triangles. By taking the triangles to be equilateral, any particular tiling implicitly provides a metric, with the number of triangles meeting at a particular vertex giving a measure of the curvature (e.g. six triangles would meet on a flat surface, so more than six triangles indicates negative curvature). Summing over all distinct tilings is thus an approximate way of summing over all distinct metrics. One can indeed implement this approximation scheme on a computer and that approach has been used to perform detailed studies of two-dimensional quantum gravity (which is what a worldsheet theory amounts to)—see e.g.[9] for a review. However, here we shall be interested in pursuing the analytical approach. In particular, one may hope that there exists a suitable continuum limit, in which the size of the polygons relative to the entire surface goes to zero and a smooth surface is recovered. We will see below that such a limit can indeed be found and that it even allows to extract analytical answers.

The crucial observation is that there exists an easy and systematic way of formally generating all possible tilings and hence, given the existence of the continuum limit, approximations to all possible worldsheets. The idea goes back to the seminal work by 't Hooft on gauge theories in the limit of large gauge group—the large N limit of gauge theories (e.g. $U(N)$ gauge theory as $N \to \infty$)—more than 30 years ago [115]. A field in the adjoint representation of the gauge group (such as gluons in ordinary QCD) can be represented as a direct product of a field in the fundamental and a field in the anti-fundamental representation. This can be represented graphically by drawing the Feynman diagram propagator as two parallel lines carrying arrows pointing in opposite directions. Gauge invariant interaction vertices are similarly 'doubled', with equal numbers of lines going in and out. In this notation, the requirement that all amplitudes be gauge invariant (and hence all indices contracted) translates into the statement that in all valid Feynman diagrams of this theory, any single line is part of an unbroken loop with all arrows pointing in the same direction (see fig. 2.1). The corresponding double-line diagrams are sometimes referred to as *fat graphs*.

Each fat graph can be drawn without self-intersection on a two-dimensional surface of some genus and the propagator lines then divide the surface into separate areas in the shape of polygonal faces. We can now form a tiling of the surface by drawing lines connecting the centroids of neightboring faces. If all interactions in the gauge theory are cubic, it is easy to convince oneself that this tiling is in fact a triangulation (see fig. 2.1). In this roundabout manner, we have thus discovered that gauge theory Feynman diagrams can be seen as generating triangulations. Performing the full gauge theory path integral thus amounts to performing exactly the type of integral we are interested in, viz. an integral over all two-dimensional geometries.

In fact, much simpler theories can generate these tilings. The example we shall consider here is that of matrix quantum mechanics (MQM) which may be considered a zero-dimensional gauge theory. This theory simply describes a quantum mechanical system in which the dynamical variables are matrices rather than numbers and the matrix indices play the role of gauge indices. We will restrict our attention to theories of $N \times N$ Hermitian matrices. If M is the matrix variable, we will demand that the theory is invariant under the similarity ('gauge') transformation

$$M \to U^{+} M U, \quad U \in U(N) \ . \tag{2.1}$$

The simplest non-trivial action for such a theory is

$$S = \frac{1}{2} \int dt \ \mathrm{Tr} \left(\dot{M}(t)^2 + M(t)^2 \right) \ , \tag{2.2}$$

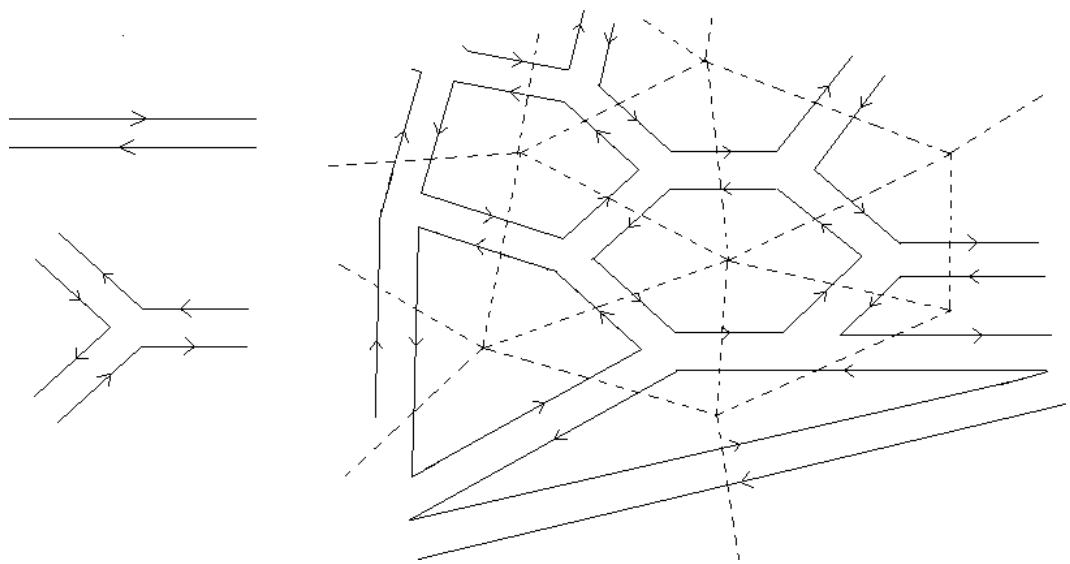

Figure 2.1: Feynamn diagrams in the double-line notation. The form of the propagator and and a cubic interaction vertex are shown on the left; on the right is an example of part of a Feynman diagram—the corresponding triangulation is indicated with dashed lines.

where $\dot{M}$ is the time-derivative of M; this is the equivalent of a free massive particle. To add interactions, we simply replace $M(t)^2/2$ with a generic polynomial potential $V(M)$. The Feynman expansion in terms of fat graphs works exactly as in the gauge case, with the quadratic part of the action giving the propagator and the higher terms in $V(M)$ giving rise to interactions. If $V(M)$ is cubic, we recover the triangular case.

Although MQM is undeniably simpler than a quantum theory of continuous surfaces, it is not obvious that the former really does provide a useful description of the latter. Two major questions remain to be answered: does MQM indeed provide a description of string theory (and if so, *which* string theory), and can we analytically solve MQM? We shall address these questions in that order.

The question of which string theory can appear in the continuum limit is best approached with some inspired guessing. The most obvious characteristic of the theory we would like to see is the dimensionality of its spacetime, i.e. the number of bosonic fields in the continuous worldsheet theory. Inspecting MQM, there is only one continuous parameter that could go over into a field in the continuum, viz. t. Let us therefore assume that the matrix time become a coordinate field X^0. With no other fields available, the most general invariant worldsheet action that involves only the worldsheet metric g_{ab} and a single power of the corresponding Ricci-scalar R is then

$$S = \frac{1}{4\pi\alpha'} \int d^2z \left(-g^{ab}\partial_a X^0 \partial_b X^0 + a + bR \right) \tag{2.3}$$

where a and b are constants. This action is certainly not invariant under Weyl rescalings of the worldsheet metric g_{ab}, i.e. the rescaling degree of freedom does not decouple. To see this explicitly, we use diffeomorphism invariance to write the metric as

$$g_{ab} = e^{cX^1} \hat{g}_{ab} \ . \tag{2.4}$$

where $\hat{g}_{ab}$ is some fiducial metric, the field X^1 is the Weyl mode, and c is a constant. After some manipulation and an appropriate choice of c, this yields the equivalent action

$$S = \frac{1}{4\pi\alpha'} \int d^2z \left(\hat{g}^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + \alpha' V_1 \hat{R} X^1 + T_0 e^{\gamma X^1} \right) \ , \tag{2.5}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1)$ and $\hat{R}$ is the Ricci scalar formed from $\hat{g}_{ab}$. In other words, this theory appears to describe a string propagating in a flat two-dimensional Minkowski spacetime, with the radial mode X^1 playing the role of the spacelike dimension. However, the presence of the two last terms shows that this is not the ordinary flat-space theory. Comparing with the generic form, we see that this action—known as the

Liouville action—in fact describes strings propagating in a flat but translationally non-invariant space in which the dilaton increases linearly with X^1 and the string coupling thus increases as $g_s \sim e^{V_1 X^1}$. There is also an exponential tachyon condensate, $T_0 e^{\gamma X^1}$, which implies that trajectories in which strings go to high X^1 are strongly suppressed. Strings coming in from $X^1 = -\infty$ will therefore bounce back from the strong-coupling regime at high X^1, due to interactions with the condensate which is therefore also referred to as a *tachyon wall*.

It is easy to check that the tachyon profile is not a solution to the $\mathcal{O}(\alpha')$ spacetime equations of motion, but it can be shown that the Liouville theory is in fact conformal (with the right choice of parameters) and therefore provides a valid string action [37, 30, 31, 29, 28]. More worrying is the fact that we have a tachyon background at all, something that would normally signal an instability of the theory. However, the tachyon mass satisfies $m_T^2 \propto (D-2)$ where D is the spacetime dimension and hence it is in fact massless in two dimensions. It is also the only string-mode present in this dimension—worldsheet diffeomorphism invariance means that two vibrational degrees of freedom are unphysical in 2D and hence only the center-of-mass mode remains. This string theory is therefore simply a theory of massless bosons in two dimensions.

The question of the existence of a continuum limit is best treated together with the second overarching question, viz. how we can analytically simplify the matrix model. To this end, let us go back to the MQM with an arbitrary potential $V(M)$, pulling out an overall factor of N for later convenience (it will play the role of $\hbar^{-1}$).

$$S = N \int dt \ \mathrm{Tr} \left(\frac{1}{2} \dot{M}(t)^2 + V(M) \right) \ . \tag{2.6}$$

Since M is Hermitian, we can use the gauge freedom to find a unitary matrix $U(t)$ at each t such that

$$\widetilde{M}(t) = U(t)^{+} M(t) U(t) = \mathrm{diag}(\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t)) \ , \tag{2.7}$$

where $\lambda_i(t)$ are the eigenvalue of $M(t)$. Substituting this back into the action and the path integral of $M(t)$, we find that that $V(M)$ simply goes into $V(\widetilde{M})$ and the kinetic term becomes

$$\frac{1}{2}\mathrm{Tr}\left(\dot{\widetilde{M}}(t)^2\right) + \frac{1}{2}\mathrm{Tr}\left(\left[\widetilde{M}, \dot{U}U^{+}\right]^2\right) \ . \tag{2.8}$$

The path integral measure takes the form

$$\mathcal{D}M(t) = \mathcal{D}U\left(\prod_{k=1}^{N} d\lambda_k\right)\Delta(\lambda)^2 \ , \tag{2.9}$$

where $\mathcal{D}U$ is the Haar measure on $U(N)$ and

$$\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j) \tag{2.10}$$

is the Vandermonde determinant. The Hamiltonian for the system now becomes

$$\hat{H} = \sum_{i=1}^{N}\left(\frac{1}{2}\hat{p}_i^2 + V(\lambda_i)\right) + \hat{A} \ , \tag{2.11}$$

where $\hat{p}_i$ is the momentum conjugate to $\hat{\lambda}_i$—it may be shown to take the form

$$\hat{p}_i = -\frac{i}{N\Delta(\lambda)}\frac{\partial}{\partial\lambda_i}\Delta(\lambda) \ . \tag{2.12}$$

The operator $\hat{A}$ roughly speaking gives the rotational kinetic energy and has eigenvalues ≥ 0. As the path integral over all of spacetime always picks out the ground state [101], only the singlet states ($A = 0$) will contribute. We will therefore focus on the single sector from here on and not include $\hat{A}$.

Having come this far, we see that our theory now looks like quantum mechanics of N non-interacting particles with one-dimensional coordinates given by λ_i, moving in a common potential V(λ). To make this analogy exact, one more step is needed. The wavefunction of the system satisfies

$$\hat{H}\Phi = E\Phi \ . \tag{2.13}$$

If we define $\Psi(\lambda_i) = \Delta(\lambda_i)\Phi(\lambda_i)$, we can rewrite this as

$$\hat{H}_F\Psi = E\Psi, \qquad \hat{H}_F = \sum_{i=1}^{N}\left(-\frac{1}{2N^2}\frac{\partial^2}{\partial\lambda_i^2} + V(\lambda_i)\right) \ . \tag{2.14}$$

We have now brought the Hamiltonian into the standard form, with $1/N$ playing the role of $\hbar$. Crucially, however, it is now a Hamiltonian for N *fermions*, since Ψ is anti-symmetric under exchange of two λs ($\Phi(\lambda_i)$ is symmetric since it is in the singlet representation of $SU(N)$). We have thus succeeded in showing that (singlet-sector) MQM is equivalent the quantum mechanics of N non-interacting fermions in a potential.

The continuum limit should, as discussed above, correspond to $N \to \infty$. Simply taking this limit, however, leaves only planar diagrams (diagrams that can be drawn on a sphere) as all higher diagrams are suppressed by factors of $1/N$. Since we are interested in string theory, we want to include triangulations of surfaces at each genus—including only the sphere corresponds to classical string theory. To rectify this, we must take the *double-scaling limit* in which we simultaneously take $N \to \infty$ and tune the parameters in the potential to make the average size (i.e. number of vertices) in the Feynman diagrams diverge. In this limit, it may be shown that every genus approaches the continuum limit and makes a contribution to the path integral

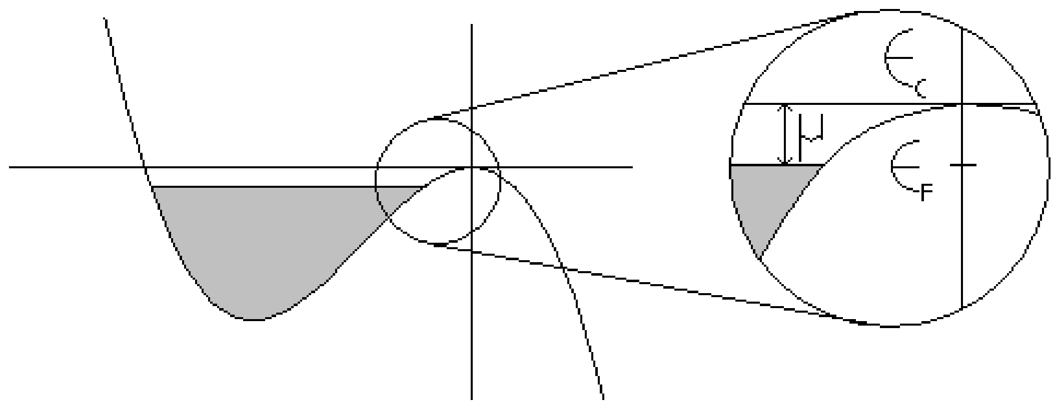

Figure 2.2: Fermion fluid in a potential. The close-up illustrates the double-scaling limit (in rescaled coordinates).

[69, 33, 50]. We can gain an intuitive understanding of this limit by returning to our free-fermion theory. The limit $N \to \infty$ is the limit of infinitely many fermions, but since $1/N$ also plays the role of $\hbar$ it is also the classical limit. The fermions prefer to sit at the minimum of the potential[1] but due to the Pauli exclusion principle they must occupy different quantum states. These states, in turn, approach each other at large N and the fermions behave like a continuous but incompressible classical fluid that fills the potential (see fig. 2.2). In what follows, this picture will be very important to us.

The energy corresponding to the boundary of the Fermi surface (fermion fluid[2] level) is the Fermi energy ϵ_F. Suppose the fermions are trapped in a single potential well whose top is below ϵ_F. We could now tune the parameters in $V(\lambda)$ such that the top of one side of the well approaches the liquid surface. At some critical value of the

[1]We assume that $V(\lambda)$ is bounded from below such that a minimum exists and the theory is stable.

[2]We deliberately avoid the term *Fermi liquid* since that typically refer to systems in which the fermions are interacting. In fact, in the language of condensed matter it would be more appropriate to use the term 'Fermi *gas*'.

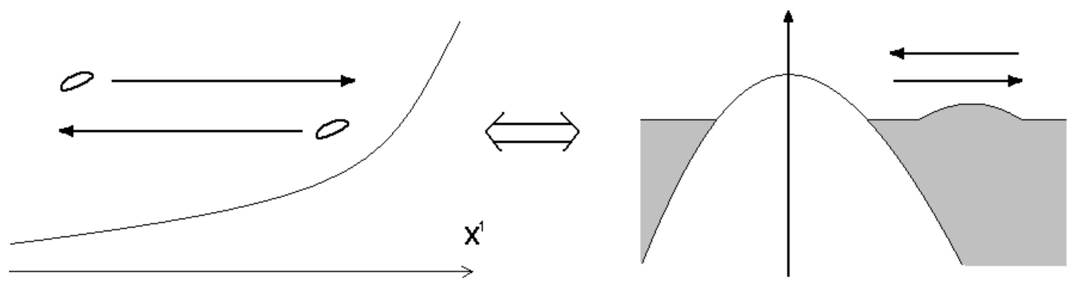

Figure 2.3: The correspondence between bouncing tachyons and reflection of Fermi sea waves.

parameters, the top will be exactly at ϵ_F and moving it further down will cause the fermion fluid to 'spill over'. Equivalently, we could consider rasing the Fermi level until it reaches the top of the well at a critical value $\epsilon = \epsilon_c$. This is exactly what happens in the fermion picture when we take limit where the average size of Feynman diagrams diverges. To take the full double-scaling limit, we must correlate this with the $N \to \infty$ limit. Defining

$$\mu = N(\epsilon_F - \epsilon_c) \ , \tag{2.15}$$

the correct combination is

$$N \to \infty, \qquad \epsilon_F \to \epsilon_c, \qquad \mu \text{ constant} \ . \tag{2.16}$$

Graphically, this corresponds to zooming in on the top of the potential well while at the same time moving the fermion fluid level closer and closer to it, such that the 'seen' gap (μ) between the Fermi sea level and the top remains the same (see fig. 2.3). Close enough to the top of the potential, the shape will be parabolic so in the double-scaled theory, the potential is always an inverted parabola.

Having gone through this technical derivation, we have come out with the suggestion that the physics of strings moving in 2D spacetime with a tachyon wall and a linear dilaton is captured by the behavior of a classical fermion liquid in a potential. At first sight, this seems a rather fanciful claim and clearly it is not something that we have established with mathematical rigor. However, beyond the heuristic derivation given above, several piece of further evidence supports this interpretation. A particularly striking example is the re-interpretation of the matrix model as an example of open-closed duality (in the spirit of the AdS/CFT correspondence [91], reviewed in e.g. [5]), established more than a decade after the discovery of the double-scaling limit [93, 88, 92]. Following the observation that the 2D string theory in question contains D0-branes trapped in the strong-coupling region [53, 116], it was pointed out that the matrix $M(t)$ could be interpreted as a *bona fide* gauge field living on a stack of N such branes. The motion of a single eigenvalue rolling down the potential and into the Fermi sea describes the decay of a D0-brane into closed strings, instantiating the general brane-decay scenario outlined by Sen [108]. The matrix-model/2D string theory correspondence seen in this way mirrors other examples of AdS/CFT, e.g. the equivalence between (closed) string theory in $AdS_5 \times S^5$ and the 4D super Yang-Mills gauge theory on a stack of N D3-branes embedded in this space.

For our purposes, a much simpler piece of evidence will be more important. If the postulated equivalence holds, we certainly expect to be able to recover the physics of tachyons bouncing of the tachyon wall in our Fermi fluid picture. This is in fact rather straightforward: consider the motion of a wave on the surface of the Fermi sea, the natural excitation of the Fermi fluid. As it hits the top of the potential, it sloshes off and returns back to ∞, just like a bouncing tachyon (see fig. 2.3). In other words, in line with the standard duality picture the perturbative degrees of freedom on one side (tachyons in spacetime) are mirrored by non-perturbative degrees of freedom on the other (collective motion of the Fermi sea surface).

2.2 Cosmologies from Matrix Models

If we follow the above derviation and take the fermion theory to provide a description of 2D string theory, we can turn the argument upside down: the theory of free fermions in a potential is completely well-defined, so we may posit that it in fact provides us with a *definition* of 2D string theory, including the otherwise elusive non-perturbative behavior.[3] Viewed this way, the matrix model is not merely a convenient technical device, but becomes a tool for asking new questions about string theory that would otherwise be beyond our reach. In particular, by considering Fermi seas other than the static filling up to a fixed ϵ_F, we can create models in which the Fermi surface evolves with time.[4] The natural spacetime interpretation of these Fermi surfaces is that they correspond to time-dependent backgrounds. This approach forms the framework of the remainder of this chapter.

Classical collective motions of the entire Fermi sea, as opposed to a motion of a single eigenvalue, were investigated in for example [95, 8]. These describe nontrivial time-dependent backgrounds for the 2D string theory and were interpreted as closed string tachyon condensation in [85]. Another class of time-dependent solutions—droplets of large but finite number of eigenvalues, corresponding to closed universe

[3]This conceptual reversal of roles is similar to the conjecture that $\mathcal{N}=4$ super Yang-Mills theory provides a non-perturbative definition of string theory on $AdS_5 \times S^5$.

[4]Since the inverted parabolic potential is not bounded from below there are, strictly speaking, no static fillings of the potential. However, we may think of the potential as being cut off at large λ by a λ^4-term that ensures stability without influencing the physics near $\lambda = 0$.

cosmologies—was proposed in [85]. Since these classical time-dependent solutions of the matrix model correspond to large motions of the Fermi surface, small fluctuations about the Fermi surface carry important information about propagation of stringy spacetime fields. As we will review below, these small fluctuations can be described in the Das-Jevicki collective field approach by a 2D effective field theory, whose action generically contains a nontrivial, time-varying metric.

A step toward understanding these time-dependent solutions was taken by Alexandrov in [6], where coordinates were found in which the metric is trivial. However, the method presented there does not extend to compact Fermi droplets. The main purpose of the rest of this chapter is to extend the construction of Alexandrov coordinates to arbitrary Fermi surfaces, including compact cases.

2.3 Collective Field Theory and Alexandrov Coordinates

In the double scaling limit, the MQM potential is an inverted parabola and hence the action simply becomes

$$S = \frac{1}{2} \int dt \ \mathrm{Tr} \left(\dot{M}(t)^2 + M(t)^2 \right) \ , \tag{2.17}$$

where M is a Hermitian matrix whose size in this limit is taken to infinity. Hence the single variable Hamiltonian is

$$\hat{H} = \frac{1}{2}(\hat{p}^2 - \hat{x}^2) \ , \tag{2.18}$$

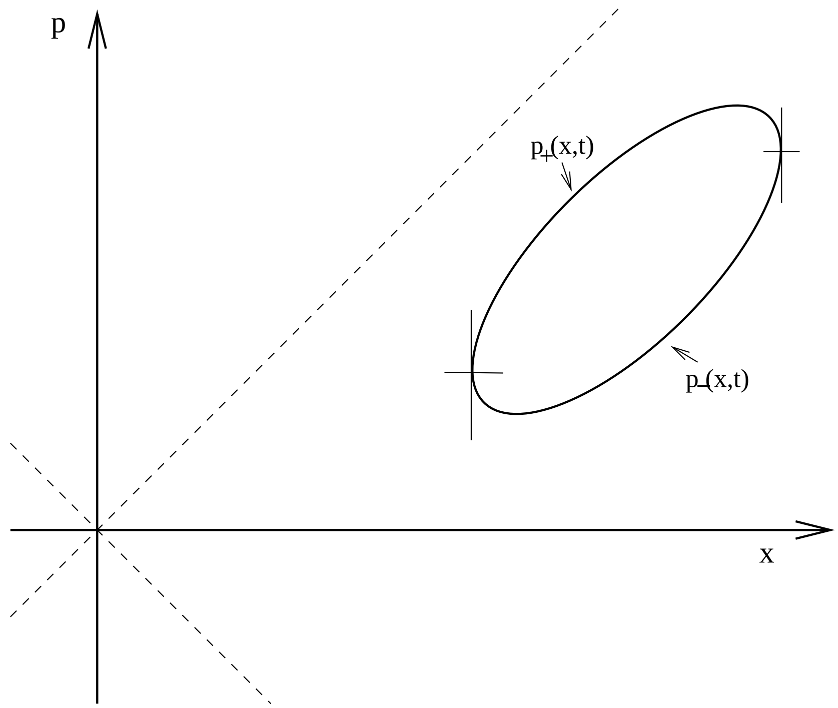

Figure 2.4: A compact Fermi surface in phase space. The upper and lower branches of the surface are labelled, and vertical points where they meet (and the collective theory becomes strongly coupled) are marked.

where we now use x to denote the spatial coordinate. Since the number of fermions is large, the classical limit of the theory is that of an incompressible fermion liquid moving in phase space (x, p) under the equations of motion given by the Hamiltonian (2.18). We will restrict our analysis here to situations where the Fermi surface (the boundary of the Fermi sea) can be given by its upper and lower branch, which we will denote with $p_{\pm}(x, t)$ (see figure 2.4). It is easy to show that $p_{\pm}(x, t)$ satisfy

$$\partial_t p_{\pm} + p_{\pm} \partial_x p_{\pm} = x \ . \tag{2.19}$$

One way to directly connect the classical limit of the matrix quantum mechanics with the collective description of fermion motion is via a procedure developed by Das and

Jevicki [41]. Define a field $\varphi(x,t)$ by

$$\varphi(x,t) = \frac{1}{\pi}\mathrm{Tr}[\delta(x - M(t))] \ , \tag{2.20}$$

so that $\varphi(x,t)$ is the density of eigenvalues at point x and time t. In the fermion description, we have the relation

$$\varphi = \frac{p_+ - p_-}{2} \ . \tag{2.21}$$

The action for the collective field is [41]

$$S = \int \frac{dt\, dx}{2\pi} \left\{ \frac{Z^2}{\varphi} - \frac{1}{3}\varphi^3 + (x^2 - 2\mu)\varphi \right\} \ , \tag{2.22}$$

where $Z = \int dx\ \partial_t \varphi$, so the equation of motion is

$$\partial_t \left(\frac{Z}{\varphi}\right) - \frac{Z}{\varphi}\partial_x \left(\frac{Z}{\varphi}\right) = \varphi \partial_x \varphi - x \ . \tag{2.23}$$

Furthermore, we have the relation [7]

$$\frac{Z}{\varphi} = -\frac{p_+ + p_-}{2} \ , \tag{2.24}$$

which allows us to verify that (2.23) is consistent with (2.19).

We want to consider a fixed solution $\varphi_0(x,t)$ of (2.23) and study the effective action for small fluctuations around this solution. In the string theory dual to the matrix model, this corresponds to studying the small fluctuations about a string background given by the solution $\varphi_0(x,t)$. Let $\partial_x \eta(x,t)$ denote the small fluctuations

$$\varphi = \varphi_0 + \sqrt{\pi}\partial_x \eta \tag{2.25}$$

and let $Z_0 = \int dx\ \partial_t \varphi_0$.

Rewriting the action and grouping terms in powers of η we find (noticing that terms linear in η vanish by the equations of motion)

$$\begin{aligned} S &= \int \frac{dt\,dx}{2\pi} \left\{ \frac{(Z_0 + \sqrt{\pi}\partial_t \eta)^2}{\varphi_0 + \sqrt{\pi}\partial_x \eta} - \frac{1}{3}(\varphi_0 + \sqrt{\pi}\partial_x\eta)^3 + (x^2 - 2\mu)(\varphi_0 + \sqrt{\pi}\partial_x \eta) \right\} \\ &= S_{(0)} + S_{(2)} + S_{int} \ , \end{aligned} \tag{2.26}$$

where $S_{(0)}$ has no η-dependence,

$$S_{(2)} = \frac{1}{2}\int \frac{dt\ dx}{\varphi_0} \left\{ \left(\partial_t \eta - \frac{Z_0}{\varphi_0}\partial_x \eta \right)^2 - \varphi_0^2 (\partial_x \eta)^2 \right\} \ , \tag{2.27}$$

and

$$\begin{aligned} S_{int} &= \frac{1}{2}\int \frac{dt\ dx}{\varphi_0} \left\{ -\frac{\sqrt{\pi}}{3}\varphi_0 (\partial_x \eta)^3 \right. \\ &\qquad \left. + \left(\partial_t \eta - \frac{Z_0}{\varphi_0}\partial_x \eta \right)^2 \sum_{n=1}^{\infty} (-\sqrt{\pi})^n \left(\frac{\partial_x \eta}{\varphi_0} \right)^n \right\} \ . \end{aligned} \tag{2.28}$$

In [6], it is proposed that there exist coordinates (τ, σ) in which $S_{(2)}$ takes the standard form of a kinetic term for a field in a flat metric

$$S_{(2)} = \int d\tau^+ d\tau^-\ \partial_{\tau^-}\eta\, \partial_{\tau^+}\eta \ , \tag{2.29}$$

where $\tau^{\pm}(x,t) = \tau \pm \sigma$ are the lightcone coordinates. We shall refer to the coordinates (τ, σ) as the *Alexandrov coordinates*. In [6], these coordinates were constructed from a specific form of the solution φ_0. In the next section, we prove (at least locally) their existence for all φ_0.

2.3.1 Alexandrov Coordinates – Existence

It is quite simple to show, using the equations of motion for the two branches of the solution (2.19), that the action (2.27) takes on the form in (2.29) as long as the coordinates $\tau^{\pm}$ satisfy

$$(\partial_t + p_{\pm}\partial_x)\tau^{\pm} = 0 \ . \tag{2.30}$$

Equation (2.30) can easily be solved, at least locally. The exact form of the solution depends on whether the slope of the solution $p_{\pm}$ is steeper or shallower than 1. The regions where $\alpha(x,t) \equiv \partial_x p_{\pm}$ satisfies $|\alpha| > 1$ will be referred to as the *steep regions*, and those were $|\alpha| < 1$ will be referred to as the *shallow regions*. In the steep regions, we have that

$$\tau^{\pm} = t - \coth^{-1}(\partial_x p_{\pm}) \ , \tag{2.31}$$

and in the shallow regions we get

$$\tau^{\pm} = t - \tanh^{-1}(\partial_x p_{\pm}) \ . \tag{2.32}$$

The solution above is not unique—a conformal change of coordinates does not change the form of the quadratic part of the action (2.29), so any change of coordinates of the form

$$\tau'^{\pm} = \tau'^{\pm}(\tau^{\pm}) \tag{2.33}$$

will provide another solution to equation (2.30). For example, the following is also a good solution

$$\tau^{\pm} = \frac{\tanh t - \partial_x p_{\pm}}{1 - \partial_x p_{\pm} \tanh t} \tag{2.34}$$

as is

$$\tau^{\pm} = \frac{\coth t - \partial_x p_{\pm}}{1 - \partial_x p_{\pm} \coth t} \ . \tag{2.35}$$

Note that if p_+ and p_- are flat on some overlapping region ($\partial_x p_{\pm} = 0$), then these coordinates will be degenerate. However, we can easily parameterize these flat regions in a nondegenerate way so that the metric is still flat.

The solutions (2.31) and (2.32) are valid locally on steep and shallow coordinate patches respectively (modulus the degenerate case mentioned above). To create a single coordinate system, we can 'glue together' the various steep, shallow, and flat patches by using the freedom of conformal coordinate changes. While our expressions guarantee the existence of Alexandrov coordinates on each patch and while there are no obvious obstacles to the gluing procedure, constructing the coordinates in this way would be very cumbersome, even in cases where the resulting coordinate systems are simple. Instead, in all the examples given in this paper, the Alexandrov coordinates are constructed by the procedure given in [6] (but see the comment at the end of section 2.4).

Another issue is that, as was shown in [6], the resulting coordinates often have boundaries (this will also be seen in section 2.4). The boundaries come in two categories. The first are timelike boundaries corresponding to the end(s) of the Fermi sea; the boundary conditions on those can be determined from the conservation of fermion number [41]. The second class of boundaries contains boundaries which are either spacelike or timelike, but for which no clear interpretation is available and the appropriate boundary conditions are unknown. We will return to the issues of boundaries in the discussion in section 2.4.

We will close this section with a simple example as an illustration. Consider a moving hyperbolic Fermi surface given parametrically by [85]

$$\begin{aligned} x &= \sqrt{2\mu}\cosh\sigma + \lambda e^t \\ p &= \sqrt{2\mu}\sinh\sigma + \lambda e^t \ . \end{aligned} \tag{2.36}$$

In this case, we have $\varphi_0 = \sqrt{(x-\lambda e^t)^2 - 2\mu} = \sqrt{2\mu}\sinh\sigma$. The Alexandrov coordinates are given simply by σ in the parametrization above and by $\tau = t$. It is a simple matter to check that the action takes the form

$$\begin{aligned} S &= \int d\tau d\sigma \left\{ \frac{1}{2}((\partial_\tau\eta)^2 - (\partial_\sigma\eta)^2) - \frac{\sqrt{\pi}}{6\varphi_0^2}(3(\partial_\tau\eta)^2(\partial_\sigma\eta) + (\partial_\sigma\eta)^3) \right. \\ &\qquad \left. + \frac{(\partial_\tau\eta)^2}{2}\sum_{n=2}^{\infty}\left(-\frac{\sqrt{\pi}(\partial_\sigma\eta)}{\varphi_0^2}\right)^n \right\} . \end{aligned} \tag{2.37}$$

Note that the coupling diverges at the point $\sigma = 0$ which corresponds to the edge of the Fermi sea, and that it does not depend on τ.

2.3.2 Alexandrov Coordinates – Special Case

In this section, we study a class of solutions (of which an example appeared at the end of the previous section) for which the Alexandrov coordinates can be written as

$$\sigma = \sigma(x,t) , \qquad \tau = \tau(t) . \tag{2.38}$$

We shall see that this leads to a very restricted class of solutions, but a class which includes both infinite and finite (compact) Fermi seas. Thus, it encompasses the two generic types of dynamic solutions.

With the coordinate ansatz above, we have

$$\begin{aligned} dt\,dx &= \frac{d\tau\,d\sigma}{|\partial_x\sigma\,\partial_t\tau|} \\ \partial_x &= \partial_x\sigma\,\partial_\sigma \\ \partial_t &= \partial_t\tau\,\partial_\tau + \partial_t\sigma\,\partial_\sigma . \end{aligned} \tag{2.39}$$

Demanding that the kinetic term take the standard flat form

$$S_{(2)} = \int d\tau\,d\sigma \left[\frac{1}{2}(\partial_\tau\eta)^2 - \frac{1}{2}(\partial_\sigma\eta)^2\right] \tag{2.40}$$

leads to the requirements that

$$\partial_t \sigma = \frac{Z_0}{\varphi_0} \partial_x \sigma \qquad \text{and} \qquad \left| \frac{\partial_t \tau}{\partial_x \sigma} \right| = \varphi_0 \ . \tag{2.41}$$

These constraints can be solved explicitly, provided that the solution is only vertical at endpoints ($\varphi_0 = 0$). Since τ depends only on t, we find that $\partial_t \tau = (\partial_\tau t)^{-1}$, $\partial_x \sigma = (\partial_\sigma x)^{-1}$, $\partial_x \tau = \partial_\sigma t = 0$,

$$\partial_\tau x = -\frac{\partial_t \sigma}{(\partial_x \sigma)(\partial_t \tau)} \ , \qquad \text{and} \qquad \partial_t \sigma = -\frac{\partial_\tau x}{(\partial_\sigma x)(\partial_\tau t)} \ . \tag{2.42}$$

Using the first equation in (2.41) we find

$$\partial_x Z_0 = -\frac{1}{\partial_\tau t} \left[\varphi_0 \partial_\tau \ln(\partial_\sigma x) + \frac{\partial_\tau x}{\partial_\sigma x} \partial_\sigma \varphi_0 \right] \ , \tag{2.43}$$

which is equal to

$$\partial_x Z_0 = \partial_t \varphi_0 = \frac{1}{\partial_\tau t} \left[\partial_\tau \varphi_0 - \frac{\partial_\tau x}{\partial_\sigma x} \partial_\sigma \varphi_0 \right] \ . \tag{2.44}$$

Comparing these two expressions, we obtain a differential equation for φ_0

$$\partial_\tau \ln(\varphi_0) = -\partial_\tau \ln(\partial_\sigma x) \tag{2.45}$$

whose solution is clearly of the form $\varphi_0 = f(\sigma)^2 (\partial_\sigma x)^{-1}$. This we can rewrite, using the second equation in (2.41), as

$$\varphi_0(\sigma, \tau) = f(\sigma) \sqrt{g(\tau)} \ , \tag{2.46}$$

where $g(\tau) = (\partial_\tau t)^{-1}$ and we assume $f(\sigma) > 0$. We also have $\partial_x \sigma = \sqrt{g}/f$.

Now we can use the equation of motion (2.23) to find the forms of f and g. Using (2.41), notice that

$$\partial_\tau = (\partial_\tau t) \partial_t + (\partial_\tau x) \partial_x = (\partial_\tau t) \left(\partial_t - \frac{Z_0}{\varphi_0} \partial_x \right) \ , \tag{2.47}$$

so the equation of motion (2.23) implies

$$\partial_\sigma \partial_\tau \left(\frac{Z}{\varphi} \right) = \partial_\sigma (\varphi \partial_x \varphi) - \frac{1}{\partial_x \sigma} \ . \tag{2.48}$$

Substituting the explicit form of φ_0 in terms of f and g into this equation, we obtain

$$2g\partial_\tau^2 g - (\partial_\tau g)^2 + 4 - 4g^2 \frac{\partial_\sigma^2 f}{f} = 0 \ . \tag{2.49}$$

Since g only depends on τ, and f only on σ, we see that $\partial_\sigma^2 f = -\alpha f$ where α is a constant.

Consider first the situation when α is positive. Then

$$f(\sigma) = f_1 \sin(\sqrt{\alpha}(\sigma - \sigma_1)) \ , \tag{2.50}$$

where f_1 and σ_1 are real numbers. To ensure that $\varphi_0 \geq 0$, we must restrict $f_1 > 0$ and $\sigma_1 \leq \sigma \leq \sigma_1 + \frac{\pi}{\sqrt{\alpha}}$. Requiring g to be real yields

$$g(\tau) = \frac{1}{\sqrt{\alpha}} \left[\sqrt{c^2+1} \, \cos(2\sqrt{\alpha}(\tau - \tau_1)) + c \right] \ , \tag{2.51}$$

where c and τ_1 are real constants of integration. If α is negative and $|c| \geq 1$, we have

$$\begin{aligned} f(\sigma) &= f_1 \sinh(\sqrt{|\alpha|}\sigma) + f_2 \cosh(\sqrt{|\alpha|}\sigma) \quad \text{and} \\ g(\tau) &= \frac{1}{\sqrt{|\alpha|}} \left[\sqrt{c^2-1} \, \cosh(2\sqrt{|\alpha|}(\tau - \tau_1)) + c \right] \ , \end{aligned} \tag{2.52}$$

while for $|c| < 1$

$$g(\tau) = \frac{1}{\sqrt{|\alpha|}} \left[\sqrt{1-c^2} \, \sinh(2\sqrt{|\alpha|}(\tau - \tau_1)) + c \right] \ . \tag{2.53}$$

Notice that positivity of $f(\sigma)$ restricts the choice of f_1 and f_2 while positivity of $g(\tau)$ in some of these cases restricts the range of τ to a finite or semi-infinite interval.

Let $F(\sigma) = \int d\sigma f(\sigma)$ so that $x(\sigma, \tau) = (F(\sigma) + k(\tau))/\sqrt{g(\tau)}$ for some function $k(\tau)$. We can also show that Z_0/φ_0 is of the form

$$\frac{Z_0}{\varphi_0} = h(\tau) - \frac{\partial_\tau g}{2\sqrt{g}} F \ . \tag{2.54}$$

The functions $h(\tau)$ and $k(\tau)$ can be computed using the equation of motion. Computing $p_\pm$ from (2.21) and (2.24), we get the following relationship

$$\alpha g^2 \left(x - \frac{k(\tau)}{\sqrt{g(\tau)}} \right)^2 + \left(p + x \frac{\partial_\tau g(\tau)}{2} + h(\tau) \right)^2 = f_1^2 g(\tau) \tag{2.55}$$

which we recognize as an ellipse (a hyperbola) if α is positive (negative). Notice that, from equation (2.51), the compact (elliptical) solutions correspond to a finite range of τ.

The interaction terms (2.28) simplify under our assumption to

$$S_{int} = \int d\tau \, d\sigma \left[\frac{1}{6} \Lambda (\partial_\sigma \eta)^3 + \frac{1}{2} (\partial_\tau \eta)^2 \sum_{n=1}^{\infty} \Lambda^n (\partial_\sigma \eta)^n \right] \ , \tag{2.56}$$

where the effective coupling constant is

$$\Lambda = -\frac{\sqrt{\pi}}{\varphi_0} \partial_x \sigma = -\frac{\sqrt{\pi}}{f(\sigma)^2} \ . \tag{2.57}$$

So we find that the coupling constant is time-independent for this class of solutions. We note that the moving-hyperbola solution (2.37) falls into this class.

As long as $|\Lambda \, \partial_\sigma \eta| < 1$, we can sum the series to get

$$S_{int} = \int d\tau \, d\sigma \left[\frac{1}{6} \Lambda \left(\partial_\sigma \eta\right)^3 + \frac{1}{2} (\partial_\tau \eta)^2 \left(\frac{\Lambda \partial_\sigma \eta}{1 - \Lambda \partial_\sigma \eta} \right) \right] \ . \tag{2.58}$$

The first interaction term diverges as $\varphi_0 \to 0$, which occurs when the width of the Fermi sea goes to zero. This mirrors the strong coupling at the tip of the static hyperbolic Fermi surface. The second interaction term diverges as $|\Lambda\,\partial_\sigma\eta| \to 1$. We have

$$\Lambda\,\partial_\sigma\eta = -\frac{\sqrt{\pi}}{\varphi_0}\partial_x\eta = \frac{\varphi_0 - \varphi}{\varphi_0}\ , \tag{2.59}$$

so the breakdown happens when the excitation size become comparable to the width of the Fermi sea (as can also been seen directly from (2.26)). In this case, the Fermi sea may pinch and split into two, so we would not expect to be able to neglect interactions between the upper and lower Fermi surface. Thus, the collective theory becomes strongly coupled exactly in the places one would expect it from general considerations.

2.3.3 An Example of a More General Solution

We have demonstrated that, under the restriction (2.38), the action takes the universal, static form (2.56). The natural question to ask is whether such a universal form of the action might exist for *all* solutions. As a partial answer to this question, we in this section explicitly analyze an example which does not fall into the class of solutions studied above. We show that even with the freedom of conformal change of coordinates it is not always possible to make the interaction term static in Alexandrov coordinates.

Returning to the general case from section 2.3.1, and, using only the property (2.30), we write the cubic part of the action as

$$\begin{aligned} S_{(3)} &= \frac{\sqrt{\pi}}{2}\int d\sigma d\tau \frac{1}{6\varphi_0|\partial_x\tau^+\partial_x\tau^-|}\Big\{ \left((\partial_x\tau^-)^3 - (\partial_x\tau^+)^3\right)\left((\partial_\sigma\eta)^3 + 3\partial_\sigma\eta(\partial_\tau\eta)^2\right) \\ &\quad - \left((\partial_x\tau^+)^3 + (\partial_x\tau^-)^3\right)\left(3(\partial_\sigma\eta)^2\partial_\tau\eta + (\partial_\tau\eta)^3\right)\Big\}\ . \end{aligned} \tag{2.60}$$

For the couplings in this action to be time independent, as in equation (2.56), $\partial_x \tau^\pm/\varphi_0$ must be a function of σ only. We will analyze this condition in a specific example.

Consider the Fermi surface given by

$$x^2 - p^2 = 1 + (x-p)^3 e^{3t} \ . \tag{2.61}$$

Parametrically, this surface is given by

$$\begin{aligned} x &= \cosh\omega + \frac{1}{2}e^{3t-2\omega} \\ p &= \sinh\omega + \frac{1}{2}e^{3t-2\omega} \ . \end{aligned} \tag{2.62}$$

Since the parametric form is similar to the one given in [6], we use the procedure given there to define the Alexandrov coordinates

$$\tau^+ = t - \omega \ , \quad \tau^- = t - \widetilde{\omega} \ , \tag{2.63}$$

where $\widetilde{\omega}$ is defined by $x(\widetilde{\omega}, t) = x(\omega, t)$ as well as $p(\omega, t) = p_+$ and $p(\widetilde{\omega}, t) = p_-$. It is possible to solve for x, t and $p_\pm$ as functions of $\tau^\pm$:

$$\begin{aligned} x(\tau^\pm) &= -\frac{e^{2\tau^+ + 2\tau^-} - e^{\tau^+} - e^{\tau^-}}{2\sqrt{e^{\tau^+ + \tau^-} - e^{2\tau^+ + 3\tau^-} - e^{3\tau^+ + 2\tau^-}}} \\ \exp(t(\tau^\pm)) &= -\frac{\sqrt{e^{\tau^+ + \tau^-} - e^{2\tau^+ + 3\tau^-} - e^{3\tau^+ + 2\tau^-}}}{e^{2\tau^+ + \tau^-} + e^{\tau^+ + 2\tau^-} - 1} \\ p_+(\tau^\pm) &= \frac{e^{2\tau^+ + 2\tau^-} + 2e^{3\tau^+ + \tau^-} + e^{\tau^-} - e^{\tau^+}}{2\sqrt{e^{\tau^+ + \tau^-} - e^{2\tau^+ + 3\tau^-} - e^{3\tau^+ + 2\tau^-}}} \\ p_-(\tau^\pm) &= \frac{e^{2\tau^+ + 2\tau^-} + 2e^{3\tau^- + \tau^+} + e^{\tau^+} - e^{\tau^-}}{2\sqrt{e^{\tau^+ + \tau^-} - e^{2\tau^+ + 3\tau^-} - e^{3\tau^+ + 2\tau^-}}} \ . \end{aligned} \tag{2.64}$$

The coordinates given here have the property that the edge of the Fermi sea ($p_+ = p_-$) is at $2\sigma = \tau^+ - \tau^- = 0$. It is now possible to compute $\partial_x \tau^\pm/\varphi_0$. Not surprisingly, this is not a function of σ only. The question is whether, by a suitable conformal change of coordinates to $\bar{\tau}^{\pm}$, this condition could be satisfied. The change of coordinates would have to map $\sigma = 0$ to itself to maintain a static Fermi sea edge in the new coordinates. Thus, the change of coordinates must be of the form $\tau^{\pm} = f(\bar{\tau}^{\pm})$, with $f(\cdot)$ an arbitrary function. Define $Q_{\pm} \equiv \partial_x \tau^{\pm}/\varphi_0$. The necessary condition is then

$$0 = \partial_{\bar{\tau}} Q_{\pm} = f'(\bar{\tau}^{+})\partial_{\tau^{+}} Q_{\pm} + f'(\bar{\tau}^{-})\partial_{\tau^{-}} Q_{\pm} \tag{2.65}$$

implying that $\partial_{\tau^-} Q_{\pm}/\partial_{\tau^+} Q_{\pm}$ is of the form

$$W_{\pm}(\tau^{+}, \tau^{-}) \equiv \frac{\partial_{\tau^-} Q_{\pm}}{\partial_{\tau^+} Q_{\pm}} = -\frac{f'(\bar{\tau}^{+})}{f'(\bar{\tau}^{-})} = -\frac{F(\tau^{+})}{F(\tau^{-})} \; . \tag{2.66}$$

Therefore,

$$W_{\pm}(\tau^{+}, \tau^{-}) W_{\pm}(\tau^{-}, \tau^{+}) = 1 \; . \tag{2.67}$$

By explicit computation, it can be checked that this condition is not satisfied. Therefore, there does not exist a coordinate transformation after which $S_{(3)}$ has no τ dependence.

2.4 Fermi Droplet Cosmology

Having considered the generalities of constructing a simple effective spacetime action, we are now finally in a position to discuss explicit examples of the class of solutions introduced in section 2.3.2—a droplet solution in which only a finite region of phase space is filled, so that the Fermi surface is a closed curve. These solutions are believed to correspond to time dependent backgrounds in the spacetime picture [85], though no precise correspondence has been found so far.

In the simplest case, the Fermi surface is a circle in phase space with radius R and center $(p,x) = (0,x_0)$ at time $t=0$. Notice that we must demand $x_0 > \sqrt{2}R$ in order for the surface not to cross the diagonals $p = \pm x$ (otherwise, some of the fermions will spill over the potential barrier as the droplet bounces off it).

It is not difficult to write down the evolution of this Fermi surface

$$e^{-2t}(x+p-x_0e^t)^2 + e^{2t}(x-p-x_0e^{-t})^2 = 2R^2 \ . \tag{2.68}$$

Solving for p we find

$$\varphi_0 = \frac{\sqrt{R^2\cosh 2t - (x - x_0\cosh t)^2}}{\cosh 2t} \ . \tag{2.69}$$

A sensible σ-coordinate is an angle parameterizing the upper surface, running from 0 to π between the points where $\varphi_0 = 0$. These are given by

$$x = x_0\cosh t \pm R\sqrt{\cosh 2t} \ , \tag{2.70}$$

so the simplest guess for an Alexandrov coordinate (which we call θ to stress its angular nature) is such that

$$x = x_0\cosh t - R\cos\theta\sqrt{\cosh 2t} \ . \tag{2.71}$$

Using the second condition in (2.41), we find

$$\partial_t\tau = \frac{1}{\cosh 2t} \ , \tag{2.72}$$

which gives

$$\tau = \tan^{-1}(\tanh t) \ . \tag{2.73}$$

Thus, τ runs over the finite range $-\pi/4 \leq \tau \leq \pi/4$. In these new coordinates, we find

$$x = \frac{1}{\sqrt{\cos 2\tau}}(x_0\cos\tau - R\cos\theta) \ , \qquad \varphi_0 = R\sqrt{\cos 2\tau}\sin\theta \ . \tag{2.74}$$

It can be checked that these coordinates do fulfill the first condition in (2.41) as well.

We see that

$$\Lambda = -\frac{\sqrt{\pi}}{\varphi_0}\partial_x\theta = -\frac{\sqrt{\pi}}{R^2\sin^2\theta} \ , \tag{2.75}$$

and

$$g(\tau) = \cos 2\tau \ , \qquad f(\theta) = R\sin\theta \ , \tag{2.76}$$

and the action (2.26) simplifies to

$$\begin{aligned} S &= \int d\tau\ d\theta\ \left\{\frac{1}{2}[(\partial_\tau\eta)^2 - (\partial_\theta\eta)^2] - \frac{\sqrt{\pi}}{6R^2\sin^2\theta}(\partial_\theta\eta)^3 \right. \\ &\left. +\frac{1}{2}(\partial_\tau\eta)^2\sum_{n=1}^{\infty}\left(-\frac{\sqrt{\pi}}{R^2\sin^2\theta}\partial_\theta\eta\right)^n\right\} \ . \end{aligned} \tag{2.77}$$

As anticipated, the theory is strongly coupled at the endpoints of the droplet where $\varphi_0 \to 0$. Note that the coordinates are smooth across the steep/shallow divide.[5]

As an aside, consider a modification to the droplet discussed above. At time $t = 0$, replace the regions $\pi/4 < \theta < 3\pi/4$ and $5\pi/4 < \theta < 7\pi/4$ by straight lines so that the droplet takes the form of a rectangle with semi-circular ends. A straightforward computation leads to the conclusion that one can find global coordinates which yield a flat kinetic term in the action. As one might expect, time is still compact as it was in the elliptical case, indicating that the compactness is not merely an accident occurring only for this particular shape.

Having constructed droplet solutions, the next question is whether they have a clear spacetime interpretation in string theory. The collective field description which we have constructed here suggests that they do indeed have an interpretation as some closed string backgrounds. The massless scalar fluctuations should correspond

[5]It is possible to explicitly reach the θ-coordinate from the generally applicable forms (2.31), (2.32) by using appropriate conformal transformations on each patch, but the computation is complicated.

to some string field, the analog of the tachyon in $c = 1$ Liouville string. The strongly coupled regions at each end of the droplet should correspond to tachyon walls—strongly coupled regions of large tachyon VEV. If such a closed string, worldsheet description could be found, it would provide an example of an open-closed string, finite N duality between a time-dependent finite universe and the matrix quantum mechanics of the $D0$ branes making up the droplet.

Unfortunately, it is not clear how to construct such a spacetime interpretation. The natural time τ is compact, corresponding to the fact that in the fermion time, t, fluctuations of a compact Fermi surface become frozen in the past and future. Also, examining the interaction term, we notice that the coupling Λ is bounded from below so that the theory does not approach a free theory in any region (though the coupling can be made arbitrarily small by taking R large). This makes it unlikely that it will be possible to define an S-matrix and hence extract information about the spacetime structure from scattering. In addition, in the standard $c = 1$ story, the matrix-to-spacetime dictionary is complicated by the presence of so-called leg pole factors, additional phases needed to match the matrix model S-matrix to its string worldsheet counterpart [43, 105, 68]. Supposedly such a complication would appear for the droplet cosmologies as well, but there is no obvious candidate for what it might be. Therefore, it seems unlikely that a spacetime analysis of tachyon scattering can be carried out as has been done in the case of the standard, static Fermi sea as well as the moving hyperbola solution (2.36) [84, 40].

Another way to view the complication introduced by the finite extent of the time τ is that in Alexandrov coordinates there appear boundaries (in this case, spacelike boundaries at $\tau = \pm\pi/4$). What the boundary condition on these should be is not clear. The appearance of boundaries is not unique to compact Fermi surfaces; boundaries of this type, both timelike and spacelike, have appeared in the analysis of noncompact Fermi surfaces in [6].

2.5 Conclusions and Outlook

By extending the methods of collective field theory and the use of standard coordinates, we have constructed models of time-evolving 2D string theories for closed universes, corresponding to finite droplets of fermion liquid moving in a potential. For a simple class of such models, we have also explicitly constructed the effective theory for excitations in such universes—the equivalent of the effective theory of tachyons in the standard scenario with a static Fermi sea—and shown that the effective coupling constants are time-independent. This allowed us to explicitly verify that the excitations exhibit the same general behavior as in the static case, e.g. that they become strongly coupled where the upper and lower branch of the Fermi surface join. The effective theory in these cases contains only derivative interactions and hence becomes free in the infrared.

While droplet solutions can be straightforwardly constructed, their interpretation is far from clear. The absence of any asymptotic spatial regions in which excitations of all energies become free makes it impossible to define an S-matrix in the usual way. This problem is compounded by the fact that it is not clear how to perform a detailed translation between collective excitations and spacetime particles, i.e. it is unclear what the appropriate generalization of leg pole factors are. To some extent, however, these problems appear to be due primarily to a failure of our standard tools of analysis (scattering and the S-matrix), rather than an intrinsic flaw in the droplet solutions. Indeed, we would not expect a generic finite universe to contain asymptotic regions without interactions; that we assume the existence of such regions for the purposes of ordinary field theory calculations is a matter of convenience and depends only on it being a good, local approximation, not on it being an exact statement about the universe. We can make similar approximations in the regime of large-radius droplets.

A more intriguing open question springs from the observation that that the natural (Alexandrov) time in the effective theory is finite for an elliptic droplet, something that also seems to apply to more general droplet forms. In other words, though the time development in the fermion picture is smooth and infinitely extended, from the point of view of the droplet, the universe has a definite beginning and end. This introduces the thorny problem of imposing boundary conditions and no simple solution presents itself. The finite time is thus both an interesting and problematic feature: interesting, because the possibility that our model universe has a definite beginning and end is an striking cosmological feature; frustrating, because the current methods do not fix what these endpoints look like.

It is a quite possible that more headway on these questions could be made by further extending the known techniques such as the leg pole factors. Indeed, there is some understanding of how these factors arise from more general objects known as *loop operators* [59], and more work has been done on the problem of cosmological boundaries [42]. A further possible path to progress is to explore the D-brane interpretation of matrix cosmologies, an approach that was also taken up in [85, 39].

Chapter 3

Spacetime Physics and Worldsheet Theories

The correspondence between 2D string theory and matrix models discussed in the previous chapter is one example of the intricate interplay between spacetime physics and worldsheet dynamics: the spacetime tachyons could be mapped to excitations in a fermion fluid that, in turn, was derived from a particular formulation of the worldsheet theory, namely MQM. Although we cannot be as explicit about this interplay for ten-dimensional string theories, there is still a very profound and even richer connection between these two complementary viewpoints in higher dimensions.

The following three chapters explore this connection, focusing on the particular case of heterotic string theories.[1] These theories have only half as much worldsheet supersymmetry as Type II strings, which makes them technically more demanding in

[1] We shall use the term 'heterotic' to refer to worldsheet theories with $(0,2)$ supersymmetry. Left-right asymmetric theories with $(2,2)$ supersymmetry can also be formed by extending the left-moving side of a symmetric $(2,2)$ models.

a number of ways. Consequently, the bulk of the work on worldsheet theory in the past 15 years has paid comparatively little attention to them. However, as we shall see in chapters 4 and 5, these technical obstacles can be overcome in a number of cases and the effort is repaid with surprising new insights, such as the existence of chiral rings in heterotic theories or the nature of the worldsheet 'image' of spacetime torsion.

Heterotic strings also remains one of the tightest (albeit still very indirect) links between string theory and observable physics. Historically, the first string theories with realistic gauge groups were heterotic and recent work has demonstrated that such theories can in fact reproduce the exact spectrum of realistic extensions of the standard model [27, 32]. Though it may seem contradictory, the recent suggestion that there exists a whole 'landscape' of possible string theory solutions should, in fact, also serve as a motivation to study heterotic theories. For while it casts doubts on the goal of writing down *the* string theory, it also forces us to consider the whole range of possible string theories. Indeed, the heterotic theories we construct in chapter 5 are closely related to the flux compactifications that gave rise to the landscape perspective. Which of these conceptually very different approaches will win out remains to be seen, but a better understanding of heterotic strings may help determine the outcome.

The present chapter serves as a foundation for the material in chapters 4 and 5 and will only review previously known results. The focus will be mainly on the conditions for *spacetime* supersymmetry and on results for theories with $\mathcal{N} = (2, 2)$ worldsheet supersymmetry (some of which we will later generalize). References to the original literature will be kept to a minimum since excellent textbook reviews of most of this material are available—three examples, in chronological order, are [63, 103, 18]. For the material on gauged linear sigma models and topological theories the comprehensive monograph [74] as well as Witten's original paper on linear models [123] are recommended. Mathematical background material can be found accessibly in [99] and exhaustively in [67].

3.1 Spacetime Supersymmetry

When working with a complicated theory, symmetries are essential since they help reign in the mathematical complexities. Often, however, symmetries can only be incorporated by imposing very unrealistic restrictions. Supersymmetry—though still unobserved in nature[2]—is a fortunate case where a powerful symmetry may in fact help make string theory *more* realistic. Indeed, many extensions of the standard model use supersymmetry to explain the seemingly unnaturally low mass of the Higgs boson (the *hierarchy problem*, see e.g. [36]). While low-energy supersymmetry by itself does not validate string theory, string theory offers a very natural framework for incorporating supersymmetry, and indeed the theory of supersymmetry has developed in close interaction with string theory.

[2]This is true as of writing (January 2007), but may change soon. The Large Hadron Collider at CERN is scheduled to commence operation in the Fall of 2007 and could deliver decisive evidence for low-energy supersymmetry.

3.1.1 Spacetime Supergravity

A field theory of interacting particles can have at most 32 supersymmetries, which corresponds to the minimal representation of the supersymmetry algebra in 11 spacetime dimensions.[3] Allowing no more than two derivatives in the action, there is unique 11-dimensional gravitational field theory incorporating this symmetry. It contains only the metric G_{MN} and a 3-form gauge field A_{MNP} and has the gauge-invariant action (here in the conventions of [103])

$$S_{11} = \frac{1}{2\kappa_{11}} \int d^{11}x \sqrt{-G}\left(R_{11} - \frac{1}{2}|F_{11}|^2\right) - \frac{1}{6}\int A_{11} \wedge F_{11} \wedge F_{11} + \text{ fermionic terms} , \tag{3.1}$$

where R_{11} is the 11-dimensional Ricci scalar and F_{11} is the 4-form field strength of A_{11}. Supergravities in all lower dimensions can be reached by Kaluza-Klein reduction of this theory, i.e. by compactifying one or more spatial dimensions and taking the compactification radius to 0, thus obtaining an effective lower-dimensional theory with extra gauge fields. In particular, the ten-dimensional theories that arise as the low-energy effective theories of supersymmetric string theories can be derived in this way. For the heterotic theory, the reduced action takes the form

$$S_{het} = \frac{1}{2\kappa_{10}}x \int d^{10}x \sqrt{-G}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H|^2 + \frac{\kappa_{10}^2}{g_{10}^2}\mathrm{Tr}(|F|^2)\right] . \tag{3.2}$$

R is the ten-dimensional Ricci-scalar, while F is a 2-form gauge field strength and Φ is the dilaton. Finally, H is the anti-symmetric 3-form tensor field strength arising from the anti-symmetric NS-NS B-field—it is often referred to as the *torsion* since that is its geometric meaning in spacetime.

[3] Appendix B in [103] provides a compact introduction to supersymmetry algebra in various dimensions and the construction of the corresponding supergravity theories.

To obtain phenomenologically viable theories, six spatial dimensions must be compactified on a six-dimensional manifold X with a characteristic scale ρ_X (typically taken to be of order $\sqrt{\alpha'}$), yielding an effective four-dimensional theory at energies below $1/\rho_X$. The geometric characteristics of the manifold X then encode many of the properties of physics in the four large dimensions, such as part of the particle spectrum.

In a seminal paper, Candelas, Horowtiz, Strominger, and Witten used this connection to describe the constraints on X that arise from demanding $\mathcal{N}=1$ supersymmetry in the extended dimensions [35]. They analyzed the low-energy theory arising from heterotic string theory compactified on $M_4 \times X$, where M_4 is four-dimensional Minkowski space. The existence of a 4D supersymmetry amounts to the existence of a supersymmetry on X, parametrized by a spinor that leaves the fermionic components of the spacetime fields invariant. Focussing on the constraints for the fields on the internal manifold X and the corresponding nowhere-vanishing spinor parameter η, we find:

$$\delta_\eta \psi_a = \left(\partial_a + \frac{1}{4}\Omega^-_{abc}\Gamma^{bc}\right)\eta = 0 \ , \tag{3.3}$$

$$\delta_\eta \chi = \left(\Gamma^a \partial_a \Phi - \frac{1}{12}\Gamma^{abc}H_{abc}\right)\eta = 0 \ , \tag{3.4}$$

$$\delta_\eta \lambda = F_{ab}\Gamma^{ab}\eta = 0 \ . \tag{3.5}$$

Here, the Γs are anti-symmetrized products of six-dimensional Dirac matrices, ψ, χ, λ are the gravitino, dilatino, and gaugino, respectively, and Ω^- is a 3-form field strength. The simplest set of solutions to these equations are obtained by setting the torsion H to zero. The resulting equations imply that X must be Calabi-Yau, i.e. a Ricci-flat Kähler manifold.

3.1.2 Compactifications with Torsion

Calabi-Yau compactifications have provided an immense number of insights into the interplay between compactified dimensions and four-dimensional physics. However, in the discussion in chapter 5 we will explicitly *not* assume $H = 0$. For examining this more general case, it will be useful to follow Strominger and recast the the above equations in an explicitly geometric form [113]. In that spirit, we will henceforth typically drop indices on fields and use geometric form notation. Using the Jacobi identity for the superalgebra, we first find that X admits an integrable complex structure, i.e. that it is a complex manifold. This complex structure can be constructed from the globally-defined Hermitian (1,1)-form $J_{a\bar{b}} = \eta^\dagger \Gamma_{a\bar{b}} \eta$, which also gives us a nowhere-vanishing Hermitian metric. Similarly, from the spinor η we can construct a nowhere-vanishing holomorphic (3,0)-form, $\Omega_{abc} = \eta^\dagger \Gamma_{abc} \eta$. It may also be shown that the conditions imply the existence of a Hermitian-Yang-Mills gauge field, $F^{(2,0)} = F^{(0,2)} = F_{mn} J^{mn} = 0$.

The existence of a nowhere-vanishing spinor on an almost-complex 3-fold implies that the frame bundle admits a connection of $SU(3)$ holonomy—i.e. that X is a special-holonomy manifold with $SU(3)$-structure. However, the connection of special holonomy need not be the Levi-Civita connection, and in general the nowhere-vanishing spinor η is *not* annihilated by the metric connection, ∇_g, but by a (unique) torsionful connection,

$$(\nabla_g + H)\eta = 0 \ . \tag{3.6}$$

H is called the *intrinsic torsion* of the $SU(3)$-structure. In the special case $H = 0$, when the nowhere-vanishing spinor is covariantly constant according to the metric connection, X admits a metric of $SU(3)$ holonomy, which is equivalent to the condition that it be Calabi-Yau. Finally, the supersymmetry conditions imply that

$$H = i(\overline{\partial} - \partial)J \ . \tag{3.7}$$

For a Kähler manifold, the right-hand side vanishes, so $H \neq 0$ implies that X is non-Kähler. Instead, X is *conformally balanced*

$$d(e^{-2\phi} J \wedge J) = 0 \ , \tag{3.8}$$

where ϕ is the Einstein-frame dilaton.

A final, crucial difference from the $H = 0$ case arises from the modified Bianchi identity for the torsion H. As shown by Green and Schwarz, the consistency of ten-dimensional heterotic supergravity is ensured by a very delicate anomaly cancellation [64]. The gauge and gravitational anomalies in ten dimension arise from hexagon diagrams with a fermion loop and six external gauge bosons/gravitons. Careful study reveals that these diagrams do *not* cancel among themselves. However, we can ascribe the B-field a non-standard transformation, which to leading order is

$$\delta B = a \, \mathrm{Tr}(\alpha F) + b \, \mathrm{tr}(\beta R) \ . \tag{3.9}$$

Here, α and β are the gauge and local Lorentz transformation parameters, respectively and a, b are constants. The gauge-invariant field strength is then, again to leading order

$$H = dB - \alpha\omega_F - \beta\omega_R \ , \tag{3.10}$$

where ω_F and ω_R are the Chern-Simons forms formed from F and R, respectively. From the point of view of string theory, these terms arise from higher-derivative interactions in the low-energy field theory approximation to the full theory. They are Wess-Zumino terms of the form

$$S_F = \int B\,\mathrm{tr}(F^4) \tag{3.11}$$

and variants involving R^4 or combinations of F and R. These terms, which enter the low-energy theory only at $\mathcal{O}(\alpha')$, lead to *tree*-diagrams involving the exchange of a single B gauge boson. These extra diagrams cancel the hexagon diagrams in what known as the *Green-Schwarz mechanism.*

The non-standard form of H gives rise to a modified form of the Bianchi identity[4]

$$dH = \alpha'(\mathrm{tr}R \wedge R - \mathrm{Tr}F \wedge F)\ . \tag{3.12}$$

This relation has several important consequences. First, any solution to this equation for which the right-hand side does not vanish must have cancellation between terms of different order in α', mirroring the cancellations of Feynman diagrams at different orders in the Green-Schwarz mechanism. That is, the solutions will have a typical scale of $\mathcal{O}\left(\sqrt{\alpha'}\right)$ and hence will *not* have a large radius limit (unlike Calabi-Yau manifolds). Therefore, supergravity perturbation theory has a finite, fixed expansion parameter and must be treated with caution. Secondly, this equation is highly non-linear, so even proving the existence of solutions is a profoundly difficult problem.

Happily, in at least one special case there exists an existence proof by Fu and Yau for solutions to the full set of conditions outlined above, including the anomaly equation [54], analogous to Yau's proof of the existence of a Ricci-flat Kähler metric

[4]The modified Bianchi identity has the unfortunate property that it very rarely occurs in exactly the same form in any pair of articles—in particular, mathematician and physicists tend to favor different versions. For our purposes, the exact normalization will not be of great importance and has hence been chosen primarily for simplicity. The sign convention is the one used in [15].

on manifolds of $SU(3)$-holonomy [125]. Unlike the Yau proof of the Calabi conjecture, however, the Fu-Yau proof begins with a very specific ansatz for the metric, torsion, and holomorphic 3-form. We will return to this construction in chapter 5.

3.2 Worldsheet Supersymmetry

A string theory with a supersymmetric worldsheet can give rise to non-supersymmetric spacetime solutions, but the reverse is not possible; as proved by Sen, $\mathcal{N} = 1$ spacetime supersymmetry implies at least $(0, 2)$ worldsheet supersymmetry, i.e. at least two left- (or right-)moving supercharges [107]. Conversely, the presence of unbroken extended supersymmetry on the worldsheet imposes constraints on the corresponding spacetime [57]. It also gives rise to a number of technical simplification that we shall exploit extensively in the following to chapters.

3.2.1 Supersymmetric Sigma Models

On the worldsheet, the spacetime configuration of the various fields is reflected in the conformal field theory. Our starting point will be the sigma model on a worldsheet Σ for a heterotic string moving on a space X with metric $G_{\mu\nu}$, as well as $B_{\mu\nu}$ and gauge-field (A_μ) backgrounds. We initially assume only the minimum supersymmetry, a single right-moving supersymmetry (i.e. $(0, 1)$ supersymmetry) which organizes the fields (ϕ^μ, ψ^μ) and (λ^A, G^A) into supermultiplets. The correct action, after integrating out the auxiliary field G^A, is then

$$S_\Sigma = \frac{1}{2\pi\alpha'}\int_\Sigma d^2z\,\{[G_{\mu\nu}(\phi)+B_{\mu\nu}(\phi)]\,\partial_z\phi^\mu\partial_{\bar{z}}\phi^\nu + G_{\mu\nu}(\phi)\psi^\mu\mathcal{D}_z\psi^\nu + \lambda^A\mathcal{D}_{\bar{z}}\lambda^B + \frac{1}{2}F^{AB}_{\rho\sigma}(\phi)\lambda^A\lambda^B\psi^\rho\psi^\sigma\Big\}\ , \tag{3.13}$$

where the covariant derivatives are given by

$$\mathcal{D}_z\psi^\mu = \partial_z\psi^\mu + \left[\Gamma^\mu_{\rho\sigma}(\phi) - \frac{1}{2}H^\mu_{\rho\sigma}(\phi)\right]\partial_z\phi^\rho\psi^\sigma\ , \tag{3.14}$$

$$\mathcal{D}_{\bar{z}}\lambda^A = \partial_{\bar{z}}\lambda^A - iA^{AB}_\mu(\phi)\partial_{\bar{z}}\phi^\mu\lambda^B\ . \tag{3.15}$$

Since the spacetime fields all depend on the coordinate fields ϕ^μ, the kinetic terms generically do not take the standard Minkowski form and the model is thus a *non-linear sigma model* (NLSM). The right-moving fermions, ψ^μ, couple to the torsion-connection of the spacetime metric, while the left-moving fermions, λ^A, couple to the spacetime gauge connection. Geometrically, this means that ψ^μ transforms in the tangent bundle of X, T_X, and λ^A transforms in a gauge bundle over X, $\mathcal{V}\to X$:

$$\lambda^A \in \Gamma\left(\sqrt{K_\Sigma}\otimes\phi^*\mathcal{V}\right),\qquad \psi^\mu\in\Gamma\left(\sqrt{\overline{K}_\Sigma}\otimes\phi^*T_X\right)\ , \tag{3.16}$$

where K_Σ is the canonical bundle on Σ and ϕ^* is the pullback of the embedding map $\phi:\Sigma\to X$ given by $\phi^\mu(z,\bar{z})$.

From this sigma model, the correct spacetime equations of motion for $G_{\mu\nu}$, $B_{\mu\nu}$ and A_μ follow by demanding that the theory be conformal. By going to higher order in the worldsheet perturbation theory, we find that this constraint also reproduces the modified Bianchi identity [34, 77, 106]. The way in which the modification of the B-field transformation appears is a particularly nice example of the interplay between worldsheet and spacetime quantities. The action (3.13) is classically invariant under a spacetime gauge transformation

$$\delta A_\mu^{AB} = D_\mu \chi^{AB}, \qquad \delta\lambda^A = i\chi^{AB}\lambda^B \ . \tag{3.17}$$

However, because the left- and right-moving worldsheet fermions are not paired up, there will generically be a (one-loop) chiral anomaly on Σ which involves the pullback of the spacetime field strength and causes S_Σ to shift by

$$\delta S \sim \int d^2z \ \mathrm{Tr}(\chi F_{\mu\nu})\partial_z\phi^\mu\partial_{\bar{z}}\phi^\nu \ , \tag{3.18}$$

where we trace over the gauge indices. This shift is exactly cancelled if we ascribe the B-field the anomalous gauge transformation (3.9). The Bianchi identity, which in spacetime arises from the cancellation of gauge and gravitational anomalies, is thus mirrored by the requirement that the worldsheet theory be anomaly-free. An extension of this idea will play a central role in chapter 5.

3.2.2 Chiral Rings, Topological Field Theory and Mirror Symmetry

Extending the amount of supersymmtry, as we must to get spacetime supersymmetry, leads to further structure on the worldsheet. Imposing $(0,2)$ supersymmetry pairs up the real fields into complex fields which are organized in $(0,2)$ multiplets, and implies the existence of a supersymmetry algebra generated by the right-moving supercharges Q_+

$$Q_+^2 = \overline{Q}_+^2 = 0, \quad \{Q_+, \overline{Q}_+\} = 2i\partial_{\bar{z}} \ . \tag{3.19}$$

The supercharges transform under a $U(1)$ R-symmetry which acts as[5]

$$[J_+, Q_+] = Q_+, \quad [J_+, \overline{Q}_+] = -\overline{Q}_+ \ . \tag{3.20}$$

If the theory is also conformal, the supersymmetry algebra is elevated to a superconformal algebra in which J_+ sits in the same $(0,2)$ supermultiplet as the right-moving stress tensor $\overline{T}(\overline{z})$ and its fermionic partner, $\overline{T}_F(\overline{z})$. Expanding in modes, the right-moving supercommutator may then be written,

$$\{G_{+(r)}, \overline{G}_{+(s)}\} = 2\overline{L}_{(r+s)} - (r-s)J_{+(r+s)} + \frac{\overline{c}}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} \ , \tag{3.21}$$

where $\overline{c}$ is the right-moving central charge. Since $\langle\psi|\{Q_+, \overline{Q}_+\}|\psi\rangle \geq 0$ for all states $|\psi\rangle$, we can set $r = 1/2$, $s = -1/2$ and derive the BPS bound,

$$\overline{\Delta} \geq \frac{1}{2} q_+ \ , \tag{3.22}$$

where Δ is the conformal weight of an operator and q_+ is its right-moving R-charge. An operator $\mathcal{O}$ saturates this right-moving BPS bound iff it is chiral primary, i.e. iff it satisfies $\overline{Q}_+\mathcal{O} = 0$; as a result, any product of chiral operators is again chiral. Similarly, an operator satisfying $Q_+\mathcal{O} = 0$ saturates the BPS bound with opposite sign, i.e. $\overline{\Delta} \geq \frac{1}{2}|q_+|$.

If the theory has $(2,2)$ supersymmetry, there exists an identical left-moving algebra with associated supercharges $Q_-, \overline{Q}_-$ and R-current J_-. Operators which are both left- and right-BPS form a particularly interesting set of operators; up to complex conjugation, there are two distinct sets, the left-chiral-right-chiral (c,c) operators

[5] For operators and their associated weights, we use the subscripts '$-$' and '$+$' to indicate left- and right-movers, respectively, and bars to indicate conjugation. The one exception is operators associated with the conformal algebra (T, L_n) and the associated quantities (c, h, Δ); there, we follow standard convention and use a bar to indicate right-moving and no bar to indicate left-moving.

and the left-anti-chiral-right-chiral (a,c) operators. Since the product of two chiral operators is again chiral, the (c,c) and (a,c) operators form rings known simply as *chiral rings* [89]. Although these operators form only a very small subset of all operators in the theory, they do in fact encode interesting information about the low-energy theory: three-point functions of operators in the ring calculate certain Yukawa-couplings in the effective IR field theory.

The set of states in the chiral ring form a very particular subset of the full Hilbert space of the sigma model, and in fact we can extract a 'sub-model' whose degrees of freedom are exactly those in the ring. The trick is to utilize the $(2,2)$ supersymmetry to isolate this particular subset [121, 122, 51]. This process involves two distinct steps which are often treated together, but which it will be important for us to separate.

The first step is what is known as *twisting* the sigma model. If we Wick-rotate the time-like coordinate on the worldsheet, the 2D Lorentz group becomes a $U(1)$ group which we shall call $U(1)_E$. Twisting now simply means redefining worldsheet spin to be the eigenvalue of a linear combination of $U(1)_E$ and the two R-symmetries, $U(1)_\pm$, rather than $U(1)_E$ alone. More specifically, we choose to measure the spin in the new theory with the shifted stress-energy tensor

$$\overline{T}^t = \overline{T} \pm \frac{1}{2} J_+ \quad \text{with eigenvalues} \quad \overline{h} = \overline{\Delta} \pm \frac{1}{2} q_+ \ , \tag{3.23}$$

and similarly for the left-movers. There are four possible choices of signs in the definitions of $\left(T^t, \overline{T}^t\right)$, but by complex conjugation only two are distinct. If we choose the two signs to be *different*, $(h, \overline{h}) = 0$ for operators in the (a,c) ring and the supercharges Q_- and $\overline{Q}_+$ become spinless (with a particular choice of overall sign). If we choose the signs to be *the same*, $(h, \overline{h}) = 0$ for operators in the (c,c) ring and the supercharges $\overline{Q}_-$ and $\overline{Q}_+$ become spinless. For sigma models, the combined R-symmetry $J_- + J_+$ is anomalous unless the target space X is Calabi-Yau, so the (c,c) ring does not exist for sigma models with more general target spaces.

The second step involves singling out the states in the chiral rings as the only physical ones. This is now relatively straightforward. Consider the (a,c) ring: after doing the appropriate twisting, we can define the spinless supercharge $Q_A = Q_- + \overline{Q}_+$ which annihilates all the elements of the ring and is nilpotent, $Q_A^2 = 0$. We now *define* the physical states to be those in the cohomology of Q_A, i.e. the physical Hilbert space is[6]

$$\mathcal{H}_{phys} = \{Q_A|\psi\rangle = 0\} \,/ \sim \ , \tag{3.24}$$

where the equivalence relation is

$$|\psi\rangle \sim |\psi\rangle + Q_A|\phi\rangle, \quad \text{all } |\phi\rangle \tag{3.25}$$

The resulting theory is know as the *topological A-model*. The theory resulting from applying the same procedure to the (c,c) ring using $Q_B = \overline{Q}_- + \overline{Q}_+$ is correspondingly called the *topological B-model*.

The chiral rings, seen as cohomologies of supercharges, have a nice geometric interpretation which is most transparent in the A-model. Since all operators in this theory must be annihilated by Q_A, we can write down the supersymmetry transformation of the NLSM fields under this particularly supersymmetry and explicitly see which combinations can give zero variation. It turns out that all covariant, Q_A-invariant

[6]This is the same as taking Q_A to be a BRST operator, in parallel with the BRST-quantization of gauge theories (see e.g. [120]).

operators are of the form

$$\mathcal{O}_A = \omega(\phi)_{ab\cdots\overline{a}\overline{b}\cdots}\chi^a\chi^b\cdots\zeta^{\overline{a}}\zeta^{\overline{b}}\cdots \ , \tag{3.26}$$

where χ and ζ are the two complex fermions fields which are spinless after twisting. Since the fermions anticommute, this is reminiscent of the component presentation of a geometric form. Indeed, we can make the formal identifications

$$\chi^a \leftrightarrow dz^a, \quad \zeta^{\overline{a}} \leftrightarrow d\overline{z}^{\overline{a}}, \quad Q_- \leftrightarrow \partial, \quad \overline{Q}_+ \leftrightarrow \overline{\partial} \ , \tag{3.27}$$

where $(z, \overline{z})$ are coordinates on the target space X (a complex manifold because of $(0,2)$ supersymmetry). With this identification, the supercharge Q_A now acts on the space of sigma model fields like the exterior derivative d acts on the space of forms on X. The cohomology of Q_A thus become identified with the de Rham cohomology of X:

$$\mathcal{H}^A_{phys} \leftrightarrow H^*_{DR}(X) \tag{3.28}$$

Taking OPEs of operators $\mathcal{O}_A$ maps to wedging forms on X. This identification, however, is only strictly correct in the infinite-radius limit of X. The NLSM is a quantum theory and receives quantum corrections which modifies the ring structure. We can therefore think of the A-model ring as giving a quantum extension of the De Rham cohomology ring. Supersymmetry rules out perturbative corrections, so the only non-vanishing quantum effects arise from instantons which are suppressed for large X. This is a concrete example of the way spacetime structure is modified by quantum effects in string theory. A similar mapping can be found for the the B-model, with the ring mapping to the Dolbeault cohomology ring (see e.g. [74]).

The A- and B-model are *topological* field theories because any physical quantity is independent of the metric on the worldsheet Σ. In particular, all correlation functions $\langle \mathcal{O}_1(z_1, \overline{z}_1), \mathcal{O}_2(z_2, \overline{z}_2), \ldots \rangle$ of operators $\mathcal{O}_i$ in the theory (i.e. in Q-cohomology) are *independent* of the insertion coordinates and therefore evaluate to numbers rather than functions of $(z_i, \overline{z}_i)$. To see this, consider changing one of the coordinates slightly, e.g. taking $z_1 \to z_1 + \epsilon$. The change in a correlation function is

$$\delta_\epsilon \langle \mathcal{O}_1(z_1, \overline{z}_1), \mathcal{O}_2(z_2, \overline{z}_2), \ldots \rangle = \epsilon \langle P_z \mathcal{O}_1(z_1, \overline{z}_1), \mathcal{O}_2(z_2, \overline{z}_2), \ldots \rangle \tag{3.29}$$

where P_z is the generator of holomorphic translations. But it may be shown that in the A- and B-models, this operator is trivial in cohomology, i.e. it can be written as (for the A-model)

$$P_z = \{Q_A, \Pi_z\} \tag{3.30}$$

for some operator Π_z. Inserting P_z in a correlation function thus amounts to inserting 0 and the variation above vanishes. Hence, the correlation function is invariant under changes in the coordinates.

Notice that the A- and B-models only differ by the relative sign of J_- and J_+ in the twisting and by the corresponding choice of Q_- versus $\overline{Q}_-$ in the supercharge. This simple observation—that the A- and B-models are related by a $\mathbb{Z}_2$ involution—is in fact a statement of the celebrated *mirror symmetry* (see [74] for an extensive review). Mirror symmetry is usually presented as a symmetry within pairs of Calabi-Yau manifolds and may be seen as a generalization of T-duality [114]. However, if we consider a mirror pair Calabi-Yaus, (X, Y), we find that the topological A-model on X is equivalent to the B-model on Y and vice versa. Thus, swapping the A- and B-models on X is equivalent to replacing X with its mirror Y. Mirror symmetry is also known to exchange the Kähler moduli of X with the complex structure moduli of Y and vice versa, and indeed it can be shown that, for a given Calabi-Yau X, the A-model encodes the information about the Kähler parameters while the B-model encodes the complex structure.

3.2.3 Gauged Linear Sigma Models

We have uncovered many nice structures in the non-linear sigma model, but we face a problem: the NLSM explicitly involves the fields $G_{\mu\nu}, B_{\mu\nu}$ etc., yet no metric for a Calabi-Yau sixfold is explicitly known (with the exception of the trivial case of T^6). This makes it very hard to do concrete calculations with an NLSM. An elegant approach that partially circumvents this problem—and has other virtues as well—was invented by Witten in the early 1990s [123]. The basic idea is quite simple: rather than working with a complicated theory, we work instead with simpler theory that reduces to the complicated one as we RG-flow into the IR. The particular theories Witten used were gauge theories with $(2, 2)$ or $(0, 2)$ supersymmetry, standard flat-space kinectic terms, and a $U(1)^s$ gauge group; he consequently dubbed them *gauged linear sigma models* (GLSMs). There is, of course, a price to pay for taking this roundabout route. Most quantities change under RG-flow, so being able to calculate them in UV using a GLSM does not tell us much about the physics of the IR NLSM. However, all the observables contained in the chiral rings may be shown to be RG-invariant and can hence be reliably calculated in the GLSM. Furthermore, by using a so-called *Born-Oppenheimer approximation*, one can in fact extract the low-energy 4D particle spectrum from a GLSM [82].

There is much interesting physics arising from these models, but to avoid being taken too far afield we will concentrate on the formalism. As in the previous sections, we shall follow the road less taken in discussing theories with $(2,2)$ and $(0,2)$ supersymmetry and start with $(0,2)$ since it will be our main interest in what follows. For a more complete discussion see the original paper [123] or the review by Distler [47].

(0,2) GLSMs

Our conventions and notation follow [123], with all factors of α' suppressed throughout. We will use $(0,2)$ superspace notation, with the coordinates $(y^+, y^-, \theta^+, \bar{\theta}^+)$ where $y^\pm = (y^0 \pm y^1)$ and $(\theta^+, \bar{\theta}^+)$ are the Grassmannian ccordinates.

We begin with the gauge multiplet. The right-moving gauge covariant superderivatives $\mathcal{D}_+, \overline{\mathcal{D}}_+$, satisfy the algebra

$$\mathcal{D}_+^2 = \overline{\mathcal{D}}_+^2 = 0, \qquad -\tfrac{i}{4}\{\, \mathcal{D}_+, \overline{\mathcal{D}}_+\} \, = \nabla_+ = \partial_+ + iQv_+ \ , \tag{3.31}$$

where Q is the charge of the field on which they act and v_+ is the $+$-component of the ordinary gauge potential. These imply that, in a suitable basis, we have the explicit expressions

$$\mathcal{D}_+ = \frac{\partial}{\partial\theta^+} - 2i\overline{\theta}^+\nabla_+, \qquad \overline{\mathcal{D}}_+ = -\frac{\partial}{\partial\overline{\theta}^+} + 2i\theta^+\nabla_+, \qquad \mathcal{D}_- = \partial_- + \tfrac{i}{2}QV_-,$$

where $V_\pm$ are real vector superfields which transform under a gauge transformation with (uncharged) chiral gauge parameter $\overline{\mathcal{D}}_+\Lambda = 0$ as $\delta_\Lambda V_- = \partial_-(\Lambda + \overline{\Lambda})$ and $\delta_\Lambda V_+ = \frac{i}{2}(\Lambda - \overline{\Lambda})$; $\nabla_\pm$ are the usual gauge covariant derivatives. This allows us to fix to Wess-Zumino gauge in which

$$V_+ = \theta^+\overline{\theta}^+ 2v_+ \qquad V_- = 2v_- - 2i\theta^+\overline{\lambda}_- - 2i\overline{\theta}^+\lambda_- + 2\theta^+\overline{\theta}^+ D.$$

Note that V_- contains a complex left-moving gaugino. Finally, the natural field strength is a fermionic chiral superfield,

$$\begin{aligned}\Upsilon &= 2[\overline{\mathcal{D}}_+, \mathcal{D}_-] = \overline{\mathcal{D}}_+(2\partial_- V_+ + iV_-) \\ &= -2\{\lambda_- - i\theta^+(D + 2iv_{+-}) - 2i\theta^+\overline{\theta^+}\partial_+\lambda_-\} \ ,\end{aligned} \tag{3.32}$$

for which the natural action is

$$S_\Upsilon = -\frac{1}{8e^2}\int \mathrm{d}^2y \, d\theta^+ d\bar{\theta}^+ \ \overline{\Upsilon}\Upsilon = \frac{1}{e^2}\int \mathrm{d}^2y \left\{2v_{+-}^2 + 2i\overline{\lambda}_-\partial_+\lambda_- + \frac{1}{2}D^2\right\}, \tag{3.33}$$

where $d^2y = dy^0 dy^1$ and we use conventions where $\int d\theta^+\theta^+ = \int \bar{\theta}^+ d\bar{\theta}^+ = 1$.

Matter multiplets are similarly straightforward. A bosonic superfield satisfying $\overline{\mathcal{D}}_+\Phi = 0$ is called a *chiral* supermultiplet and contains a complex scalar and a right-moving complex fermion $\Phi = \phi + \sqrt{2}\theta^+\psi_+ - 2i\theta^+\bar{\theta}^+\nabla_+\phi$. Under gauge transformations $\Phi \to e^{-iQ(\Lambda+\overline{\Lambda})/2}\Phi$. The gauge invariant Lagrangian is given by

$$\begin{aligned}S_\Phi &= -i\int \mathrm{d}^2y \, \mathrm{d}^2\theta \ \overline{\Phi}\mathcal{D}_-\Phi, \\ &= \int \mathrm{d}^2y \Big\{ -|\nabla_\alpha\phi|^2 + 2i\overline{\psi}_+\nabla_-\psi_+ - iQ\sqrt{2}\overline{\phi}\lambda_-\psi_+ + iQ\sqrt{2}\phi\overline{\psi}_+\overline{\lambda}_- + QD|\phi|^2\Big\} \ ,\end{aligned} \tag{3.34}$$

where the metric is given by $\eta^{+-} = -2$.

Left-moving fermions transform in their own supermultiplet, the *Fermi* supermultiplet, which satisfies the chiral constraint

$$\overline{\mathcal{D}}_+\Gamma = \sqrt{2}E \tag{3.35}$$

and has component expansion $\Gamma = \gamma_- - \sqrt{2}\theta^+ G - 2i\theta^+\bar{\theta}^+\nabla_+\gamma_- - \sqrt{2}\bar{\theta}^+ E$, where E is a bosonic chiral superfield with the same gauge charge as Γ and satisfying $\overline{\mathcal{D}}_+E = 0$.

The action for Γ is given by

$$\begin{aligned} S_\Gamma &= -\frac{1}{2}\int \mathrm{d}^2y\, d^2\theta\ \overline{\Gamma}\Gamma \\ &= \int \mathrm{d}^2y\ \left\{2i\overline{\gamma}_-\nabla_+\gamma_- + |G|^2 - |E|^2 - \left(\overline{\gamma}_-\frac{\partial E}{\partial\phi_i}\psi_{+i} + \overline{\psi}_{+i}\frac{\partial\overline{E}}{\partial\overline{\phi}_i}\gamma_-\right)\right\}\ . \end{aligned} \tag{3.36}$$

In general, we can add superpotential terms to our Lagrangian. Since these are integrals over a single supercoordinate, the superpotential can be written as a sum of Fermi superfields Γ_m times holomorphic functions $J^m(\Phi)$ of the chiral superfields,

$$\begin{aligned} S_{\mathcal{W}} &= \frac{1}{\sqrt{2}}\int \mathrm{d}^2y\, d\theta^+\ \Gamma_m J^m|_{\bar\theta^+=0} + \text{h.c.}, \\ &= -\int \mathrm{d}^2y\ \left\{G_m J^m(\phi_i) + \gamma_{-m}\psi_{+i}\frac{\partial J^m}{\partial\phi_i}\right\} + \text{h.c.}\ . \end{aligned} \tag{3.37}$$

Since Γ_m is not an honest chiral superfield but satisfies (3.35), we need to impose the condition

$$E\cdot J = 0 \tag{3.38}$$

to ensure that the superpotential is chiral. Finally, since Υ is a chiral fermion, we can also add a Fayet-Iliopoulos (FI) term of the form

$$S_{\text{FI}} = \frac{it}{4}\int \mathrm{d}^2y\, d\theta^+\ \Upsilon|_{\bar\theta^+=0} + \text{h.c.} = \int d^2y\,(-rD + 2\theta v_{+-})\ , \tag{3.39}$$

where $t = r + i\theta$ is the complexified FI parameter.

To see how these elements come together, let us construct a model that flows to an NLSM for a vector bundle $\mathcal{V}\to S$ over a $K3$ hypersurface S in a resolved weighted projective space $W\mathbb{P}^3$. The associated GLSM includes the gauge group $G = U(1)^s$ with s gauge field-strengths Υ_a, $(3+s)$ chiral scalars $\Phi_{i=1,\ldots 3+s}$ with charges $Q_i^a > 0$, a set of c neutral scalars $\Sigma_{A=1\ldots c}$, a single chiral scalar Φ_0 with charges $-d^a < 0$, r Fermi multiplets $\Gamma_{m=1,\ldots r}$ with charges q^a_m satisfying the constraints $\overline{\mathcal{D}}_+\Gamma_m = \sqrt{2}\Sigma_A E^A_m(\Phi)$, and a single chiral fermion Γ_0 with charges $-m^a$. We finally add *spectator fields* as needed to ensure vanishing of the one-loop tadpole for D^a—they otherwise decouple entirely in the IR and we shall generally suppress them [48, 47]. The Lagrangian now takes the form

$$\mathcal{L} = -\int d^2\theta \left[\frac{1}{8e_a^2}\overline{\Upsilon}_a\Upsilon_a + \frac{1}{2}\overline{\Gamma}_m\Gamma_m + i\overline{\Sigma}_A\partial_-\Sigma_A + i\overline{\Phi}_i(\partial_- + \tfrac{i}{2}Q^a_i V_{-a})\Phi_i\right] + \frac{1}{\sqrt{2}}\int d\theta^+ \left[\Gamma_0 G(\Phi_i) + \Gamma_m\Phi_0 J^m(\Phi_i) + i\frac{\sqrt{2}}{4}t^a\Upsilon_a\right] + \text{h.c.}\ . \tag{3.40}$$

Integrating out the auxiliary fields results in a scalar potential

$$U = \sum_a \frac{e_a^2}{2}(\textstyle\sum_i Q^a_i|\phi_i|^2 - d^a|\phi_0|^2 - r^a)^2 + |G(\phi)|^2 + \sum_m \left(|\phi_0|^2|J^m(\phi)|^2 + \left|\textstyle\sum_{A=1}^c\sigma_A E^A_m(\phi)\right|^2\right)\ . \tag{3.41}$$

Given an appropriate choice of r^a,[7] as we RG-flow to the IR the model effectively becomes confined to the minimal-energy subspace corresponding to $U = 0$ (and hence unbroken supersymmetry). This subspace becomes the target space of the corresponding NLSM. Setting the first term in U to zero forces the ϕ_i to take values on a bundle over a resolved weighted projective space $W\mathbb{P}^3$. The simplest case is that of a single $U(1)$ ($s = 1$), in which case the bundle is simply $\mathcal{O}(-d) \to \mathbb{P}_3$ (for $r > 0$), with $\phi_{1\ldots\ldots 4}$ being projective coordinates on the $\mathbb{P}_3$ and ϕ_0 providing the fiber coordinate.

Demanding that the remaining terms vanish imposes a further set of constraints,

[7] We have glossed over the fact that a GLSM can in fact give rise to very different types of low-energy behavior (which Witten referred to as *phases*), of which a purely geometric NLSM is only one example. For a generic choice of r^a, the IR theory will be in a hybrid phase, with some fields acting like coordinates in an NLSM (ϕ_i taking values on a continuous space) and some resembling field in *Landau-Ginsburg* models (isolated vacua in ϕ_i-space with an associated superpotential governing fluctuations around it). These intricacies are well covered in [74].

the solutions to which form a compact surface in the resolved projective space. Non-singularity of this surface requires $G(\Phi_i)$ and $J^m(\Phi_i)$ to be transverse,

$$\begin{aligned} G = \frac{\partial G}{\partial \phi_1} = \frac{\partial G}{\partial \phi_2} = \cdots = 0 \qquad &\Longleftrightarrow \qquad \forall i : \phi_i = 0 \ , \\ G = J^1 = J^2 = \cdots = 0 \qquad &\Longleftrightarrow \qquad \forall i : \phi_i = 0 \ . \end{aligned}$$

In the relevant geometric phase, the Yukawa interactions

$$\begin{aligned} \mathcal{L}_{Yuk} \;\; = \;\; & -\Bigg[\gamma_{-m}\left(\psi_{+0}J^m + \phi_0\psi_{+i}\frac{\partial J^m}{\partial \phi_i}\right) + \sum_i \sqrt{2} i Q_i^a \overline{\phi}_i \lambda_{-a}\psi_{+i} \\ & +\overline{\gamma}_{-m}\left(\eta_{+A}E_m^A + \psi_{+i}\sigma_A\frac{\partial E_m^A}{\partial \phi_i}\right) + \gamma_{-0}\psi_{+i}\frac{\partial G}{\partial \phi_i}\Bigg] + \text{h.c.} \end{aligned}$$

give masses to various linear combinations of the right- and left-moving fermions. The massless right-moving fermions couple to a bundle which fits into two exact sequences. For instance, for a single $U(1)$ we have

$$\begin{aligned} & 0 \to \mathcal{O}_{W\mathbb{P}} \xrightarrow{Q_i\phi_i} \oplus_i \mathcal{O}_{W\mathbb{P}}(Q_i) \to T_{W\mathbb{P}} \to 0 \ , \\ & 0 \to T_S \to T_{W\mathbb{P}}|_S \xrightarrow{\partial_{\phi_i} G} \mathcal{O}_S(d) \to 0 \ , \end{aligned} \tag{3.42}$$

so the massless right-moving fermions couple to T_S. Similarly, the bundle $\mathcal{V}_S$ to which the massless left-moving fermions couple, also fits into a pair of short exact sequences,

$$\begin{aligned} & 0 \to \oplus_A \mathcal{O}_{W\mathbb{P}} \xrightarrow{E_m^A} \oplus_m \mathcal{O}(q_m) \to \mathcal{V}_{W\mathbb{P}} \to 0 \ , \\ & 0 \to \mathcal{V}_S \to \mathcal{V}_{W\mathbb{P}}|_S \xrightarrow{J^m} \mathcal{O}(m) \to 0 \ . \end{aligned} \tag{3.43}$$

The function $G(\phi)$ thus determines the form of the NLSM tangent bundle, while $E_m^A(\phi)$ and $J^m(\phi)$ shape the gauge bundle. By judicious choices of these function, a range of low-energy gauge groups can be obtained [48, 47].

(2,2) GLSMs

A special class of $(0,2)$ theories have enhanced $(2,2)$ supersymmetry. To describe these theories, we enlarge our superspace by adding two further fermionic coordinates, $(y^+, y^-, \theta^+, \bar{\theta}^+, \theta^-, \bar{\theta}^-)$, and introducing the corresponding supercovariant derivatives

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - 2i\bar{\theta}^\pm\partial_\pm \ , \qquad \overline{D}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} + 2i\theta^\pm\partial_\pm \ . \tag{3.44}$$

Unlike the $(0,2)$ case, there are two kinds of $(2,2)$ chiral multiplets, *chiral* multiplets satisfying

$$\overline{D}_+\Phi_{2,2} = \overline{D}_-\Phi_{2,2} = 0 \ , \tag{3.45}$$

and *twisted chiral* multiplets satisfying

$$\overline{D}_+Y_{2,2} = D_-Y_{2,2} = 0 \ . \tag{3.46}$$

Both have the field content of one $(0,2)$ chiral and one $(0,2)$ Fermi multiplet,

$$\Phi_{2,2} = \Phi + \sqrt{2}\theta^-\Gamma_- - 2i\theta^-\bar{\theta}^-\partial_-\Phi \ , \qquad Y_{2,2} = Y + \sqrt{2}\bar{\theta}^-F + 2i\theta^-\bar{\theta}^-\partial_-Y \ .$$

The kinetic term for these fields are written as integrals over all of $(2,2)$ superspace:

$$\int d^4\theta \ \overline{\Phi}\Phi \ , \qquad \int d^4\theta \ \overline{\Upsilon}\Upsilon \tag{3.47}$$

For a charged chiral field, the action becomes

$$\int d^4\theta \ \overline{\Phi}e^{2QV_{2,2}}\Phi \ . \tag{3.48}$$

Here $V_{2,2}$ is the $(2,2)$ vector, whose field strength is a twisted chiral multiplet $\Sigma = \frac{1}{\sqrt{2}}\overline{D}_+D_-V_{2,2}$. It is built out of an uncharged $(0,2)$ chiral multiplet Σ_0 and a $(0,2)$ vector multiplet $V_\pm$ as

$$V_{2,2} = V_+ + \theta^-\bar{\theta}^-V_- + \sqrt{2}\bar{\theta}^+\theta^-\Sigma_0 + \sqrt{2}\bar{\theta}^-\theta^+\overline{\Sigma}_0 \ , \tag{3.49}$$

where $\Sigma_0 = \sigma - i\sqrt{2}\theta^+\overline{\lambda}_+ - 2i\theta^+\overline{\theta^+}\partial_+\sigma$ for agreement with [123], and $\delta_g V_{2,2} = \frac{i}{2}\left(\Lambda_{2,2} - \overline{\Lambda}_{2,2}\right)$. A $(2,2)$ superpotential is given by a holomorphic function $W(\Phi)$ of the chiral fields and takes the form

$$\mathcal{L}_W = \int d^2\theta \; W(\Phi) + \text{h.c.} \tag{3.50}$$

Finally, the standard FI-term is

$$\mathcal{L}_{FI} = \frac{-t}{2\sqrt{2}} \int d^2\tilde{\theta} \; \Sigma + \text{h.c.} = -rD + 2\theta v_{+-} \; , \tag{3.51}$$

where $t = r + i\theta$ and $d^2\tilde{\theta} = d\theta^+ d\bar{\theta}^-$.

Lastly, we note that a $(2,2)$ chiral multiplet with $U(1)$-charge Q reduces to a charged $(0,2)$ chiral multiplet Φ and a charged Fermi multiplet Γ satisfying

$$\overline{\mathcal{D}}_+\Gamma = \sqrt{2}E \tag{3.52}$$

in $(0,2)$ notation, and where E is given by

$$E = \sqrt{2}Q\Sigma_0\Phi \; . \tag{3.53}$$

We will omit the subscripts “2,2” in what follows as it should always be clear from the context to which supersymmetry we refer.

Chapter 4

Topological Heterotic Rings

The existence of chiral rings in $\mathcal{N}=(2,2)$ sigma models and the construction of the corresponding topological models relied heavily on the large amount of supersymmetry—indeed, we defined the physical states in the rings by cohomologies of supercharges. The supercharges Q_A and Q_B, in turn, were constructed as combination of left- and right-moving supersymmetry. *A priori*, it would thus seem unlikely that this delicate mathematical structure should persist if we were to cut the symmetry down to half and consider $\mathcal{N} = (0, 2)$.

As is shown in this chapter, this expectation in fact turns out to be incorrect: chiral rings exist quite generically in heterotic theories [2]. The argument is surprisingly simple and parallels that used by Lerche, Vafa, and Warner to prove the existence of the $(2, 2)$ rings in their original paper on the topic [89]. That chiral rings persist in heterotic models is an indication that they are more amenable to explicit analysis than one might have assumed, and even hints at the possibility that there could exists heterotic topological theories.

4.1 Introduction

$\mathcal{N}$=(2, 2) sigma models have been used extensively to study the quantum geometry of spacetime using the connections described in the previous chapter (for a partial list of reference see [121, 12, 11, 10, 118, 78] and references therein). Their chiral rings are among their most remarkable features, reducing to classical cohomology rings of the target manifold X in the large volume limit and defining *quantum* cohomology rings at finite radius where worldsheet instantons correct the classical result [89, 122].

Since generic (2, 2) models may be smoothly deformed into (0, 2) models, it is natural to wonder if there is a (0, 2) generalization of the quantum cohomology ring which reduces to the finite-dimensional (a,c) ring on the (2, 2) locus. As discussed above, this seems unlikely at first glance since the space of chiral operators is infinite dimensional in the absence of left-moving supersymmetry. Moreover, while (2, 2) supersymmetry ensures the non-singularity of the OPEs of (a,c) chiral operators (from which topological invariance of their correlators follows), general right-chiral operators in (0, 2) models have singular (and thus metric-dependent) OPEs. Finally, worldsheet instanton effects can be much more dangerous and uncontrolled than in the more well-understood (2, 2) models—indeed, until recently [112, 13, 14], it was widely believed that most (0, 2) models were destabilized by instantons and could not be defined non-perturbatively [45, 46]. But in fact, finite-dimensional apparently topological rings *have* been identified in several (0, 2) theories, both by mirror symmetry [1] and through explicit construction in certain exactly solved (0, 2) models [23]. The problem thus appears to be not *if* they exist, but *when*.

Classical geometry provides a hint in the case of (0, 2) NLSMs on holomorphic vector bundles over Kähler targets, $\mathcal{V} \to X$, where the right-moving supercharge maps to the Dolbeault operator on X twisted in the bundle $\mathcal{V}$ to which the left-moving fermions couple. In these models, while the cohomology of the right-moving supercharge is in general infinite dimensional (and is related to the elliptic cohomology of X), $H^*(X, \wedge^* \mathcal{V})$ forms a finite dimensional sub-algebra. When $\mathcal{V}$ is a smooth deformation of T_X, this is a smooth deformation of the de Rham cohomology ring of X, and the usual trace, given by integration over X, is maintained. For more general $\mathcal{V}$, integration on X provides a natural trace if $\wedge^{top}\mathcal{V} = K_X^*$, where K_X is the canonical bundle over X; the existence of a trace makes our ring a Frobenius ring. Suggestively, $\wedge^{top}\mathcal{V} = K_X^*$ implies the preservation of a left-moving $U(1)$ current algebra on the worldsheet; if the $(0,2)$ theory is a deformation of a $(2,2)$ theory by an element of $H^1(X, \mathrm{End}(T_X))$, this $U(1)$ is the unbroken left-moving R-symmetry. It is thus natural to suppose that the ring of operators which computes the de Rham cohomology in the classical limit of a $(2,2)$ model persists as a ring away from $(2,2)$ loci.

The main result of this chapter is a proof of existence for finite-dimensional topological rings in $(0,2)$ theories containing conserved left-moving $U(1)$ currents. While the A and B twists are only quasi-topological away from $(2,2)$ loci, their ground rings are fully topological on open sets in the $(0,2)$ moduli space and sometimes globally, reducing to the (a,c) and (c,c) rings at $(2,2)$ loci and to classical sheaf cohomology rings at large radius. They thus define a quantum deformation of sheaf cohomology.

Let us first briefly outline the argument. We begin by defining the set of (a,c) and (c,c) operators away from $(2,2)$ loci. The familiar definition of the (a,c) and (c,c) operators as the cohomology of left and right moving supercharges clearly does not generalize. Happily, a familiar stratagem from Hodge theory suggests a natural definition which *does* generalize: since, at the $(2,2)$ point, $\{\overline{Q}_-, Q_-\} = L_0 - \frac{1}{2}J_{-0}$ and $Q_-^2{=}0$, where Q_- is the left-moving supercharge of the $(2,2)$ point, the kernel of $L_0 - \frac{1}{2}J_{-0}$ is in one-to-one correspondence with cohomology classes of Q_-. We thus focus attention on the set of states satisfying $\Delta = \pm\frac{1}{2}q_-$ within right-moving Q_+-cohomology, which we refer to as the A and B operators.

We then prove that the OPE of these operators is non-singular on open patches of the bundle moduli space, and globally under certain conditions. The argument is quite simple. First, quantization of worldsheet spin ensures that left- and right-moving conformal dimensions vary in lock step as we vary bundle moduli[1] i.e. $\Delta - \overline{\Delta} = n \in \mathbb{Z}$ (we will see that the spins are always integer in the twisted models we work with). As a result, the holomorphic (i.e. left-moving) dimension of a right-chiral operator in a completely generic $(0,2)$ model is bounded from below despite the absence of a left-moving BPS bound. By working around special points in moduli space where quantum corrections may be controlled, e.g. $(2,2)$ loci or large radius, we will be able to forbid singular terms in the OPEs of A or B-ring operators for finite motions in moduli space, ensuring that their correlators are independent of insertion points and that the A- and B-rings close under OPE. Local results in hand, the left-moving $U(1)$ current-algebra provides global statements: in the case of $(0,2)$ CFTs with bundles of rank less than 8, unitarity will actually forbid the appearance of singular terms globally on the moduli space; when the rank is 8 or greater, it puts powerful

[1]We will assume throughout that the spectrum varies smoothly as we vary bundle moduli—importantly, this is the case at generic points in the moduli space of good string compactifications. At branch points of the moduli space, the story is modified, as we shall discuss below.

constraints on the form such operators must take.

It is important to emphasize that these arguments rely on $(0, 2)$ supersymmetry and the existence of a (possibly anomalous) left-moving $U(1)$ current algebra, but not on sigma model perturbation theory—they are exact to all orders and non-perturbatively in the sigma model coupling. In particular, these rings are not violated by worldsheet instantons. This fits nicely with conjectures for the quantum cohomology of $(0, 2)$ models derived via mirror symmetry [1], in which both perturbative and worldsheet instanton contributions respected the ring structure of the A and B models.

4.2 The A and B Operators in $(0, 2)$ Theories

Our aim is to identify topological rings in $(0, 2)$ models which deform to the (a,c) and (c,c) rings at $(2, 2)$ loci. As such, it is useful to observe that, since the definition of the A and B twists makes no reference to supersymmetry but only to non-anomalous left- and right-moving $U(1)$ currents by which to deform the Lorentz generators, any $(0, 2)$ model with a good left-moving $U(1)$ current admits the A (and, if $c_1(X) = c_1(\mathcal{V}) = 0$, B) twist as defined in the previous chapter.

Of course, the (a,c) and (c,c) operators of $(2, 2)$ theories are conventionally defined as those annihilated by the appropriate right *and* left moving supercharges, as reviewed above; in the twisted models, they correspond to the cohomology of the left and right moving scalar supercharges. This definition clearly does *not* generalize to $(0, 2)$ models, in which the left-moving supercharges are entirely absent, leaving us with a single right-moving scalar supercharge whose cohomology is the infinite dimensional space of right-chiral operators—while a beautiful mathematical object in its own right, related to the elliptic cohomology of X, this is not the ring we are looking for.[2]

There is, however, an alternate definition of the (a,c) and (c,c) operators which *does* generalize, as mentioned above. The basic strategy is to pullback to the world-sheet the usual Hodge Theory relation between the Dolbeault (or sheaf) cohomology, $H^p(X, \wedge^q T_X^*)$, and the zero eigenspace of the elliptic operator $\overline{\partial}^\dagger\overline{\partial} + \overline{\partial}\overline{\partial}^\dagger$, i.e. the set of harmonic (p,q)-forms: since (after twisting) $\overline{Q}_-{}^2 = 0$ and $\{\overline{Q}_-, Q_-\} = L_0^t$, states of $L_0^t = 0$ are in one to one correspondence with $\overline{Q}_-$-cohomology classes. We may thus define the set of (a,c) and (c,c) operators in twisted $(2,2)$ theories as the sub-set of $\overline{Q}_+$-cohomology satisfying $h = \Delta - \frac{1}{2}|q_-| = 0$. While equivalent to the conventional definition of (a,c) and (c,c) operators at $(2,2)$ points, this definition generalizes naturally to any $(0,2)$ model with a conserved left-moving $U(1)$ by which we can twist the left-movers.[3] Quantum mechanically, while these operators clearly form a sub*space*, it is not entirely obvious that they form a sub*ring*. Proving that will be the task of the next section; the task of the remainder of this section will be to make our definition precise. We begin by reviewing some details of $(0,2)$ non-linear sigma-models, which will be our main examples throughout.

[2]In fact, while this work was being completed, two preprints addressing precisely this topic appeared on the arXiv [83, 124]; in particular, both texts seek to provide a physical interpretation of the relatively well-developed mathematical theory of chiral de Rham operators in terms of the full $\overline{Q}_+$-cohomology in the A-twist of $(2,2)$ model or half-twist of a (0,1) or $(0,2)$ theory, respectively. Our interest differs from these extremely interesting papers both in studying finite-dimensional sub-rings sharing many of the properties of the familiar (a,c) or (c,c) ring, and in studying $(0,2)$ models with left-moving fermions coupling to interesting vector bundles.

[3]In what we hope will be a forgivable abuse of terminology, we will refer to these as the A and B operators, and the rings which they may (or may not) form as the A- and B-rings.

4.2.1 The Sigma Model

While our basic results will obtain for all suitably well-behaved $(0,2)$ CFTs, we will use $(0,2)$ NLSMs as the basic example throughout; it will thus be helpful in what follows to describe in some detail a number of particular features, making explicit use of the supersymmetry generator which will become the scalar supercharge after twisting.

Our generic example will be the non-linear σ-model on a rank-r holomorphic bundle over a Kähler manifold, $\mathcal{V} \to X$. Anomaly cancellation requires $c_2(\mathcal{V}) = c_2(T_X)$ and $c_1(\mathcal{V}) = c_1(T_X)$; as mentioned above and discussed in some detail below, we will actually require the slightly stronger constraint,

$$\wedge^r \mathcal{V} = K_X^* \ .$$

This ensures the existence of a natural inner product on $H^*(X, \wedge^* \mathcal{V})$.

The fields of the NLSM include coordinates ϕ^i on X, their right-moving fermionic superpartners ρ^i which couple to the tangent bundle T_X, and left-moving fermions λ^a (plus their auxiliary superpartners, l^a) which couple to the holomorphic vector bundle $\mathcal{V}$,

$$\lambda^a \in \Gamma\left(\sqrt{K_\Sigma} \otimes \phi^* \mathcal{V}\right) \qquad \rho^i \in \Gamma\left(\sqrt{\overline{K}_\Sigma} \otimes \phi^* T_X\right)$$
$$\lambda_a \in \Gamma\left(\sqrt{K_\Sigma} \otimes \phi^* \mathcal{V}^*\right) \qquad \rho^{\bar{\imath}} \in \Gamma\left(\sqrt{\overline{K}_\Sigma} \otimes \phi^* \overline{T}_X\right)$$

and mix under the right-moving supersymmetry, $\overline{Q}_+$, as

$$\begin{aligned}
\delta\phi^{\bar\imath} &= \rho^{\bar\imath} & \delta\phi^{i} &= 0 \\
\delta\rho^{\bar\imath} &= 0, & \delta\rho^{i} &= \partial_{\bar z}\phi^{i} \\
\delta\lambda_a &= l_a, & \delta\lambda^a &= 0 \\
\delta l_a &= 0, & \delta l^a &= D_{\bar z}\lambda^a + F^a{}_{bi\bar\jmath}(\phi)\lambda^b\rho^i\rho^{\bar\jmath}\ ,
\end{aligned} \tag{4.1}$$

where $F^a{}_{bi\bar\jmath} = -A^a{}_{bi,\bar\jmath}$ is the curvature form of $\mathcal{V}$ and

$$D_{\bar z}\lambda^a = \partial_{\bar z}\lambda^a + A^a{}_{bi}(\phi)\partial_{\bar z}\phi^i\lambda^b.$$

is the covariant derivative.[4] The action can then be written as

$$S = \int d^2z\ \{\overline{Q}_+, \chi\} + S_\omega\ , \tag{4.3}$$

where

$$\chi = g_{\bar\imath j}(\phi)\partial_z\phi^{\bar\imath}\rho^j + \lambda_a l^a \tag{4.4}$$

and

$$S_\omega = \frac{1}{2}\int d^2z\ g_{\bar\imath j}(\phi)(\partial_{\bar z}\phi^{\bar\imath}\partial_z\phi^j - \partial_z\phi^{\bar\imath}\partial_{\bar z}\phi^j) + i\int \phi^* B \tag{4.5}$$

[4]The second supersymmetry, Q_+, is

$$\begin{aligned}
\tilde\delta\phi^{i} &= \rho^{i}, & \tilde\delta\lambda^a &= l^a + A^a{}_{bi}\lambda^b\rho^i \\
\tilde\delta\rho^{i} &= 0, & \tilde\delta l^a &= -A^a{}_{bi}l^b\rho^i \\
\tilde\delta\phi^{\bar\imath} &= 0, & \tilde\delta\lambda_a &= 0 \\
\tilde\delta\rho^{\bar\imath} &= \partial_{\bar z}\phi^{\bar\imath}, & \tilde\delta l_a &= \partial_{\bar z}\lambda_a\ .
\end{aligned} \tag{4.2}$$

The supersymmetry algebra is satisfied, provided the $(2,0)$ part of the curvature vanishes:

$$A^a{}_{b[i,j]} - A^a{}_{c[i}A^c{}_{bj]} = 0\ .$$

Note that we have introduced a shift in the definition of the l^a, so as to make all of the $\overline{Q}_+$ supersymmetry variations gauge-covariant. This greatly simplifies many formulæ in the twisted model. In the untwisted model, one might prefer a more symmetrical choice.

is (i times) the pullback of the complexified Kähler form, $B + i\omega$. (To avoid the concomitant complexities, we take the 2-form, B, to be closed). In its full component glory, then,

$$S = \int d^2z \ \frac{1}{2} g_{\bar{\imath}j}(\phi)(\partial_z\phi^{\bar{\imath}}\partial_{\bar{z}}\phi^j + \partial_z\phi^j\partial_{\bar{z}}\phi^{\bar{\imath}}) - g_{\bar{\imath}j}(\phi)\rho^j\left(\delta^{\bar{\imath}}_{\bar{k}}\partial_z + \Gamma^{\bar{\imath}}_{\bar{k}\bar{l}}(\phi)\partial_z\phi^{\bar{l}}\right)\rho^{\bar{k}}$$
$$+ \lambda_a D_{\bar{z}}\lambda^a + F^a{}_{bi\bar{\jmath}}(\phi)\lambda_a\lambda^b\rho^i\rho^{\bar{\jmath}} + l_a l^a + i\int \phi^* B \ . \qquad (4.6)$$

Classically, this model possesses a right-moving R-symmetry and a left-moving flavour symmetry forming a $U(1)_+ \times U(1)_-$ global symmetry group under which ρ^i and $\rho^{\bar{\imath}}$ have charges $(\pm 1, 0)$, λ^a and λ_a have charges $(0, \pm 1)$, and l^a and l_a have charges $(\pm 1, \pm 1)$. Classically, then, we may shift the spins of all fields by a linear combination of their charges; in the model twisted by $J = (1-2s)J_- + (2\bar{s}-1)J_+$, the "fermion" transform as

$$\lambda^a \in \Gamma\left(K_\Sigma^{(1-s)} \otimes \phi^*\mathcal{V}\right) \ , \qquad \rho^i \in \Gamma\left(\overline{K}_\Sigma^{\bar{s}} \otimes \phi^* T_X\right) \ , \qquad (4.7)$$
$$\lambda_a \in \Gamma\left(K_\Sigma^{s} \otimes \phi^*\mathcal{V}^*\right) \ , \qquad \rho^{\bar{\imath}} \in \Gamma\left(\overline{K}_\Sigma^{(1-\bar{s})} \otimes \phi^* \overline{T}_X\right) \ .$$

Here, s and $\bar{s}$ label the spin of the left- and right-moving fermions; the untwisted theory has $s = \bar{s} = \frac{1}{2}$. The auxiliary bosonic fields transform as

$$l^a \in \Gamma\left(K_\Sigma^{(1-s)} \otimes \overline{K}_\Sigma^{\bar{s}} \otimes \phi^*\mathcal{V}\right) \ , \qquad l_a \in \Gamma\left(K_\Sigma^{s} \otimes \overline{K}_\Sigma^{(1-\bar{s})} \otimes \phi^*\mathcal{V}^*\right) \ . \qquad (4.8)$$

Quantum-mechanically, these $U(1)$ symmetries are anomalous, with the charge-violation on a genus-g surface given by

$$\delta q_- = (1-g)(1-2s)r + \phi^* c_1(\mathcal{V})$$
$$\delta q_+ = (1-g)(2\bar{s}-1)d + \phi^* c_1(T_X) \ .$$

In the untwisted ($s = \overline{s} = \frac{1}{2}$) model, the first terms vanish. To ensure that the twisted model has a non-anomalous Lorentz symmetry, we should twist only by a non-anomalous combination of global currents.[5] Since, in the models we consider, $c_1(T_X)$=$c_1(\mathcal{V})$, the current $J_+ - J_-$ is always nonanomalous, allowing us to twist by this $U(1)$ to obtain the $\overline{s} = s = 1$ A-model, in which the fermions transform as

$$\lambda^a \in \Gamma\left(\phi^*\mathcal{V}\right) \qquad \rho^i \in \Gamma\left(\overline{K}_\Sigma \otimes \phi^* T_X\right) \tag{4.9}$$
$$\lambda_a \in \Gamma\left(K_\Sigma \otimes \phi^*\mathcal{V}^*\right) \qquad \rho^{\bar{\imath}} \in \Gamma\left(\phi^* T_X^*\right) \ .$$

If $c_1(T_X)$=$c_1(\mathcal{V}) = 0$, i.e. X is Calabi-Yau, J_+ and J_- are separately nonanomalous and other twists are possible. For instance, the $s = 0$, $\overline{s} = 1$ B-model involves twisting by $J_- + J_+$, while the $s = \frac{1}{2}$, $\overline{s} = 1$ half-twisted model involves twisting by J_+ alone.[6]

Even in the non-Calabi-Yau case, we might be tempted to consider these other twists, or relax the condition $\wedge^r\mathcal{V} = K_X^*$. In doing so, however, we pay a price (in addition to giving up the existence of a trace on the algebra, as discussed below): while the local physics of these more general theories looks fairly familiar in σ-model perturbation theory, the $U(1)$ by which we twist is almost invariably violated by worldsheet instantons, changing the physics radically.[7] Consider, for instance, the $\mathbb{P}^1$

[5] While we require the twisted Lorentz symmetry to be non-anomalous, the global symmetry may, and generally will, pick up an anomaly after twisting.

[6] Note that the difference between the various twisted models is less dramatic in $(0,2)$ than in $(2,2)$ theories—in particular, *all* models are subject to worldsheet instanton corrections. That said, exchanging the roles of λ_a and λ^a while reversing the sign of J_- (and changing the l^a-dependence of (4.2)(4.1), which is trivial on-shell) maps $A(X,\mathcal{V})$ into $B(X,\mathcal{V}^*)$, imposing interesting constraints on the form of instanton corrections.

[7] On a flat worldsheet, the theory, 4.6, suffers from a σ-model anomaly, unless

$$ch_2(\mathcal{V}) - ch_2(T_X) = 0 \ .$$

On a curved worldsheet, there is an additional contribution the anomaly 4-form,

$$\frac{1}{2}c_1(\Sigma)\big((2s-1)c_1(\mathcal{V}) - (2\overline{s}-1)c_1(T_X)\big) \ .$$

model, with $\mathcal{V}=0$, discussed in [124]. In σ-model perturbation theory, there is a rich spectrum of operators in the $\overline{Q}_+$-cohomology (provided one does not restrict oneself to scaling dimension zero). However, worldsheet instantons correct the $\overline{Q}_+$-action in such a way that all the operators pair up, and the $\overline{Q}_+$-cohomology of the exact theory is empty.

Aside from the existence of other twists, there is another distinction between the Calabi-Yau case and more general "massive" σ-models with $c_1 \neq 0$ which will be very important for us below. At the classical level, both are conformally-invariant: $T_{z\overline{z}}=0$ and the other components of the stress tensor (for the A-model),

$$\begin{aligned} T_{zz} &= -g_{i\overline{\jmath}}(\phi)\partial_z\phi^i\partial_z\phi^{\overline{\jmath}} - \lambda_a D_z\lambda^a \\ T_{\overline{z}\overline{z}} &= -g_{i\overline{\jmath}}(\phi)\partial_{\overline{z}}\phi^i\partial_{\overline{z}}\phi^{\overline{\jmath}} + g_{i\overline{\jmath}}(\phi)\rho^i(\partial_{\overline{z}}\rho^{\overline{\jmath}} + \Gamma^{\overline{\jmath}}_{\overline{k}\overline{l}}\partial_{\overline{z}}\phi^{\overline{k}}\rho^{\overline{l}}) \end{aligned} \tag{4.10}$$

satisfy

$$\begin{aligned} [\overline{Q}_+, T_{zz}] &= -l_a D_z\lambda^a + \left[-g_{i\overline{\jmath}}\left(\partial_z\rho^{\overline{\jmath}} + \Gamma^{\overline{\jmath}}_{\overline{k}\overline{l}}\partial_z\phi^{\overline{k}}\rho^{\overline{l}}\right) + F^a{}_{b i\overline{\jmath}}\lambda_a\lambda^b\rho^{\overline{\jmath}}\right]\partial_z\phi^i \\ &= 0 \qquad \text{on-shell} \\ T_{\overline{z}\overline{z}} &= \{\overline{Q}_+, -g_{i\overline{\jmath}}\rho^i\partial_{\overline{z}}\phi^{\overline{\jmath}}\}\ . \end{aligned} \tag{4.11}$$

Thus $T_{\overline{z}\overline{z}}=0$ in $\overline{Q}_+$-cohomology, while T_{zz} descends to an operator on the $\overline{Q}_+$-cohomology.

The fact that T_{zz} is not $\overline{Q}_+$-exact, even classically, means that the $(0,2)$ A-model is a 2D *conformal* field theory, rather than a 2D *topological* field theory. Our interest in the ground ring is that it forms a "topological subsector" of this conformal field theory.

The untwisted theory ($s=\overline{s}=1/2$) is non-anomalous, but twisted theories are only sensible when this quantity vanishes.

Quantum mechanically, the conformal structure is violated by the one-loop β-function. Renormalization adds to the action a term of the form,

$$\Delta\chi_{1-\text{loop}} = \kappa_1\ R_{\bar{\imath}j}\partial_z\phi^{\bar{\imath}}\rho^j + \kappa_2\ g^{\bar{\imath}j}F^a{}_{b\bar{\imath}j}\lambda_a l^b$$

for some divergent constants $\kappa_{1,2}$ (with a concomitant shift of S_ω by the pullback of the Ricci form). In the Calabi-Yau case, we can make the conformal anomaly *vanish* by choosing the Ricci-flat metric and a solution to the Uhlenbeck-Yau equation, $g^{\bar{\imath}j}F^a{}_{b\bar{\imath}j} = 0$. In the "massive models", however, conformal invariance is necessarily lost, and there *is* nontrivial RG running. It is, however, $\overline{Q}_+$-trivial, and so does not affect the correlation functions of operators in the $\overline{Q}_+$-cohomology.[8] More precisely, $T_{z\bar{z}}$, while no-longer vanishing, remains $\overline{Q}_+$-exact,

$$T_{z\bar{z}} \propto \{\overline{Q}_+, \Delta G_{z\bar{z}}\}\ ,$$

and $T_{\bar{z}\bar{z}}$ remains $\overline{Q}_+$-exact, so, on the level of the $\overline{Q}_+$-cohomology, we are in almost as good shape as before. However, there is a fly in the ointment. Back in 4.11, we noted that, classically, $[\overline{Q}_+, T_{zz}]$ closed onto the equations of motion. In the massive A-model, this fails quantum mechanically:

$$[\overline{Q}_+, T_{zz}] = \partial_z V \neq 0\ , \tag{4.12}$$

where $V = R_{\bar{\imath}j}\partial_z\phi^j\rho^{\bar{\imath}} + \ldots$ and its derivative does *not* vanish by the equations of motion. As a result, general changes of holomorphic coordinate do *not* preserve $\overline{Q}_+$-cohomology classes, so the A-model is *not* conformal, though conservation of the stress

[8] In perturbation theory, that statement was precisely correct. At the level of worldsheet instantons, the result is to trade the exponential of the pullback of the Kähler form for the dimensionful scale, via dimensional transmutation.

tensor, $\partial_{\bar{z}} T_{zz} = -\partial_z T_{z\bar{z}}$, does ensure that the left-moving stress tensor is holomorphic up to $\overline{Q}_+$-trivial terms, $\partial_{\bar{z}} T_{zz} \sim 0$.

Fortunately, our arguments do not depend on full conformal invariance; as we shall see in the next section, all we really need is that holomorphic scaling dimension and momentum remain good quantum numbers in $\overline{Q}_+$-cohomology, i.e. that L_0 and L_{-1} commute[9] with $\overline{Q}_+$. This is easily verified. For example, 4.12 and the holomorphy of T_{zz} imply $[\overline{Q}_+, L_{-1}] = 0$. Similarly, since $\overline{Q}_+$ is by construction spinless after twisting,

$$0 = [S, \overline{Q}_+] = i[L_0, \overline{Q}_+] - i[\overline{L}_0, \overline{Q}_+] = i[L_0, \overline{Q}_+] \ ,$$

as $\overline{L}_0$ is a $\overline{Q}_+$-commutator (since $T_{\bar{z}\bar{z}}$ is). L_0 thus preserves cohomology class. This is enough for our purposes.

Summing up the last two paragraphs, the massive A-model is a holomorphic field theory invariant under global dilatations and translations, though not general holomorphic coordinate transformations. As we shall see, this provides just enough control to ensure the existence of the A- and B-rings in massive $(0, 2)$ models, at least in open balls around special points in the moduli space. We will return to this subtlety in our discussion of massive $(0, 2)$ models in the next section; for now we will restrict attention to models which were already conformal *before* twisting.

4.2.2 Bundle Moduli

It is instructive to work out the "integrated vertex operators" which represent infinitesimal deformations of the moduli of the bundle of our $(0, 2)$ σ-model. Infinites-

[9]Since $\partial_z T_{z\bar{z}}$ is nonzero, T_{zz} is not holomorphic so it does not make sense to speak of its Laurent coefficients, L_n. However, since $T_{z\bar{z}}$ is $\overline{Q}_+$-exact, so is the non-holomorphic dependence of T_{zz}. Thus, it does make sense to talk about the Laurent coefficients, L_n, when working modulo $\overline{Q}_+$-exact operators.

imal deformations of the holomorphic structure of the vector bundle, $\mathcal{V}$, correspond to elements $h \in H^1(X, \mathrm{End}\mathcal{V})$. Explicitly, these are $h^a{}_{b\bar{\imath}}(\phi)$, which are $\overline{\partial}$-closed and traceless,

$$h^a{}_{b[\bar{\imath},\bar{\jmath}]} = \delta^b{}_a h^a{}_{b\bar{\imath}} = 0$$

modulo those which are $\overline{\partial}$-exact. We can write down a vertex operator,

$$V = [h^a{}_{b\bar{\imath}}\partial_{\bar{z}}\phi^{\bar{\imath}} + h^a{}_{b\bar{\imath};j}\rho^j\rho^{\bar{\imath}}]\lambda_a\lambda^b + h^a{}_{b\bar{\imath}}\rho^{\bar{\imath}}\lambda_a l^b \ , \tag{4.13}$$

where

$$h^a{}_{b\bar{\imath};j} = h^a{}_{b\bar{\imath},j} + A^a{}_{cj}h^c{}_{b\bar{\imath}} - h^a{}_{c\bar{\imath}}A^c{}_{bj} \ .$$

This represents a deformation of holomorphic structure of $\mathcal{V}$, which—when referred to the original basis of local sections, λ^a—adds a (0,1) component to the connection, while preserving the fact that the curvature is of type (1,1) and preserving its trace. The second term in (4.13), involving the auxiliary field l^a, was added for convenience; it vanishes on-shell. If we demand that the curvature of the deformed bundle satisfy the Uhlenbeck-Yau equation, we should choose a harmonic representative, $g^{\bar{\imath}j}h^a{}_{b\bar{\imath};j} = 0$, for this cohomology class.

A short computation shows that

$$[\overline{Q}_+, V] = \partial_{\bar{z}}(h^a{}_{b\bar{\imath}}\rho^{\bar{\imath}}\lambda_a\lambda^b) + \dots \ ,$$

where '...' are terms proportional to

$$\partial_{\bar{z}}\lambda_a - A^c{}_{aj}\lambda_c\partial_{\bar{z}}\phi^j - F^c{}_{aj\bar{k}}\lambda_c\rho^j\rho^{\bar{k}}$$

and to l_a, both of which vanish on-shell. So $\int d^2z\ V$ is a $\overline{Q}_+$-invariant deformation of the σ-model.

As with deformations of the holomorphic structure of $\mathcal{V}$, infinitesimal deformations of the complex structure can be viewed as deformations of the $\overline{\partial}$ operator on X which preserve $\overline{\partial}^2 = \{\overline{\partial}, \partial\} = 0$. Such a deformation can be written as $\overline{\partial} \to \overline{\partial} + d\phi^{\bar{\imath}} h_{\bar{\imath}}{}^{j} \partial_j$, where $h_{\bar{\imath}}{}^{j}(\phi) \in H^1(X, T_X)$. The vertex operator which implements this is somewhat lengthy to write down in full. It can most succinctly be written as

$$V = \tilde{\delta} U, \quad \text{where } U = h_{\overline{k}}{}^{j}(g_{\bar{\imath}j}\partial_z\phi^{\bar{\imath}} - F^a{}_{bj\overline{m}} l_a \lambda^b \rho^{\overline{m}})\rho^{\overline{k}} \ , \tag{4.14}$$

where we have used the second supersymmetry, (4.2). By construction, we have $\{\overline{Q}_+, V\} = \partial_{\bar{z}} U + ...$, where "..." are terms which vanish by the equations of motion.

Finally, deformations of the complexified Kähler structure represent another set of $\overline{Q}_+$-invariant deformations of the action which are even easier to understand. As we saw, the dependence of the action (4.3) on the complexified Kähler class is given, up to $\overline{Q}_+$-trivial terms, by S_ω (4.5). Shifting $B + i\omega$ by a complex, closed (1,1)-form, b, shifts S by $i \int \phi^* b$.

In the $(2, 2)$ case, A-model correlation functions are independent of the complex structure moduli, while B-model correlators, which do not receive world-sheet instanton corrections, are independent of the Kähler moduli. In the $(0, 2)$ context, the story is, *a priori*, more complicated: to begin, we have a third class of moduli, the deformations of the holomorphic vector bundle, $\mathcal{V}$, on which correlators may depend; more troublingly, both A *and* B twisted models now receive instanton corrections, and thus depend on both Kähler and complex structure moduli, as well as the bundle moduli. One powerful constraint, that we can see from the explicit construction of the deformations, is that the dependence on these moduli is *holomorphic*. Further restrictions arise for $(0, 2)$ models in which $\mathcal{V}$ is a deformation of the tangent bundle, as they must reproduce the familiar results at the $(2,2)$ locus. However, away from such loci, or in general $(0,2)$ theories without $(2,2)$ loci on their moduli spaces, simplifications appear few and far between.

That said, explicit computations often reveal that the most general possible dependence on the moduli does not, in fact, arise. Rather, one finds intriguing hints of various "non-renormalization" theorems ensuring that the ring relations remain independent of certain moduli. The full implications of these observation are, however, beyond the scope of this thesis.

4.2.3 Right-chiral Ground States in $(0,2)$ Models

Operators in the twisted theory are mapped to bundle-valued forms in the target space. We will drop the auxiliary fields to simplify the notation, as they will not contribute to correlation functions at non-coincident points, their propagators being trivial.

Since our single scalar supercharge acts as $\overline{Q}_+ \sim \rho^{\bar{\imath}} \frac{\delta}{\delta \phi^{\bar{\imath}}}$, operators in $\overline{Q}_+$-cohomology take the form,

$$\mathcal{O}_{\bar{\imath}_1 \ldots \bar{\imath}_p} \rho^{\bar{\imath}_1} \ldots \rho^{\bar{\imath}_p}$$

with

$$\mathcal{O}_{\bar{\imath}_1 \ldots \bar{\imath}_p} = \mathcal{O}_{\bar{\imath}_1 \ldots \bar{\imath}_p}(\phi, \partial_z \phi, \partial_z^2 \phi, \ldots ; \overline{\phi}, \partial_z \overline{\phi}, \partial_z^2 \overline{\phi}, \ldots ; \lambda^a, \partial_z \lambda^a, \ldots ; \lambda_a, \partial_z \lambda_a, \ldots) \ ,$$

where we have taken the liberty of using the equations of motion for $\rho^{\bar{\imath}}$ to trade z-derivatives of $\rho^{\bar{\imath}}$ for the other fields and their derivatives. $\overline{Q}_+$-closedness implies holomorphy in (the constant mode of) ϕ, but is otherwise not very contraining.

Restricting to operators with $L_0^t = 0$ simplifies this structure dramatically: since $\partial_z^k \phi$, $\partial_z^k \overline{\phi}$ and λ_a all contribute positively to the (twisted) dimension, operators of dimension $h = 0$ take the beautifully simple form

$$\mathcal{O}_{p,q} \sim \mathcal{O}_{\bar{\imath}_1 \ldots \bar{\imath}_p; a_1 \ldots a_q}(\phi) \rho^{\bar{\imath}_1} \ldots \rho^{\bar{\imath}_p} \lambda^{a_1} \ldots \lambda^{a_q} \ .$$

Modding out by $\overline{Q}_+$-trivial operators, these are in 1-to-1 correspondence with elements of the sheaf cohomology,

$$\mathcal{O}_{p,q} \in H^p(X, \wedge^q \mathcal{V}^*) \ .$$

(Kodaira-)Serre duality then provides a trace on the ring iff the dualizing sheaf $\wedge^r \mathcal{V} \otimes K_X$ is trivial:

$$H^p(X, \wedge^q \mathcal{V}^*) = H^{d-p}(X, \wedge^{r-q} \mathcal{V}^* \otimes \wedge^r \mathcal{V} \otimes K_X)^* = H^{d-p}(X, \wedge^{r-q} \mathcal{V}^*)^* \ . \qquad (4.15)$$

Of course, $\wedge^r \mathcal{V} \equiv K_X^*$ implies $c_1(T_X) = c_1(\mathcal{V})$, which was already required to have a nonanomalous left-moving $U(1)$ by which to twist; it is pleasing that this slightly stronger condition also guarantees (classically) the existence of a trace on our ring.

4.2.4 Correlators of $\overline{Q}_+$-Cohomology Classes in the Twisted Models

Correlators of right-chiral operators in the twisted model satisfy several very important properties which will be crucial in what follows. First, since the twisted vacua are annihilated by the supercharges, correlators including an insertion of a $\overline{Q}_+$ commutator vanish,

$$\langle \mathcal{O}_1 \ldots \{\overline{Q}_+, M\} \ldots \mathcal{O}_s \rangle = 0 \ .$$

Since the right-moving stress tensor is trivial in $\overline{Q}_+$-cohomology, $\overline{T} = \{\overline{Q}_+, G_+\}$, correlators of $\overline{Q}_+$-chiral operator with insertions of the stress tensor automatically vanish,

$$\langle \overline{T} \prod_i \mathcal{O}(z_i, \overline{z}_i) \rangle \ = \ \langle \{\overline{Q}_+, G_+\} \prod_i \mathcal{O}(z_i, \overline{z}_i) \rangle \ = \ 0 \ .$$

Correlators of $\overline{Q}_+$-chiral operators are thus completely independent of $\overline{z}$, depending only holomorphically on their insertion points on the worldsheet,

$$\langle \prod_i \mathcal{O}(z_i, \overline{z}_i) \rangle \ = \ \langle \prod_i \mathcal{O}(z_i) \rangle \ .$$

Scaling invariance and conservation of the left-moving $U(1)$ thus ensure that the OPE of two $\overline{Q}_+$-chiral operators takes the form

$$\mathcal{O}_a(z)\mathcal{O}_b(0) = \sum_{q_c = q_a + q_b} \frac{f_{abc}}{z^{h_a + h_b - h_c}} \mathcal{O}_c(0) \ , \tag{4.16}$$

where $\mathcal{O}_c$ is necessarily $\overline{Q}_+$-chiral. Now suppose one could show that there existed a subset of the $\overline{Q}_+$-cohomology whose OPEs were completely non-singular,

$$\mathcal{O}_a(z)\mathcal{O}_b(0) = \sum_{q_c = q_a + q_b} f_{abc}\mathcal{O}_c(0) + O(z) \ .$$

On good physical grounds, we do not expect the correlation function to diverge as $z \to \infty$, so it can be extended analytically to the Riemann sphere. Since the only holomorphic function on a compact Riemann surface without a pole is the constant function, correlators of such magical operators would be completely independent of insertion points, and thus of the worldsheet metric, forming an extremely simple topological ring,

$$\mathcal{O}_a\mathcal{O}_b = f_{abc}\mathcal{O}_c \ .$$

As we shall see in the next section, under very mild conditions, the ground operators of A and B twisted $(0,2)$ models introduced above satisfy precisely such a condition, their non-singular OPEs providing a ring structure and ensuring the topological character of their correlators. Let us prove it.

4.3 A and B Rings in $(0,2)$ Models

As we shall see, $(0,2)$ superconformal symmetry together with a left-moving $U(1)$ current algebra satisfying simple conditions will suffice to ensure the existence of rings (in fact, finite dimensional algebras) of topological operators closing under non-singular OPE and forming the ground rings of the A and B twisted models. We begin by considering deformations of $(2,2)$ models, then generalize to intrinsically $(0,2)$ SCFTs, and finally address massive $(0,2)$ models.

4.3.1 Local Results for $(0,2)$ Deformations of $(2,2)$ Models

Let us begin with a flanking maneuver. Consider the (c,c) ring of a $(2,2)$ SCFT. Left and right chirality ensures that these operators saturate both left and right twisted BPS bounds, $h = 0$ and $\overline{h} = 0$. Note that worldsheet conformal invariance implies that the twisted worldsheet spin is quantized, $h - \overline{h} = s \in \mathbb{Z}$.

Now deform this theory by a marginal operator which preserves both $(0,2)$ and the left-moving $U(1)$ R-current, i.e. a dimension (1,1) operator with R-charge $(0,2)$ (in the case of an NLSM, this would correspond to an element of $H^1(X, \mathrm{End}(T_X))$ of the form discussed in the previous section). Denote by α the deformation parameter. We make the (quite reasonable) assumption that the spectrum varies smoothly under this marginal deformation, i.e. that we do not begin with or approach a singular CFT.

Since this deformation preserves worldsheet conformal invariance, $h(\alpha)-\overline{h}(\alpha) = s$ continues to hold in the deformed theory. This allows us to translate the antiholomorphic BPS bound into an effective bound on the holomorphic weights. Explicitly, since the dimensions and spin of operators remain well-defined and vary smoothly under our deformation, the left-moving conformal dimension of every operator in the theory is pegged to vary in lock-step with its right-moving conformal dimension,

$$\delta_\alpha h = \delta_\alpha \overline{h} \ .$$

The unbroken anitholomorphic BPS bound $\overline{h}(\alpha) \geq 0$ thus translates into a bound on the holomorphic dimension *away* from the $(2,2)$ locus,

$$h(\alpha) = \overline{h}(\alpha) + s \geq s \in \mathbb{Z} \ .$$

In other words, the holomorphic weight of every operator is bounded by the amount by which the operator failed to saturate the antiholomorphic BPS bound in the undeformed theory. In particular, the holomorphic weight of a right-chiral operator cannot decrease as we turn on the deformation—and all this despite the absence of any left-moving supersymmetry!

This bears repeating. The deformed theory does not have a holomorphic BPS bound, and there will in general exist operators with $h(\alpha) < 0$. However, since conformal invariance implies that $h(\alpha) - \overline{h}(\alpha) = s \in \mathbb{Z}$, operators which were right-chiral at the $(2,2)$ point (e.g. the (a,c) or (c,c) operators) continue to satisfy $h(\alpha) \geq 0$ even away from the $(2,2)$ locus. By the same token, operators which were *not* right-chiral at the $(2,2)$ locus may flow down, keeping $h(\alpha) - \overline{h}(\alpha) = s$ fixed, until they saturate the right-BPS bound; this puts a bound on the amount by which the holomorphic weight can flow, $h(\alpha) \geq s$; we shall return to this possibility below.

First, though, let us study the consequences of this bound on OPEs of A-model groundstates. Let $\{\mathcal{O}_a\}$ be the subset of $\overline{Q}_+$-cohomology with holomorphic weight $h_a = 0$, forming the groundstates of the A-model; by the above arguments, these operators continue to satisfy $h(\alpha) = 0$ away from the $(2,2)$ locus. Conformal invariance and left-moving $U(1)$ conservation thus imply

$$\begin{aligned}\mathcal{O}_a(z)\mathcal{O}_b(0) &= \sum_{q_c=q_a+q_b} \frac{f_{abc}}{z^{h_a(\alpha)+h_b(\alpha)-h_c(\alpha)}}\mathcal{M}_c(0) \\ &= \sum_{q_c=q_a+q_b} \frac{f_{abc}}{z^{-h_c(\alpha)}}\mathcal{M}_c(0) \ ,\end{aligned}$$

Note that $\mathcal{M}_c$ does *not*, in general, obey $h_c = 0$, i.e. the OPE does *not*, in general, close on the ground operators *within* $\overline{Q}_+$-cohomology. However, as discussed at the end of the last section, to ensure that correlators of A and B operators remain completely independent of insertion point and continue to define a topological ring it is sufficient to show that their OPEs remain non-singular away from the $(2,2)$ loci.

When, then, can singular terms arise? The appearance of poles in the OPE requires the existence of a right-chiral operator with

$$h_c(\alpha) < 0 \ ,$$

i.e. the ring relations can only become singular if there exists a right chiral operator violating the erstwhile left-moving BPS bound. By the above, this operator must have flowed from an operator which was *not* right-chiral at the $(2,2)$ locus – otherwise it would continue to respect the left-moving BPS bound away from $(2,2)$ locus, $h_c(\alpha) \geq 0$.

By quantization of worldsheet spin and continuity of the spectrum under marginal deformations, this operator can only have entered the right-chiral ring after a finite deformation away from the $(2,2)$ locus. This ensures that the OPEs of ground-ring elements remain closed and non-singular in at least an open neighborhood of the $(2,2)$ locus, proving that the twisted ground-ring exists, as a topological ring, even away from $(2,2)$ loci.

In fact, this argument gives us much more. In an *arbitrary* $(0,2)$ σ-model, the engineering dimensions of all the operators in the twisted theory are non-negative. To find a "dangerous" operator with $(h(\alpha),\overline{h}(\alpha)) = (-|s|,0)$, we need to start with an operator with non-negative engineering dimension, $(h(\alpha),\overline{h}(\alpha)) = (h, h+|s|)$, which picks up a large negative anomalous dimension under RG flow. That clearly cannot happen while the σ-model remains weakly coupled. Remarkably, even far from weak coupling, there are constraints, to which we now turn.

4.3.2 Global Results for $(0,2)$ SCFTs

If the $(0,2)$ model was superconformal *before* twisting, then unitarity of the untwisted model provides further powerful constraints. The stress tensor of a unitary $(0,2)$ SCFT with a r left-moving fermions counted by a left-moving $U(1)$ current algebra can be put in Sugawara form,

$$T = T' + \frac{1}{2r} :J^2(q_-): \ .$$

Since T' is the stress tensor of the (unitary!) coset conformal field theory, its spectrum of conformal weights is non-negative. Thus we have a bound relating the $U(1)$ charge and the (untwisted) dimension,

$$\Delta \geq \frac{q_-^2}{2r} \ . \tag{4.17}$$

Now twist. In the A-model (B-model), the conformal weights are

$$h = \Delta \mp \frac{1}{2} q_-$$
$$\overline{h} = \overline{\Delta} + \frac{1}{2} q_+ \ .$$

We are interested in operators with $\overline{h} = 0$, i.e. $\overline{\Delta} = \frac{1}{2} q_+$. At the same time, we wish to drive $h = \Delta \mp \frac{1}{2} q_-$ sufficiently negative. But (4.17) implies

$$h \geq \frac{q_-^2}{2r} \mp q_-/2 = \frac{q_-(q_- \mp r)}{2r} \ .$$

The RHS is minimized for $q_- = \mp r/2$, so we have the bound

$$h \geq -\frac{r}{8} \ .$$

To get a pole, we need $h = s \in \mathbb{Z} < 0$; this requires $r \geq 8$. For $r < 8$, *unitarity of the untwisted theory forbids a large negative anomalous dimension*, even if the σ-model is strongly coupled. Thus, in the absence of any candidate "dangerous" operators, the ground ring must persist even deep into the $(0,2)$ moduli space.

4.3.3 Massive $(0,2)$ Theories

As we saw at the end of Section 3.1, in A-models for "massive" $(0,2)$ theories, $T_{z\overline{z}}$ and $T_{\overline{z}\overline{z}}$ are $\overline{Q}_+$-exact, while L_0 and L_{-1} are $\overline{Q}_+$-closed, ensuring that, for fixed but arbitrary metric, these A-models are holomorphic field theories invariant under global dilitations and translations. Operators in $\overline{Q}_+$-cohomology thus carry well-defined holomorphic scaling dimensions and spin, and are invariant under translations of their insertion points.

This is just enough to apply the arguments of the previous subsections to A-twisted massive models. It is simple to verify that OPEs of spinless operators are again non-singular in the neighborhood or large-radius or $(2,2)$ points, and that their correlators are regular at worldsheet infinity. Again, operators can flow down and enter the ring in pairs, but this can only happen after some finite bundle deformation.

More explicitly, to define

$$\langle \mathcal{O}_1(z_1)\dots\mathcal{O}_n(z_n)\rangle$$

we need to specify a metric; for simplicity, we will use the round metric on the sphere,

$$ds^2 = \frac{4|dz|^2}{(1+|z|^2/R^2)^2}. \tag{4.18}$$

As we have argued, correlation functions of ground ring operators are both holomorphic[10] and invariant under dilatations, and thus independent of the parameter R in the round metric (4.18). As before, we need to investigate the possibility of poles as $z_i \longrightarrow z_j$.

The massive model has a dimensionful scale, Λ, and we have a dimensionless parameter, $\epsilon = R\Lambda$ at our disposal. At finite ϵ, we need to use the curved-space Green's functions associated to the metric (4.18), rather than the flat space ones. But all we are interested in is whether there is a pole as $z_i \longrightarrow z_j$. This is entirely governed by the operator product expansion of $\mathcal{O}_i$ and $\mathcal{O}_j$. Again, the danger is

[10] Since we are using the round metric, they are also invariant under $SO(3)$ rotations of the sphere – recall that the ground ring operators are spinless—so they inherit an *accidental* $PSL(2,\mathbb{C})$ symmetry.

that there exists a $\overline{Q}_+$-invariant operator of spin N, with large negative anomalous dimension, which might contribute a singular term to the OPE. But, for small enough ϵ, the theory is weakly coupled, and the anomalous dimensions are all small. Hence, there can be no such singular contribution to the correlation function. Since it is a globally holomorphic function on the sphere, it must be a constant.

4.4 Examples

While $(0,2)$ ground rings have previously not been systematically investigated, special cases have cropped up in the literature. Here we review two examples where the rings have been explicitly constructed, providing concrete examples of the abstract structures discussed above.

4.4.1 Explicitly Solvable Conformal Models

Much work has gone into the exact solution of various $(0,2)$ models, focusing largely on Landau-Ginsburg orbifolds and their deformations [82, 48, 81, 23]. In these cases, the ring structure may be extracted by inspection. In a remarkable paper, Blumenhagen, Schimmrigk and Wißkirchen (BSW) did precisely that, identifying chiral sub-rings in a series of $(0,2)$ deformation of Gepner models [23]. Notably, their construction relies crucially on the existence of a left-moving $U(1)$ current; in retrospect, we may twist by this $U(1)$ and check that the resulting ground ring is precisely the ring they identify. The interpretation as the cohomology ring $H^p(X, \wedge^q V)$ was also pointed out.

Consider, for example, their "(80,0) model", a deformed $3^{\otimes 5}$ Gepner model with

K_1	$\langle K_1\xi_1\xi_2\rangle = 1$	$\langle K_1\xi_2\xi_3\rangle = 1$	$\langle K_1\xi_3\xi_4\rangle = 1$	
K_2	$\langle K_2\xi_1\xi_1\rangle = 1$	$\langle K_2\xi_1\xi_3\rangle = 1$	$\langle K_2\xi_2\xi_2\rangle = \kappa^2$	$\langle K_2\xi_2\xi_3\rangle = \kappa$
	$\langle K_2\xi_3\xi_3\rangle = 1$	$\langle K_2\xi_3\xi_4\rangle = \kappa$	$\langle K_2\xi_4\xi_4\rangle = \kappa^2$	
$K_3/\widetilde{K}_3$	$\langle K_3\xi_1\xi_5\rangle = 1$	$\langle K_3\xi_1\xi_6\rangle = 1$	$\langle K_3\xi_2\xi_6\rangle = \kappa$	$\langle K_3\xi_3\xi_6\rangle = 1$
	$\langle K_3\xi_3\xi_7\rangle = \kappa$			
$K_4/\widetilde{K}_4$	$\langle K_4\xi_1\xi_7\rangle = 1$	$\langle K_4\xi_2\xi_5\rangle = 1$	$\langle K_4\xi_2\xi_6\rangle = 1$	$\langle K_4\xi_3\xi_7\rangle = 1$
	$\langle K_4\xi_4\xi_6\rangle = 1$			
$K_5/\widetilde{K}_5$	$\langle K_5\xi_1\xi_6\rangle = 1$	$\langle K_5\xi_2\xi_6\rangle = \kappa$	$\langle K_5\xi_2\xi_7\rangle = \kappa^2$	$\langle K_5\xi_3\xi_5\rangle = 1$
	$\langle K_5\xi_3\xi_6\rangle = 1$	$\langle K_5\xi_3\xi_7\rangle = \kappa$	$\langle K_5\xi_4\xi_6\rangle = \kappa$	$\langle K_5\xi_4\xi_7\rangle = \kappa^2$
$K_6/\widetilde{K}_6$	$\langle K_6\xi_3\xi_6\rangle = \kappa$	$\langle K_6\xi_4\xi_7\rangle = \kappa^3$		

Table 4.1: *Nonvanishing ring relations for the (80,0) B-model ground-ring.* K_n *and* $\widetilde{K}_n$ *transform in the* ***10*** *of* $SO(10)$ *and* ξ_a *and* $\widetilde{\xi}_a$ *in the* ***16****;* κ *is a numerical constant. Where a* $\widetilde{K}$ *appears in the left-hand column, the* $\widetilde{K}$*-relation is the same as that for* K*, but with* ξ_5, ξ_6, ξ_7 *replaced with their* $\widetilde{\xi}$ *counterparts (adapted from [23]).*

a spacetime $SO(10)$ gauge group, whose geometric phase corresponds to a rank four bundle over a complete intersection in the weighted projective space $W\mathbb{P}_{(1,1,1,1,2,2)}$ (the familiar two-parameter Calabi-Yau "Example 2" from [98], also studied for example in [48]). BSW explicitly checked for the existence of the ring by calculating the massless Yukawa couplings with the $SO(10)$ representation structure $\langle \mathbf{10}\cdot\mathbf{16}\cdot\mathbf{16}\rangle$ that define the B-ring. The corresponding vertex operators saturate the chiral bound $\Delta \geq \frac{1}{2}|q_-|$. The complete ring structure is given in Table 1. They also checked that this ring agrees with the coordinate ring derived from the superpotential of the associated gauged linear sigma model, i.e.

$$\mathcal{R} = \frac{\mathbb{C}[\Phi_i]}{\{J_a(\Phi_i) = 0\}} ,$$

where the $W_{(0,2)} = \int d\theta\, \Gamma^a J_a(\Phi_i)$ is the $(0,2)$ superpotential. Since $J_a = \partial_a W$ at $(2,2)$ loci, this matches the usual B-ring at $(2,2)$-points; at generic points, this is precisely the B-ring of the (80,0) $(0,2)$ theory.

The sufficiently curious reader might find it entertaining to return to the literature on exactly solved $(0,2)$ models and identify a slew of topological ground rings; these might turn out to be very handy in the study of $(0,2)$ mirror symmetry.

4.4.2 Mirror Symmetry and a Massive Model

The worldsheet construction of mirror symmetry of $(0,2)$ models [1][11] also provides strong evidence for the existence of these $(0,2)$ rings in both conformal and massive models—more precisely, mirror symmetry has led to the prediction of such rings in numerous systems. The most well-studied example involves the deformation of the tangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$, a massive $(0,2)$ model whose A-ring was computed via mirror symmetry in [1] and checked via direct computation of the intersection form on the associated instanton moduli space by Katz and Sharpe [86] (see also [110]).

The basic strategy of [1] involved the extension of Morrison-Plesser/Hori-Vafa [97, 76] worldsheet dualization techniques to construct dual pairs of $(0,2)$ models related by the mirror automorphism, $J_- \leftrightarrow -J_-$ and $Q_- \leftrightarrow \overline{Q}_-$, with the mirror superpotential effectively summing the instantons of the original σ-model. The ring derived from the mirror superpotential was thus interpreted as the mirror of a quantum cohomology of the original σ-model.

In the case of $T[\mathbb{P}^1 \times \mathbb{P}^1]$, the resulting ring relations turn out to be[12]

$$\widetilde{X}^2 = \exp(it_2), \qquad X^2 - (\epsilon_1 - \epsilon_2)X\widetilde{X} = \exp(it_1) \ ,$$

where t_1, t_2 are the Kähler parameters of the two $\mathbb{P}^1$s, while ϵ_1, ϵ_2 parametrize certain elements in $H^1(X, \mathrm{End}(T_X))$, the bundle moduli. Notice that the latter parameters need not be perturbatively small.

[11] See also e.g. [25, 26, 111] for alternate approaches to $(0,2)$ avatars of mirror symmetry.
[12] Using the notation of [86].

Katz and Sharpe checked this argument by explicitly computing the cohomology products this ring was expected to reproduce [86], roughly generalizing the Gromov-Witten counting of rational curves to the $(0, 2)$ context and reducing the problem to precise computations in sheaf cohomology on the instanton moduli space. Explicitly, since the anomaly ensures that each correlation function receives corrections from only specific worldsheet instanton sectors, by computing the appropriate intersection numbers on the instanton moduli space, Katz and Sharpe, in a remarkable and powerful computation, constructed by hand the two- and four-point functions[13] of the generators of the ground ring, finding precise agreement with the ring relations predicted in [1].

There was one important caveat: [86] computed the intersection form on the moduli space of instantons with 2 or 4 marked points, but could not ensure that the associated A-model correlators were independent of the positions of the marked points. Importantly, the results presented above imply that the OPE of A-ring operators is non-singular, ensuring that these correlators are in fact independent of the insertion points. The results of [86] are thus in precise agreement with [1].

4.5 The Reservations at the End of the Chapter

In this chapter, we have verified the existence of topological ground-rings in A and B twisted $(0, 2)$ models in open balls around classical points in the moduli space and globally under relatively generic assumptions, generalizing the quantum cohomology rings of $(2, 2)$ theories to the sheaf-cohomological context more natural to

[13] Including, more recently, even the coefficients [109].

heterotic compactifications. The key ingredient we needed was a left-moving $U(1)$ global symmetry by which to twist—in particular, we did not require the theory to be a deformation of a $(2, 2)$ model, nor even that it be geometrical: all arguments obtain at the level of the CFT, and thus cannot be destabilized by worldsheet instantons in the special case of non-linear σ-models.

It is quite remarkable that the topological rings of $(2, 2)$ theories persist as rings in $(0, 2)$ models under such mild constraints. This leads to a host of natural questions. What is the geometry of the malevolent operators which may destroy the ring when $rk(\mathcal{V}) \geq 8$—are they related to conifolds, where the chiral ring also degenerates badly, or perhaps to some small heterotic instanton transition? How are "bundle flops" realized in the chiral ring? What are the generalizations of the periods, physically and mathematically? In the case of massive models, can we make any global statements, or nail down the smooth behavior at infinity of A-model OPEs? And finally, can we explicitly construct topological $(0, 2)$ theories whose degrees of freedom are exactly the elements of the rings defined above? These and other questions we leave to future work.

Chapter 5

Linear Models for Flux Vacua

Calabi-Yau compactifications have formed the backbone of much of string theory in the past two decades, but they ultimately constitute a very small part of the full moduli-space, a part that also has some distinct drawbacks. However, the step in going from Kähler to non-Kähler compactification manifolds by setting $H \neq 0$ is arguably even bigger than the step in going from left-right symmetric theories to heterotic ones. Even for heterotic strings, compactifying on a Calabi-Yau manifold immediately guarantees that we have a wide range of analytic tools at our disposal because of the highly constrained geometry. Equally important, setting $H = 0$ implies that the Bianchi identity (3.12) can be trivially solved by embedding the spin connection in the gauge connection, causing the traces on the right-hand side to cancel. Without this simplification, solutions are notoriously difficult, a property that is well illustrated by the fact that there was a 20-year gap between the explicit statement (due to Strominger) of the geometric problem of finding supersymmetric compactification with $H \neq 0$ [113], and the first rigorous solution by Fu and Yau [54]. Since the geometric problem only reflects the $\mathcal{O}(\alpha')$ equations of motion [3], even a rigorous solution does not prove that there exists a solution to the full quantum-corrected equations.

In this chapter, we shall show that gauged linear sigma models can be applied to illuminate these issue [3]. We will construct a GLSM that flows to the Fu-Yau geometry in the IR, in the process describing a new class of models, *torsion linear sigma models (TLSMs)*. In relation to the Fu-Yau solution, this construction has two major implications: it is strong evidence that an all-order extension of the solution does indeed exist, and it gives us a powerful computational tool which can presumably extend the techniques developed for ordinary $(0, 2)$ models [48, 82]. On a more general level, it suggests a new route to a deeper understanding of string theory vacua with background fluxes in general, and the Type II flux solutions which are dual to the Fu-Yau solutions in particular.

5.1 Introduction

It is a a beautiful and frustrating fact of life that Calabi-Yau manifolds have interesting moduli spaces. On the one hand, the topology and geometry of their moduli spaces govern the low-energy physics of string theory compactified on a Calabi-Yau, so understanding their structure teaches us about four-dimensional stringy physics. On the other, the resulting massless scalar particles are a phenomenological disaster. This difficulty has long been appreciated and significant amounts of work has been devoted to surmounting it. Much of it has centered around using compactifications in which the Calabi-Yau supports non-trivial fluxes of gauge or other fields. At the level of type II supergravity, beautiful work of KKLT and others demonstrated that a judicious choice of fluxes and branes wrapping suitable cycles in a fiducial Calabi-Yau can generate a scalar potential which fixes all moduli of the underlying CY. Foundational work on this approach can be found in [94, 72, 80, 79], while two reviews with further references are [61, 49]. However, since these type II flux vacua necessarily involve RR fluxes and other effects which are not amenable to worldsheet analysis, it is difficult to construct a microscopic description for them, and a sufficiently hard-nosed physicist could rationally wonder whether these vacua, in fact, exist.

Duality provides a powerful hint about how to address this question. For a large class of flux vacua, such as the KST models of [80], there exists [16] a duality frame involving a heterotic compactification on a non-Kähler manifold of $SU(3)$-structure with non-trivial gauge and NS-NS 3-form flux, $H{\neq}0$, all of which is in principle amenable to worldsheet analysis. A microscopic description of heterotic flux vacua would thus provide a microscopic description of the dual KST vacua.

Of course, there are excellent reasons that most work has focused on Kähler compactifications, which necessarily have $H = 0$. In particular, only for Kähler manifolds does Yau's Theorem ensure the existence of solutions to the tree-level supergravity equations; the beautiful results of Gross & Witten [70] and Nemachamsky & Sen [100] then ensure that these classical solutions extend smoothly to solutions of the exact string-corrected equations. When $H \neq 0$, the story is much more complicated, due in part to the absence of effective computational tools analogous to Hodge theory or special geometry for non-Kähler manifolds, and in part to the tremendous analytic complexity of the Bianchi identity, which we repeat here for convenience

$$dH = \alpha'(\mathrm{tr}R \wedge R - \mathrm{Tr}F \wedge F) \ . \tag{5.1}$$

Moreover, since the Bianchi identity scales inhomogeneously with the global conformal mode, any solution has total volume-modulus fixed near the string scale, so such compactifications can *not* be described by conventional, weakly-coupled NLSMs. Whether these Fu-Yau solutions, like Calabi-Yaus, can be smoothly extended to solutions of the exact string equations has thus remained very much unclear.

The purpose of this chapter is to develop tools with which to study heterotic compactifications with non-vanishing H, i.e. holomorphic vector bundles over non-Kähler manifolds with intrinsic torsion satisfying 5.1. Motivated by Fu and Yau, we focus on torus bundles over Kähler bases, $T^m \to X \to S$, with gauge bundle $\mathcal{V}_X$ and NS-NS flux H turned on over the total space X. When $m = 2$ and $S = K3$, this is precisely the Fu-Yau compactification.[1]

Our strategy closely parallels the familiar gauged linear sigma model (GLSM) approach to Calabi-Yau compactifications [123]: we build a massive 2D gauge theory which flows in the IR to an interacting CFT with all the properties that we expect of a Fu-Yau compactification. In the Calabi-Yau case, the GLSM flows to a NLSM whose large-radius limit is the chosen Calabi-Yau. This is not possible in the Fu-Yau case as no large-radius limit exists; however, the classical moduli space of the one-loop effective potential of our GLSM will precisely reproduce the Fu-Yau geometry. We thus take the CFT to which our torsion linear sigma model (TLSM) flows to provide

[1] While we refer to these geometries as *Fu-Yau* geometries, it should be emphasized that Strominger's elaboration of the precise equations to be solved was crucial to the eventual construction of solutions by Fu and Yau, which also used studies of the underlying manifolds by Goldstein and Prokushkin [60].

a microscopic definition of the Fu-Yau compactification.

A central ingredient in these models is a two-dimensional implementation of the Green-Schwarz mechanism. The $ch_2(T_S) - ch_2(\mathcal{V}_S)$ anomaly of a $(0,2)$ non-linear sigma model on $\mathcal{V}_S \to S$ is contained[2] in the gauge anomaly of $(0,2)$ GLSMs for S. In compactifications with intrinsic torsion, this sum does not vanish even in cohomology. To restore gauge invariance, we introduce a novel $(0,2)$ multiplet containing a doublet of axions whose gauge variation precisely cancels the gauge anomaly. The one-loop geometry of the resulting model is easily seen to be a T^2 fibration X over the Calabi-Yau S—a Fu-Yau geometry—with the anomaly cancellation conditions of the TLSM reproducing the conditions for the existence of a solution to the Bianchi identity.

Crucial to our construction is a manifest $(0,2)$ supersymmetry with non-anomalous R-current and a non-anomalous left-moving $U(1)$. These ensure the perturbative non-renormalization of the superpotential and are necessary for the existence of a chiral GSO projection. The worry, as usual in a $(0,2)$ theory, is that worldsheet instantons may generate a non-perturbative superpotential [45, 46]. The power of a gauged linear description is that the moduli space of worldsheet instantons is embedded within the moduli space of gauge theory instantons, which is manifestly compact; without a direction along which to get an IR divergence, it is thus impossible to generate the poles required for the generation of a spacetime superpotential [112, 14]. Such arguments have been used to rigorously forbid the existence of non-perturbative superpotentials for $(0,2)$ gauged linear sigma models of Calabi-Yau geometries; while some technical details differ so that we cannot present a direct proof, these results appear to extend unproblematically to our torsion linear models.

[2] In fact, this is a somewhat subtle story, as we shall elaborate below.

Along the way we will construct a number of $(2,2)$ TLSMs for generalized Kähler geometries, including non-compact models built out of chiral and twisted chiral multiplets and more intricate models in which we gauge chiral currents built out of semi-chiral multiplets. While not our main interest here, these models provide useful guidance in our construction of $(0,2)$ models with torsion and are worth studying on their own merits.

Since the arXiv-publication of the paper [3] on which this chapter is based, progress on these questions has also been made from a complementary point of view: part of the moduli space of Fu-Yau geometries has been constructed by explicitly considering the massless deformation of the spacetime solutions [19]. This approach verified the observation, also made in [3], that the complex structure on the T^2-fibres is fixed. Conversely, it should be possible to check the new results result obtained in [19] by explicit calculations in the TLSM. Indeed, a self-contained follow-up to the material presented here, including examples and details omitted below, is in preparation [4].

5.2 The Fu-Yau Geometry

Though linear models may be able to describe a much broader class of compactifications, the Fu-Yau solution of Strominger's equations provides a very direct motivation and it will guide much of the construction below. We therefore start with an introduction to this geometry, focussing on aspects that will be particularly important for what follows.

The underlying manifold satisfying all of the supersymmetry constraints unrelated to the gauge bundle was first constructed by Goldstein and Prokushkin (GP), who noted that heterotic string solutions constructed by string dualities took the form of T^2-bundles of Calabi-Yau 2-folds [60]. Their solution involved explicitly constructing a complex 3-fold as a T^2 bundle over a T^4 or $K3$ base. Fu and Yau (FY) used this underlying manifold and constructed a gauge bundle satisfying the remaining supersymmetry constraints as well as the modified Bianchi identity, a monumental accomplishment given the complexity of the underlying differential equation [54]. We shall start by explaining the GP manifold.

Let S be a complex Hermitian 2-fold and choose[3]

$$\frac{\omega_P}{2\pi}, \frac{\omega_Q}{2\pi} \in H^2(S;\mathbb{Z}) \cap \Lambda^{1,1}T^*_S \ . \tag{5.2}$$

where ω_P and ω_Q are anti self-dual forms. Being elements of integer cohomology, there are two $\mathbb{C}^*$-bundles over S, call them P and Q, whose curvature 2-forms are ω_P and ω_Q, respectively. We can then restrict to unit-circle bundles S^1_P and S^1_Q of P and Q respectively, and take the product of the two circles over each point in S to form a T^2 bundle over S which we will refer to as X ($T^2 \to X \xrightarrow{\pi} S$).

Given this setup, Goldstein and Prokushkin showed that if S admits a non-vanishing, holomorphic $(2,0)$-form, then X admits a non-vanishing, holomorphic $(3,0)$-form. Furthermore, they showed that if ω_P or ω_Q are nontrivial in cohomology on S, then X admits *no* Kähler metric. They constructed the non-vanishing holomorphic $(3,0)$-form and a Hermitian metric on X from data on S.

The curvature 2-form ω_P determines a non-unique connection ∇ on S^1_P (and similarly for ω_Q on S^1_Q). A connection determines a split of T_X into a vertical and horizontal subbundle—the horizontal subbundle is composed of the elements of T_X

[3] Actually, Goldstein and Prokushkin only required that $\omega_P + i\omega_Q$ have no $(0,2)$-component, but Fu and Yau used the restriction that we have stated.

that are annihilated by the connection 1-form, the vertical subbundle is then, roughly speaking, the elements of T_X tangent to the fibres. Over an open subset $U \subset S$, we have a local trivialization of X and we can use unit-norm sections, $\xi \in \Gamma(U; S^1_P)$ and $\zeta \in \Gamma(U; S^1_Q)$, to define local coordinates for $z \in U \times T^2$ by

$$z = (p, e^{i\theta_P}\xi(p), e^{i\theta_Q}\zeta(p)), \tag{5.3}$$

where $p = \pi(z) \in U$. The sections ξ and ζ also define connection 1-forms via

$$\nabla\xi = i\alpha_P \otimes \xi \qquad \text{and} \qquad \nabla\zeta = i\alpha_Q \otimes \zeta \ , \tag{5.4}$$

where $\omega_P = d\alpha_P$ and $\omega_Q = d\alpha_Q$ on U, and the α_i are necessarily real to preserve the unit-norms of ξ and ζ.

The complex structure is given on the fibres by $\partial_{\theta_P} \to \partial_{\theta_Q}$ and $\partial_{\theta_Q} \to -\partial_{\theta_P}$ while on the horizontal distribution it is induced by projection onto S.[4] Given a Hermitian 2-form ω_S on S, the 2-form

$$\omega_u = \pi^* \left(e^u \omega_M\right) + \left(d\theta_P + \pi^*\alpha_P\right) \wedge \left(d\theta_Q + \pi^*\alpha_Q\right) \ , \tag{5.5}$$

where u is some smooth function on S, is a Hermitian 2-form on X with respect to this complex structure. The connection 1-form

$$\vartheta = (d\theta_P + \pi^*\alpha_P) + i(d\theta_Q + \pi^*\alpha_Q) \tag{5.6}$$

annihilates elements of the horizontal distribution of T_X while reducing to $d\theta_P + id\theta_Q$ along the fibres. These data define the complex Hermitian 3-fold (X, ω_u), which we

[4] Actually, this just gives an almost complex structure, but Goldstein and Prokushkin proved that it is integrable [60]

call the *GP manifold* [60]. Explicitly,

$$
\begin{aligned}
ds^2_X &= \pi^*\left(e^u ds^2_S\right) + (d\theta_P + \pi^*\alpha_P)^2 + (d\theta_Q + \pi^*\alpha_Q)^2 \\
J_X &= \pi^*\left(e^u J_S\right) + \frac{1}{2}\vartheta \wedge \overline{\vartheta} \\
\Omega_X &= \pi^*\left(\Omega_S\right) \wedge \vartheta \\
H &= \sum_{i=P,Q} (d\theta_i + \pi^*\alpha_i) \wedge \pi^*\omega_i \ ,
\end{aligned}
$$

where Ω_S is the nowhere-vanishing, holomorphic $(2,0)$-form on S ($K3$ or T^4). It is straightforward check that all the supersymmetry constraints are satisfied by this ansatz; however for a valid heterotic compactifications a gauge bundle still needed to be constructed to satisfy the Bianchi identity.

Fu and Yau undertook the problem of proving the existence of gauge bundles over the GP manifold with Hermitian-Yang-Mills connections satisfying the Bianchi identity (5.1). They used the Hermitian form (5.5) and converted the Bianchi identity into a differential equation for the function u. Under the assumption

$$\left(\int_{K3} e^{-4u}\frac{\omega^2_{K3}}{2}\right)^{1/4} \ll 1 = \int_{K3}\frac{\omega^2_{K3}}{2} \ , \tag{5.7}$$

they showed that there exists a solution u to the Bianchi identity for *any* compatible choice of gauge bundle $\mathcal{V}_X$ and curvatures ω_P and ω_Q such that the gauge bundle $\mathcal{V}_X$ over X is the pullback of a stable, degree 0 bundle $\mathcal{V}_{K3}$ over $K3$, $\mathcal{V}_X = \pi^*\mathcal{V}_{K3}$ [54]; this is what we call the *Fu-Yau (FY) geometry.*

Note that by a “compatible” choice of gauge bundle and ω_i's we mean the following: choose the gauge bundle $\mathcal{V}_X$ and the curvature forms to satisfy the integrated Bianchi identity

$$\chi(S) - \mathrm{Tr}F^2 = \int_S \sum_i \omega_i^2 \ . \tag{5.8}$$

In particular, note that the right-hand side and $\mathrm{Tr}F^2$ are manifestly non-negative, since $*_S F = -F$ and F is anti-Hermitian. Hence, the only possible solution for a T^4 base is to take the gauge bundle *and* the T^2 bundle to be trivial, leaving us with a Calabi-Yau solution $T^2 \times T^4$ [15, 54]. This is in agreement with arguments from string duality ruling out the Iwasawa manifold as a solution to the heterotic supersymmetry constraints [58].

5.3 Torsion in $(2,2)$ GLSMs

We expect that the presence of non-vanishing torsion will also have a significant impact on the form of the worldsheet theory. While a number of $(2,2)$ gauged linear sigma models with non-trivial NS-NS flux have been studied in the literature—most notably the (4,4) H-monopole GLSM [117]—the structure of general models has received relatively little attention. In this section, we will review the incorporation of NS-NS flux into $(2,2)$ models, emphasizing features which will generalize to the more complicated $(0,2)$ examples studied below.

Let us start with a standard $(2,2)$ GLSM for some toric variety V built out of chiral and vector supermultiplets. The IR geometry of such models is necessarily Kähler. What we seek is a way to introduce non-trivial $H = dB \neq 0$ into a standard $(2,2)$ GLSM. Since H is an obstruction to Kählerity, we are also looking for a construction of non-Kähler geometries via $(2,2)$ GLSMs. It has long been known that sigma models built entirely out of chiral multiplets are necessarily Kähler [57], so we would seem to need to introduce non-chiral multiplets. However, since a $(2,2)$ gauge field minimally coupled to chiral multiplets cannot be minimally coupled to twisted chirals while preserving $(2,2)$, there would seem to be a no-go argument forbidding minimally-coupled GLSMs for non-Kähler geometries with non-vanishing H.

As is often the case, this no-go statement tells us exactly where to go. Recall that B appears in the GLSM through the imaginary parts of the complexified FI parameters $t^a = r^a + i\theta^a$ appearing in the twisted chiral superpotential,

$$-\frac{1}{2\sqrt{2}}\int d^2\widetilde{\theta}\; t^a\Sigma_a + \text{h.c.} \;=\; -\,r^a D_a + 2\theta^a v_{+-a}\;. \tag{5.9}$$

More precisely, t^a are the restriction of the complexified Kahler class $\mathcal{J} = J + iB$ to the hyperplane classes $H_a \in H^2(V)$ corresponding to the gauge fields Σ_a, i.e. $B = \theta^a H_a$. To get $H \neq 0$ we must promote some of the θ^a, say m of them, to dynamical fields. Note that this *adds* dimensions to the geometry, so we are no longer working with a sigma model on V, but with a geometry with local product structure $V \times (S^1)^m$.

For the moment, consider promoting a single FI parameter to a dynamical field. Since the FI parameter appears in the twisted chiral superpotential, $(2,2)$ supersymmetry requires that it be promoted to a twisted chiral multiplet Y with action

$$-\frac{N^a}{2\sqrt{2}}\int d^2\widetilde{\theta}\, Y\Sigma_a + \text{h.c.} - \frac{1}{8}\int d^4\theta\, k^2(Y+\overline{Y})^2 = -k^2[(\partial r)^2 + (\partial\theta)^2] - N^a\,[rD_a - 2\theta\, v_{+-a}] + \ldots\,, \tag{5.10}$$

where $k \in \mathbb{R}$ and $y = r + i\theta \in \mathbb{C}^*$ is the scalar component of Y. The geometry is thus a complex manifold with local product structure, $W \sim V \times \mathbb{C}^*$, and NS-NS potential $B = \theta N^a H_a$ on the total space W that is no longer closed,

$$H = d\theta \wedge N^a H_a \neq 0\;.$$

The resulting IR geometry is non-Kähler, evading the no-go statement above by coupling the gauge supermultiplet minimally to chirals and *axially* to twisted chirals.

Note that the resultant H-flux has two legs along V and one along the S^1 coordinatized by θ. Note, too, that this is precisely the form of the relevant couplings in the (4,4) H-monopole GLSM.

In some sense, what we have done by promoting the FI parameter/Kähler modulus t to a dynamical field Y is to take the variety V and construct a new variety W as a fibration of V over a complex line in the Kähler moduli space of V. This should give us pause; the moduli space includes points where the original variety V goes singular, so this fibration is degenerate. How do we know that the total space of the fibration is, in fact, smooth?

Consider, for example, the resolved conifold $F = xy - wz - r = 0$ in $\mathbb{C}^4$, and let W be the fibration of the conifold over the complex line r. The point $r = 0$ is a very singular point—even the CFT is singular—and it is natural to wonder if W is singular at $r = 0$. In fact, it is straightforward to see that W is completely well behaved at $r = 0$. Like V, W is the vanishing locus of F, now viewed as a function on $\mathbb{C}^5$. However, since $\partial_r F = -1$, F is strictly transverse, so the hypersurface $W = F^{-1}(0)$ is everywhere smooth. By virtue of the linear nature of the axial coupling, a similar result can be argued to obtain for all $(2,2)$ models in which the FI parameter is promoted to a dynamical field.

T-dualizing the dynamical FI parameter is revealing. Consider a GLSM with gauge group $U(1)^s$, $(N+s)$ chirals Φ_I, and m axially coupled twisted chirals, Y_l, with Lagrangian,

$$\mathcal{L} = \int d^4\theta \left[-\frac{1}{4e_a^2}\overline{\Sigma}_a\Sigma_a + \frac{1}{4}\overline{\Phi}_I e^{2Q_I^a V_a}\Phi_I - \frac{1}{8}k_l^2(\overline{Y}_l + Y_l)^2 \right] - \frac{1}{2\sqrt{2}}\int d^2\widetilde{\theta}\, M_l^a Y_l \Sigma_a + \text{h.c.}\ . \tag{5.11}$$

Dualizing all the twisted chirals Y_l into chiral multiplets $\mathcal{P}_l$ results in a simple model,

$$\tilde{\mathcal{L}} = \int d^4\theta \, \left[-\frac{1}{4e_a^2}\overline{\Sigma}_a\Sigma_a \; + \frac{1}{4}\overline{\Phi}_I e^{2Q_I^a V_a}\Phi_I \; + \frac{1}{8k_l^2}(\overline{\mathcal{P}}_l + \mathcal{P}_l + 2M_l^a V_a)^2 \right] \, . \tag{5.12}$$

All matter fields are now chiral, so the classical moduli space is automatically Kähler. But with which Kähler metric and on which space? We can clearly eat the imaginary component of all m fields $\mathcal{P}_l$ to make m of the gauge fields massive (provided $s \geq m$), and use their real components to solve m of the D-terms. However, integrating out the massive vectors and scalars deforms the Kähler potential for the $(N+s)$ fields Φ_I. The surviving $(s-m)$ gauge fields then effect a Kähler quotient of $\mathbb{C}^{N+s}$, but now starting with a deformed Kähler structure. The IR geometry is thus an $N+m$ dimensional variety whose topology is controlled by the charges Q_I^a of the Φ_I under the surviving gauge fields, but with deformed Kähler structure [75]. This can be used to construct GLSMs for, say, squashed spheres. T-dualizing with this squashed metric then gives non-trivial B, which was what we found above.

It is fun to note in passing that we could just as well have dualized the *chiral* multiplets in our torsion model to get a theory of only *twisted chiral* multiplets, all axially coupled to an otherwise free gauge multiplet. As emphasized by Morrison & Plesser [97] and by Hori & Vafa [76], the resulting theory has a non-perturbative superpotential of the form $W = e^{-Z_I}$, where Z_I are the twisted chirals dual to the original Φ_I. The resulting theories end up looking like complicated generalizations of Liouville theories coupled to a host of scalars.

Going back to our strategy of axially coupling twisted chirals to the gauge multiplets of a chiral GLSM, and vice versa, a little play leads us to the very general form,

$$\mathcal{L} = \mathcal{L}_V(\Phi, \Sigma) + \mathcal{L}_W(Y, S) \ + \int d^2\widetilde{\theta}\ \Sigma G(Y) \ + \int d^2\theta\ SF(\Phi) + \text{h.c.}\ , \qquad (5.13)$$

where $\mathcal{L}_{V,W}$ are the Lagrangians for standard chiral (twisted chiral) GLSMs on V (W), F and G are gauge invariant analytic functions of the chiral and twisted chiral fields, repectively, and S is the chiral field-strength of the gauge field in $\mathcal{L}_W$. The resulting geometry has an obvious local product structure, $M \sim V \times W$, but is globally non-trivial—this is a simple extension of the fibration structure discussed above. One annoying feature of all such models is that any model of this form, which has trivial one-loop running of the D-term (i.e. all the Ricci-flat manifolds), appears to be, at first blush, non-compact: it is simply impossible to build a non-trivial coupling of this form when V and W are both compact Calabi-Yaus. Something remains missing.

Note that the models described above evaded the "no-go" statement by coupling a $(2,2)$ vector minimally to chiral matter and axially to twisted chirals or vice vera. While these models have a particularly simple presentation, they are by no means the most general $(2,2)$ models one can construct—in particular, there are many more representations than simply chiral and twisted chiral. In fact, as has only recently been proven [90], the most general off-shell $(2,2)$ NLSM can only be written by including semi-chiral multiplets anihilated by a single supercharge. It is reasonable to ask if the same is true of GLSMs.

As it turns out, a large class of generalized geometries [73, 71] only admit gauged linear descriptions using semi-chiral superfields. Suppose we want to couple a $(2,2)$ gauge field to a conserved current; of necessity, that current must be either a chiral or a twisted chiral current. However, the matter fields which appear in the current do not have to be chiral or twisted chiral, only the total current is so constrained. This suggests a simple strategy for constructing a $(2,2)$ GLSM out of semi-chiral fields: begin with a theory of free semi-chiral fields and identify a chiral isometry of this free theory under which the semi-chiral matter fields rotate by a chiral phase. Then, couple the associated current to a canonical $(2,2)$ gauge supermultiplet. The result is a manifestly $(2,2)$ GLSM which, in general, does not reduce to a theory of chirals.

There are many fun $(2,2)$ torsion linear sigma models one can build, with interesting geometric and algebraic properties, but our interests here lie with the heterotic string, so we now turn to $(0,2)$ models, leaving a thorough discussion of the $(2,2)$ case (and the intriguing liminal $(1,2)$ case) to another publication.

5.4 Non-Compact $(0,2)$ Models and the Bianchi Identity

Suppose we are handed a well-behaved $(0,2)$ GLSM for a vector bundle $\mathcal{V}_S$ over some happy Kähler manifold S. The FI parameters of the GLSM, t^a, parameterize some of the complexified Kähler moduli of S. As in the $(2,2)$ cases discussed above, introducing non-trivial H into this $(0,2)$ GLSM is a simple matter of promoting some subset of the FI parameters t^a to dynamical fields $Y_{l=1\ldots m}$ in the GLSM. The FI coupling in a $(0,2)$ model is again a superpotential interaction, so the requisite promotion is

$$\frac{i}{4}\int d\theta^+ \; t^a\Upsilon_a + \text{h.c.} \;\rightarrow\; \frac{i}{4}\int d\theta^+ \; N_l^a Y_l \Upsilon_a + \text{h.c.} \; - i\int d^2\theta \; \overline{Y}_l \partial_- Y_l \; , \qquad (5.14)$$

where $y_l = r_l + i\theta_l \in \mathbb{C}^*$, and with $N_l^a \in \mathbb{Z}$ to ensure single-valuedness of the action. This results in non-trivial NS-NS 3-form flux,

$$B = N_l^a \theta_l H_a \quad \Rightarrow \quad H = N_l^a d\theta_l \wedge H_a \ , \tag{5.15}$$

not on S, but on a non-compact fibration $(\mathbb{C}^*)^m \to \widetilde{X} \to S$, with H having two legs along S and one along the fibre. (Here, H_a is the a^{th} hyperplane class on S.)

This model has two major limitations. First and foremost is the fact that the Bianchi identity is solved rather trivially: $dH = 0$ by construction, since both $d\theta_l$ and H_a lift trivially to closed forms on the total space of the $(\mathbb{C}^*)^m$-fibration, $\widetilde{X}$. What we are after is an interesting solution to the Bianchi identity. Secondly, the classical moduli space, $\widetilde{X}$, is non-compact. Since the non-compactness is due to the unconstrained real part of the dynamical FI parameters, we might try to simply lift them, leaving the imaginary part dynamical as required for non-trivial H-flux.[5] Unfortunately, this explicitly breaks $(0,2)$ supersymmetry. In the remainder of this section we will focus on correcting the triviality of the Bianchi identity—the thorny problem of compactification we defer to the next section.

To begin, note a curious difference from the $(2,2)$ case above. In a $(0,2)$ gauge theory, the FI parameter does not appear in a twisted chiral superpotential—indeed, there *is* no twisted chiral representation of $(0,2)$— but in a *chiral* superpotential, so the dynamical FI parameters in a $(0,2)$ theory are *chiral*, just like the minimally coupled scalars. This raises an interesting possibility: since supersymmetry no longer forbids the minimal coupling of the gauge fields to the Y_l, we can couple Y_l both

[5] Indeed, this is what happens in the Goldstein-Prokushkin construction [60], whose compact non-Kähler manifolds arise as the unit-circle sub-bundles of two $\mathbb{C}^*$-bundles over a base Calabi-Yau, as described in section 5.2.

axially *and* minimally in a completely supersymmetric fashion:

$$\mathcal{L} = -\frac{1}{2}\int d^2\theta\ (\overline{Y}_l + Y_l + 2M_l^a V_{+a})(i\partial_-[Y_l - \overline{Y}_l] - M_l^a V_{-a}) + \frac{i}{4}\int d\theta^+\ N_l^a Y_l \Upsilon_a + \text{h.c.}\ , \tag{5.16}$$

where the M_l^a are integers (we will discuss their quantization later). Unfortunately, under a gauge transformation $Y_l \to Y_l - iM_l^b\Lambda_b$, the superpotential transforms as

$$\delta_\Lambda \mathcal{L} = \frac{1}{4}\int d\theta^+\ M_l^b N_l^a \Lambda_b \Upsilon_a + \text{h.c.}\ , \tag{5.17}$$

which is not a total derivative, so this Lagrangian does not appear to be terribly useful.

However, this gauge variation has the familiar form of the *gauge anomaly* of a $(0,2)$ GLSM. Consider a GLSM for a holomorphic vector bundle $\mathcal{V}_S$ over a Calabi-Yau base, S, built out of chiral superfields Φ_I and Fermi superfields Γ_m. While the classical Lagrangian is manifestly gauge invariant, the measure generically suffers from a set of one-loop exact chiral gauge anomalies[6] of the form

$$\mathcal{D}[\Phi,\Gamma] \xrightarrow{\delta_\Lambda} \mathcal{D}[\Phi,\Gamma]\exp\left(-\frac{iA^{ab}}{8\pi}\int d^2y\left[\int d\theta^+ \Lambda_b \Upsilon_a + \text{h.c.}\right]\right)\ , \tag{5.18}$$

where A^{ab} is a quadratic form built out of the gauge charges Q_I^a and q_m^a of the right- and left-moving fermions,

$$A^{ab} = \sum_I Q_I^a Q_I^b\ - \sum_m q_m^a q_m^b\ . \tag{5.19}$$

This can be easily derived by examining the loop diagram with two external gauge bosons. This anomaly, a familiar feature of $(0,2)$ GLSM building, has a natural

[6]Such gauge anomalies are strictly absent in $(2,2)$ models, where left- and right-handed fermions are paired up in $(2,2)$ chiral multiplets to give an overall non-chiral theory; in a $(0,2)$ model, by contrast, left- and right-moving fermions transform in different supersymmetry multiplets and may thus transform differently under the gauge symmetry, leading to the gauge anomaly advertised above.

geometric interpretation. Recall that the right-handed fermions transform as sections of a sheaf $\mathcal{F}_V$ over the ambient toric variety V which restricts over S to the tangent bundle, T_S. Meanwhile, the left-handed fermions transform as sections of a sheaf $\mathcal{V}_V$ which restricts to the gauge bundle, $\mathcal{V}_S$. The gauge anomaly measures

$$\mathcal{A} \propto ch_2(\mathcal{F}_V) - ch_2(\mathcal{V}_V) \ . \tag{5.20}$$

Since the Bianchi identity is just the restriction of $\mathcal{A}$ to S, the vanishing of the gauge anomaly[7] ensures that the IR NLSM satisfies the heterotic Bianchi identity with $dH = 0$. This connection will be better explored in section 5.5.3.

These two effects—the gauge variance of the classical action and the one-loop gauge anomaly—dovetail beautifully. Consider a GLSM for $\mathcal{V}_S \to S$ with $ch_2(T_S) \neq ch_2(\mathcal{V}_S)$. On its own, this model is anomalous. Now promote some subset of FI parameters to dynamical fields Y_l with axial couplings N_l^a and charges M_l^a. Under a gauge variation, the effective action ($S_{\mathit{eff}} = \frac{1}{4\pi}\int d^2y\, \mathcal{L}_{\mathit{eff}}$) picks up classical terms from the axions and one-loop terms from the anomaly,

$$\delta_\Lambda \mathcal{L}_{\mathit{eff}} = \frac{1}{2}\int d\theta^+ \ \left[\frac{1}{2}M_l^b N_l^a - Q_I^a Q_I^b + q_m^a q_m^b\right] \Lambda_b \Upsilon_a + \text{h.c.} \ . \tag{5.21}$$

Thus, for every solution of the Diophantine equation

$$\frac{1}{2}\sum_l M_l^b N_l^a = \sum_I Q_I^a Q_I^b - \sum_m q_m^a q_m^b \tag{5.22}$$

[7]Note that the gauge anomaly may fail to vanish even when the classical moduli space of the GLSM has vanishing ch_2 anomaly. For example, consider a $(0,2)$ model for an elliptic curve in $\mathbb{P}^2$ with trivial left-moving bundle. A NLSM on an elliptic curve cannot have a ch_2 anomaly—nonetheless, the GLSM has a gauge anomaly. What is going on? The point is that the gauge anomaly computes the non-vanishing self-intersection number of the hyperplane class in $\mathbb{P}^2$, an intersection which does not restrict to the hypersurface (indeed, there is no four-cohomology on T^2). This is a somewhat familiar fact in $(0,2)$ model building: many geometries for which a NLSM analysis is perfectly consistent do not seem to admit GLSM descriptions due to uncanceled gauge anomalies.

we have a non-anomalous $(0,2)$ quantum field theory. Since the superpotential of this $(0,2)$ theory is not renormalized beyond one loop in perturbation theory, and since the anomaly is one-loop exact, the path integral remains gauge invariant to all orders in perturbation theory.[8] Note that the ch_2 anomaly in the NLSM is also one-loop exact. We shall refer to a $(0,2)$ GLSM which implements the above cancellation mechanism as a *torsion linear sigma model* (TLSM).

Notice what has happened. First, we have replaced the Kähler geometry S with a non-Kähler $(\mathbb{C}^*)^m$-fibration $\widetilde{X}$ over S such that the curvature 2-forms of the $(\mathbb{C}^*)^m$-fibration are trivial in $H^2(\widetilde{X},\mathbb{Z})$, the cohomology of the total space. It is important to distinguish $ch_2(T_S) - ch_2(\mathcal{V}_S)$, the anomaly on S, from the very different quantity $ch_2(T_{\widetilde{X}}) - ch_2(\mathcal{V}_{\widetilde{X}})$, the anomaly on the $(\mathbb{C}^*)^m$-fibration $\widetilde{X}$ *over* S. At the end of the day, the physical Bianchi identity lives on $\widetilde{X}$ and says that $dH = ch_2(T_{\widetilde{X}}) - ch_2(\mathcal{V}_{\widetilde{X}})$, so in cohomology on $\widetilde{X}$, $ch_2(T_{\widetilde{X}}) = ch_2(\mathcal{V}_{\widetilde{X}})$. However, since $\widetilde{X}$ is a non-trivial fibration over S, cohomology classes do not trivially lift, or descend (think about the Hopf map). The upshot it that Bianchi identity does *not* imply that $ch_2(T_S) = ch_2(\mathcal{V}_S)$, even in cohomology. However, the 3-form flux $H = N_l^a d\theta_l \wedge H_a$ on the the total space, $\widetilde{X}$, was constructed precisely so as to solve the Bianchi identity when pushed down the fibres—this is what led us to introduce the gauge-variant axial coupling in the first place.

This graceful mechanism of anomaly cancellation—a one-loop gauge anomaly canceling the gauge variation of an axionic coupling in the classical Lagrangian—is simply a 2D avatar of the Green-Schwarz anomaly in the target space.

[8]We will discuss non-perturbative effects below.

5.5 Compact $(0,2)$ Models and the Torsion Multiplet

Let us summarize the story so far. We begin with a conventional $(0,2)$ GLSM for a Calabi-Yau S equipped with a generic holomorphic bundle $\mathcal{V}_S$. The $ch_2(T_S) \neq ch_2(\mathcal{V}_S)$ anomaly of the associated NLSM is realized in the GLSM as a gauge anomaly. To cancel the gauge anomaly, we promote some of the FI parameters to dynamical axions carrying charges chosen such that the gauge variation of the classical action cancels the one-loop gauge anomaly in a 2D version of the Green-Schwarz mechanism. The IR geometry of the resulting non-anomalous $(0,2)$ GLSM is a non-compact $(\mathbb{C}^*)^m$-fibration $\widetilde{X}$ over S,

$$\begin{array}{ccc} (\mathbb{C}^*)^m & \longrightarrow & \widetilde{X} \\ & & \downarrow \\ & & S \end{array} \quad ,$$

where the curvature two-forms of the $\mathbb{C}^*$-bundles are $M_l^a H_a|_S \in H^2(S,\mathbb{Z})$. Threading this geometry is a non-trivial NS-NS 3-form flux, $H = N_l^a d\theta_l \wedge H_a$, which satisfies the Bianchi identity non-trivially. For simplicity of presentation, we will focus on the special cases $S = K3$ or T^4 with $m = 2$; the generalization to higher dimension and other geometries is straightforward.

Not coincidentally, this is enticingly close to the compact Fu-Yau geometry—all we need to do is restrict to the T^2 sub-bundle of the $(\mathbb{C}^*)^2$ bundle by lifting the real direction along each $\mathbb{C}^*$ fibre. What could be easier?

5.5.1 Decoupling of Radial Fields

In fact, this turns out to be rather non-trivial. The issue is supersymmetry. The target space of any sigma model with a linearly realized $\mathcal{N}=2$ is a complex manifold, and the specific presentation of the $\mathcal{N}=2$ corresponds to a specific choice of complex structure. Under the particular $\mathcal{N}=2$ respected by our GLSM, the real directions along the $\mathbb{C}^*$ fibre, r_l, are paired with the S^1 angles, θ_l, so removing only the radial coordinates would explicitly break our $(0,2)$ supersymmetry to an all-but-useless $(0,1)$ subgroup (which we are not allowed to lose since this $(0,1)$ will be gauged when we couple our matter theory to heterotic worldsheet supergravity). The situation appears to be grim.

To reassure ourselves that there *should* be a $(0,2)$ on the T^2 sub-bundle, note that

$$(\mathbb{C}^*)^2 = \mathbb{C} \times T^2$$

if the coordinates $y_l = r_l + i\theta_l$ on $(\mathbb{C}^*)^2$ are reorganized into the coordinates $r = r_1 + ir_2$ on $\mathbb{C}$ and $\theta = \theta_1 + i\theta_2$ on T^2. The IR geometry thus must admit an $\mathcal{N}=2$ corresponding to this choice of complex structure, pairing the two angles into one supermultiplet and the two lines into another. Unfortunately, an extensive search for such an $\mathcal{N}=2$ in our UV gauge theory quashes our high expectations.

Let us explore this apparent failure more explicitly. The relevant terms in the action are, in components,

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{K3} - k_l^2(\partial r_l)^2 - k_l^2(\partial\theta_l + M_l^a v_a)^2 + 2ik_l^2\overline{\chi}_l\partial_-\chi_l + 2N_l^a\theta_l v_{+-a} \\ &\quad + \left(2k_l^2 M_l^a - N_l^a\right)\left[r_l D_a + \frac{i}{\sqrt{2}}\chi_l\lambda_a\right] + \ldots\ .\end{aligned}$$

Meanwhile, under the linearly realized $(0,2)$ supersymmetry

$$\delta_\epsilon \lambda_a = i\epsilon(D_a + 2iv_{+-a}) \ . \tag{5.23}$$

Now suppose we attempt reorganize the Y_l superfields into superfields that respect the $\mathbb{C}\times T^2$ complex structure: $R \sim r_1 + ir_2 + \ldots$, $\Theta \sim \theta_1 + i\theta_2 + \ldots$. The problem is that the variation of λ_a yields terms of the form $\epsilon\chi_l D_a$ and $\epsilon\chi_l v_{+-a}$. The only way to cancel these terms is for the variation of both r_l and θ_l to include terms of the form $\epsilon\chi_l$. This makes it appear impossible to split r_l and θ_l into two separate supermultiplets for generic charges.

The key word here is "generic". Note that our troublesome terms are both proportional to $(2k_l^2 M_l^a - N_l^a)$, where $M_l^a, N_l^a \in \mathbb{Z}$ and $k_l \in \mathbb{R}$. If we fix k_l and N_l^a so that $N_l^a = 2k_l^2 M_l^a$, these terms disappear from the action! Repeating our analysis, we find that there *is* a $(0,2)$ supersymmetry with exactly the desired properties:

$$R = (r_1 - ir_2) + i\sqrt{2}\theta^+(\chi_1^I + i\chi_2^I) + \ldots \qquad \Theta = (\theta_1 + i\theta_2) + \sqrt{2}\theta^+(\chi_1^R - i\chi_2^R) + \ldots \ , \tag{5.24}$$

where R and I superscripts refer to the real and imaginary parts of the fermions, respectively. In fact, the R-multiplet is free and *entirely decouples!* What is more, since k_l, which measures the radius of the T^2 in string units, is fixed in terms of two integers, the volume of the fibre is quantized in terms of the torsion flux, just as it is in Fu-Yau.[9]

[9]Since $\int d^2y\, v_{+-a} \in \pi\mathbb{Z}$, θ_l is automatically periodic, $\theta_l \sim \theta_l + 2\pi L_l$, such that $N_l^a L_l \in 2\mathbb{Z}$. Fixing $N_l^a = 2k_l^2 M_l^a$ then implies that $k_l^2 M_l^a L_l = n_a \in \mathbb{Z}$, so M_l^a is quantized in terms of k_l and L_l. Meanwhile, the anomaly cancellation condition implies that that $\frac{n_a^2}{k_l^2 L_l^2}$ should be an integer, since the Q_I and q_m are integers. Since the physical radius of the S^1 is $k_l L_l$, this means that the radius is quantized as claimed. For the rest of this chapter, we will work with $k_l = L_l = 1$ for simplicity.

Life is now sweet and easy. Based on the above, we define

$$\begin{aligned} &\theta = \theta_1 + i\theta_2 \qquad \chi = \chi_1^R - i\chi_2^R \qquad N^a = 2k^2 M^a = 2k^2(M_1^a + iM_2^a) \\ &r = r_1 - ir_2 \qquad \tilde{\chi} = i\chi_1^I - \chi_2^I \qquad \nabla_\pm\theta = \partial_\pm\theta + M^a v_{\pm a} \ , \end{aligned} \tag{5.25}$$

which transform under $\mathcal{N} = 2$ supersymmetry as

$$\begin{aligned} &\delta_\epsilon\theta = -\sqrt{2}\epsilon\chi \qquad \delta_\epsilon\chi = 2\sqrt{2}i\bar{\epsilon}\ \nabla_+\theta \\ &\delta_\epsilon r = -\sqrt{2}\epsilon\tilde{\chi} \qquad \delta_\epsilon\tilde{\chi} = 2\sqrt{2}i\bar{\epsilon}\ \partial_+ r \ . \end{aligned} \tag{5.26}$$

In these coordinates, the action reduces to

$$\begin{aligned} \mathcal{L} \ = \ & \mathcal{L}_{K3} + 2\nabla_+\bar{\theta}\nabla_-\theta + 2\nabla_+\theta\nabla_-\bar{\theta} + 2i\overline{\chi}\partial_-\chi + 2(N^a\bar{\theta} + \overline{N}^a\theta)v_{+-a} \\ & -2|\partial r|^2 + 2i\overline{\tilde{\chi}}\partial_-\tilde{\chi} \ . \end{aligned} \tag{5.27}$$

We may now drop the radial supermultiplet $R = r + \sqrt{2}\theta^+\tilde{\chi} - 2i\theta^+\bar{\theta}^+\partial_+ r$, as it is entirely decoupled.

It is important to verify that the truncated Lagrangian is invariant under the $(0,2)$ supersymmetry defined above. However, $\delta_{susy}^2 = \delta_{gauge}$ in WZ gauge, so the gauge variance of the classical action rears its stupefying head and some care is required. Under a supersymmetry transformation, the classical action transforms non-trivially,

$$\delta_\epsilon\mathcal{L} = 2(M^a\overline{M}^b + \overline{M}^a M^b)v_{+b}(i\epsilon\overline{\lambda}_a + i\bar{\epsilon}\lambda_a) \ . \tag{5.28}$$

This is not a disaster because the gauge transformation needed to return us to WZ gauge (which we have been using throughout), $\alpha_a = -4i\theta^+\bar{\epsilon}v_{+a}$, induces a shift in the effective action from the anomalous measure:

$$\delta_{WZ}\mathcal{D}[\Phi,\Gamma] = \mathcal{D}[\Phi,\Gamma]\exp\left\{\frac{A^{ab}}{\pi}\int d^2y\, v_{+a}(i\epsilon\overline{\lambda}_b + i\bar{\epsilon}\lambda_b)\right\} \ . \tag{5.29}$$

Anomaly cancellation ensures that this cancels the supersymmetry variation of the action.

At this point, we can play various games to simplify the presentation of the theory. For example, we can build a superfield out of the T^2 multiplet,

$$\Theta = \theta + \sqrt{2}\theta^+\chi - 2i\theta^+\bar{\theta}^+\nabla_+\theta\ .$$

This looks a lot more convenient than it actually is. While it has the usual field content, this is *not* a standard chiral multiplet: the gauging is complex, with both real and imaginary components of θ shifting under gauge transformations. Since no other superfield transforms in the same strange way, gauge multiplet included, it is extremely hard to build gauge covariant or invariant operators out of Θ. In fact, the only gauge-invariant dressed field is $(\partial_-\Theta + \frac{1}{2}M^aV_{-a})$. Meanwhile, the only chiral operator we can build out of Θ is $(\Theta + iM^aV_{+a})$, which we cannot add to the superpotential in a gauge invariant fashion. Indeed, it is impossible to build a supersymmetric and gauge invariant action for this multiplet alone since the supersymmetry variation of the kinetic terms cancels against the variation of the axial superpotential. To emphasize its peculiar role, we call Θ a *torsion multiplet*.

5.5.2 The IR Geometry

Setting $N_l^a = 2k^2M_l^a$ has decoupled the R multiplet, leaving us with a non-Kähler T^2 sub-bundle $X \subset \widetilde{X}$ with torsionful $SU(3)$-structure induced from $\widetilde{X}$. In other words, the semi-classical IR geometry of our TLSM is a compact holomorphic T^2 fibration X over a Calabi-Yau S, endowed with a Hermitian metric, a stable holomorphic sheaf $\mathcal{V}_X = \pi^*\mathcal{V}_S$ pulled back from S, and an NS-NS 3-form H satisfying the Bianchi identity on $\mathcal{V}_X \to X$. Moreover, the radii of the T^2 fibres are fixed to discrete values in terms of the integral curvatures of the T^2-bundles, given as integer classes on the base $K3$. Up to uninteresting changes of coordinates, this is the Fu-Yau construction.

It is revealing to derive this IR geometry explicitly from the final TLSM. Let us begin by writing out the component Lagrangian in all its majesty. To simplify our lives, we will call all the chiral multiplets ϕ_I whether their charges are positive, negative, or zero, and leave all obvious sums implicit. This is easy to unpack when we focus on specific models. After integrating out the auxillary fields, the kinetic terms are,

$$\begin{aligned}\mathcal{L}_{kin} &= -|(\partial + iQ_I^a v_a)\phi_I|^2 + 2i\overline{\psi}_I(\partial_- + iQ_I^a v_{-a})\psi_I \\ &\quad +4(\partial_+\theta_l + M_l^a v_{+a})(\partial_-\theta_l + M_l^a v_{-a}) + 4M_l^a\theta_l v_{+-a} + 2i\overline{\chi}_l\partial_-\chi_l \\ &\quad +2i\overline{\gamma}_m(\partial_+ + iq_m^a v_{+a})\gamma_m + \frac{2}{e_a^2}\left[(v_{+-a})^2 + i\overline{\lambda}_a\partial_+\lambda_a\right] ,\end{aligned}$$

and the scalar potential is

$$U = \sum_m \left(|E_m|^2 + |J^m|^2\right) + \sum_a \frac{e_a^2}{2}\left(\sum_I Q_I^a|\phi_I|^2 - r^a\right)^2 \tag{5.30}$$

where $\overline{\mathcal{D}}_+\Gamma_m = \sqrt{2}E_m(\Phi)$ and $J^m(\Phi)$ is a $(0,2)$ superpotential satisfying $\sum_m E_m J^m = 0$. For completeness, the Yukawa terms are

$$\mathcal{L}_{Yuk} = -\sqrt{2}iQ_I^a\lambda_a\psi_I\overline{\phi}_I - \overline{\gamma}_m\psi_I\frac{\partial E_m}{\partial\phi_I} - \gamma_m\psi_I\frac{\partial J^m}{\partial\phi_I} + \text{h.c.} \ . \tag{5.31}$$

As in the case of $(2,2)$ GLSMs on Kähler geometries, the Hermitian geometry of the Higgs branch of our TLSM may be computed by integrating out the massive vectors and scalars in the gauge theory to derive a Born-Oppenheimer effective action on the classical moduli space. However, since the classical action of our TLSM is not gauge invariant, the story is slightly more subtle than usual.

Suppose, for example, that we simply integrate out the massive vector as usual—let us work in polar variables where $\phi_I = \rho_I e^{i\varphi_I}$. This replaces the gauge connection v_μ with a non-trivial implicit connection $v_\mu(\rho_I, \varphi_I, \theta_l, \ldots)$ on the classical moduli space. The chiral fermion content then leads to an anomaly in the resulting non-linear sigma model—an anomaly which cancels against the classical variation of the action due to the torsion multiplet. This presentation has the advantage of making the role of the anomalous gauge transformation in the NLSM manifest, but it complicates the computation of the effective metric.

Alternatively, we can take a lesson from Fujikawa and change coordinates in field space to work with uncharged fermions *before* integrating out the massive vector [55, 56]. The Jacobian of this field redefinition introduces a gauge variant operator to the action whose gauge variation cancels against that of the classical torsion terms, leaving the action gauge invariant. We can then integrate out the massive vector and massive scalars to compute the effective metric on moduli space.

Let us take the second approach and change variables to gauge invariant fermions. For each right-moving fermion ψ_I, there is a natural choice of uncharged dressed fermion $\tilde{\psi}_I = e^{-i\varphi_I}\psi_I$; for the left-movers, there is generically no model-independent choice, so we choose an arbitrary linear combination $\hat{\varphi}_m = l^I_m \varphi_I$ of phases with the correct charges to make the dressed fermion $\tilde{\gamma}_m = e^{-i\hat{\varphi}_m}\gamma_m$ gauge neutral, i.e. such that $\delta_\alpha \hat{\varphi}_m = -q^a_m \alpha_a$. The Jacobian for this change of variables shifts the action by a simple term

$$\mathcal{L}_{Jac} = -4\omega^a v_{+-a}, \qquad \omega^a \equiv Q^a_I \varphi_I - q^a_m \hat{\varphi}_m \equiv T^a_I \varphi_I \ , \tag{5.32}$$

whose gauge variation is just the familiar anomaly,

$$\delta_\alpha \mathcal{L}_{Jac} = 4\left(Q_I^a Q_I^b - q_m^a q_m^b\right)\alpha_a v_{+-b} \ . \tag{5.33}$$

We notice that this is highly reminiscent of the addition of Chern-Simon forms to the NLSM action, cf. equation (3.10). The total axial coupling is thus

$$\mathcal{L}_{axial} = 4\left(M_l^a \theta_l - \omega^a\right) v_{+-a} \ , \tag{5.34}$$

which is gauge invariant by construction. The typical next step is to fix a gauge. However, since the Faddeev-Popov measure for the simplest gauge choice, $\theta_l = 0$, is trivial, it is just as easy to work in gauge unfixed presentation; the decoupled longitudinal mode will simply cancel the volume of the gauge group in the path integral.

With the action and measure now both independently gauge invariant, we can consistently integrate out the massive vector. Since the action is quadratic in the vector, this is straightforward. Solving the classical EOM for the two components of our massive vector, and splitting them into fermionic and bosonic components, yields

$$\begin{aligned} v_{-a} &= (\Delta^{-1})^{ab}\left(\frac{1}{2}\bar{\tilde{\gamma}}_m \tilde{\gamma}_m q_m^b - \rho_I^2 \partial_- \varphi_I Q_I^b + \partial_- \omega^b - 2M_l^b \partial_- \theta_l\right) = v_{-a}^F + v_{-a}^B \\ v_{+a} &= (\Delta^{-1})^{ab}\left(\frac{1}{2}\bar{\tilde{\psi}}_I \tilde{\psi}_I Q_I^b - \rho_I^2 \partial_+ \varphi_I Q_I^b - \partial_+ \omega^b\right) = v_{+a}^F + v_{+a}^B \ , \end{aligned}$$

where we define

$$\Delta^{ab} \equiv \rho_I^2 Q_I^a Q_I^b + M_l^a M_l^b \equiv \Delta_Q + \Delta_M \ ,$$

which is naturally symmetric in the gauge indices. It is easy to check that both components of v transform covariantly under gauge transformations.

Thus prepared, we are finally ready to compute the effective metric on the Higgs branch. After a tedious but miserable calculation, the bosonic effective action reduces to

$$\begin{aligned}\mathcal{L}^B_{kin} &= 4\partial_+\rho_I\partial_-\rho_I + 4\partial_+\varphi_I\partial_-\varphi_J\left[\rho_I^2\delta_{IJ} - \rho_I^2\rho_J^2(\Delta^{-1})_{ab}Q_I^aQ_J^b\right] + 4(\Delta^{-1})_{ab}\partial_+\omega^a\partial_-\omega^b \\ &\quad +4\partial_+\theta_l\partial_-\theta_l - 8(\Delta^{-1})_{ab}\left(\rho_I^2\partial_+\varphi_IQ_I^a + \partial_+\omega^a\right)\partial_-\theta_lM_l^b - 8(\Delta^{-1})_{ab}\rho_I^2Q_I^a\partial_{[+}\omega^b\partial_{-]}\varphi_I\ ,\end{aligned}$$

and the fermionic effective action to

$$\begin{aligned}\mathcal{L}^F_{kin} &= 2i\overline{\tilde{\psi}}_I(\partial_- + iQ_I^av^B_{-a} + i\partial_-\varphi_I)\tilde{\psi}_I + 2i\overline{\chi}_l\partial_-\chi_l + 2i\overline{\tilde{\gamma}}_m(\partial_+ + iq_m^av^B_{+a} + i\partial_+\hat{\varphi}_m)\tilde{\gamma}_m \\ &\quad -(\Delta^{-1})_{ab}\overline{\tilde{\psi}}_I\tilde{\psi}_I\overline{\tilde{\gamma}}_m\tilde{\gamma}_mQ_I^aq_m^b\ ,\end{aligned}$$

where $A_{[+}B_{-]} \equiv \frac{1}{2}(A_+B_- - A_-B_+)$, $A_{(+}B_{-)} = \frac{1}{2}(A_+B_- + A_-B_+)$. We will also find it useful to define $\Delta_2^{-1} \equiv \Delta^{-1} - \Delta_Q^{-1} = -\Delta_Q^{-1}\Delta_M\Delta^{-1}$, and to make a habit of suppressing gauge indices, representing them instead by matrix multiplication.

Since one of the features we would like to make manifest is the natural complex structure on the total space X, it is natural to return to complex variables ϕ_I and θ, as well as $M^a \equiv M_1^a + iM_2^a$. It is also natural to split the Lagrangian into terms symmetric and anti-symmetric in the derivatives, corresponding to the pullback to the worldsheet of the metric and B-field, respectively. The symmetric terms we will refer to as ds^2, where we will also use the shorthand

$$dAdB \equiv \partial_{(+}A\partial_{-)}B \qquad dA\wedge dB \equiv \partial_{[+}A\partial_{-]}B\ ,$$

remembering that the "differentials" dA and dB are symmeterized without the $\wedge$.

Using these conventions and the definition of Δ_2^{-1}, we can easily factor out the usual kinetic terms for the ambient variety V:

$$ds_V^2 = 4|d\phi_I|^2 - 4(\overline{\phi}_Id\phi_I)(\phi_Jd\overline{\phi}_J)Q_I^T\Delta_Q^{-1}Q_J\ .$$

The metric can then be written as

$$\begin{aligned} ds^2 &= ds_V^2 - 4|\phi_I|^2|\phi_J|^2(d\ln\overline{\phi}_I d\ln\phi_J)Q_I^T\Delta_2^{-1}Q_J - \left[d\ln\frac{\phi_I}{\overline{\phi}_I}\right]\left[d\ln\frac{\phi_J}{\overline{\phi}_J}\right]T_I^T\Delta^{-1}T_J \\ &\quad +4|d\theta|^2 + 2i\left[d\ln\frac{\phi_I}{\overline{\phi}_I}\right]\left(|\phi_I|^2Q_I^T + T_I^T\right)\Delta^{-1}\left(Md\bar{\theta} + \overline{M}d\theta\right) , \end{aligned} \tag{5.35}$$

where we have used $dr^a = \sum_I Q_I^a(\phi_I d\overline{\phi_I} + \overline{\phi}_I d\phi_I) = 0$ to simplify the expression. Working patchwise on V makes the geometry somewhat more transparent. We can cover V by patches on which s of the homogeneous coordinates, say $\phi_{\sigma=N+1,\ldots,N+s}$, are nonzero and for which Q^a_σ is an invertible $s\times s$ matrix. We can then define gauge invariant coordinates on each patch,

$$z_A \equiv \phi_A \prod_{\sigma=N+1}^{N+s} \phi_\sigma^{-(Q^{-1})^\sigma_a Q^a_A}, \qquad \zeta \equiv \theta + i(Q^{-1})^\sigma_a M^a \ln\phi_\sigma ,$$

where $A = 1,\ldots,N$.

All of these coordinates transform holomorphically as we move from one patch of V to another. Furthermore, from the gauge variant coordinates it is clear that there are no fixed points of the T^2 action (complex shifts of θ). Thus, as long as $S\subset V$ is smooth, our construction will yield a principal holomorphic T^2 bundle over S *à la* Goldstein and Prokushkin. In these manifestly holomorphic coordinates, the metric can be written in Hermitian form,

$$\begin{aligned} ds_H^2 &= ds_V^2 + 4\left|d\zeta - \frac{iM^T}{2}\left(\partial P - \Delta^{-1}(Q_A|\phi_A|^2 + T_A)d\ln z_A\right)\right|^2 \\ &\quad + \left(\partial P^T\Delta_M + d\ln z_A\, T_A^T\right)\left(2\Delta^{-1} - \Delta^{-1}M\overline{M}^T\Delta^{-1}\right)\left(\Delta_M\overline{\partial}P + T_B\, d\ln\overline{z}_B\right) , \end{aligned}$$

where $P_a \equiv \sum_\sigma (Q^{-1})^\sigma_a \ln|\phi_\sigma|^2$ and

$$ds_V^2 = 4|\phi_A|^2|d\ln z_A|^2 - 4\left(|\phi_A|^2Q_A^T\Delta_Q^{-1}Q_B|\phi_B|^2\right)[d\ln z_A]\,[d\ln\overline{z}_B]$$

is the analog of the Fubini-Study metric for V (and reduces to it in the case of $\mathbb{P}^N$). A similarly tedious but straightforward computation gives the resulting B-field,

$$\begin{aligned} B &= 2i\left[d\ln\frac{z_A}{\bar{z}_A}\right]\wedge(|\phi_A|^2Q_A^T+T_A^T)\Delta^{-1}\left[Md\bar{\zeta}+\overline{M}d\zeta\right] \\ &\quad -2(|\phi_A|^2Q_A^T\Delta^{-1}T_B)\left[d\ln\frac{z_A}{\bar{z}_A}\right]\wedge\left[d\ln\frac{z_B}{\bar{z}_B}\right] \\ &\quad -(\overline{M}^aM^T-M^a\overline{M}^T)\Delta^{-1}(Q_A|\phi_A|^2+T_A)\left[d\ln\frac{z_A}{\bar{z}_A}\right]\wedge dP_a\ . \end{aligned}$$

We thus have a manifestly Hermitian metric on a smooth principal holomorphic T^2-bundle over S, with non-vanishing H threading the total space. This is precisely the geometry we were expecting to find.

5.5.3 The Bianchi Identity

As we sketched in section 3, the one-loop exact spacetime Bianchi identity is realized in the TLSM by the one-loop exact gauge anomaly. However, the gauge anomaly is independent of the superpotential and thus naturally lives on the ambient toric variety V, while the Bianchi identity lives on the space X, so the connection between the Bianchi identity and the gauge anomaly requires some work to explicate.

Their relationship is most transparent when the Bianchi identity is pushed down to the base, S. In the Fu-Yau case, it has been shown on purely geometric grounds that [54, 15]

$$dH=\pi^*\left(\omega\wedge *_S\overline{\omega}\right)+\ldots, \qquad ch(T_X)=\pi^*\left(ch(T_S)\right)+\ldots\ , \tag{5.36}$$

where $\omega=\omega_1+i\omega_2$ is the anti-self-dual[10] (1,1) curvature form of the T^2 bundle, and the omitted terms are all exact forms on S and thus vanish in cohomology on

[10]Strictly speaking, there can also be a self-dual $(2,0)$ ω-form, but it is automatically absent in the TLSM construction.

the base. Meanwhile, by construction, $ch(\mathcal{V}_X) = \pi^*(ch(\mathcal{V}_S))$, so the Bianchi identity reduces to a simple equation in the cohomology of S:

$$\omega \wedge *_S\overline{\omega} = -\omega_1^2 - \omega_2^2 = 2ch_2(\mathcal{V}_S) - 2ch_2(T_S)\ . \tag{5.37}$$

All the quantities in this equation can now be written in terms of the defining charges of the TLSM. The second Chern characters can be calculated from the short exact sequences 3.42 and 3.43 to be

$$\begin{aligned} ch_2(T_S) &= \frac{1}{2}\sum_{a,b}\left(\sum_i Q_i^a Q_i^b - d^a d^b\right)(H_a \wedge H_b)|_S\,, \\ ch_2(\mathcal{V}_S) &= \frac{1}{2}\sum_{a,b}\left(\sum_m q_m^a q_m^b - m^a m^b\right)(H_a \wedge H_b)|_S\ , \end{aligned}$$

Meanwhile, the curvature ω of the T^2 fibration can be expressed as $\omega = (M_1^a + iM_2^a)H_a$, so the Bianchi identity pushes down to S to give

$$\sum_{a,b}\left(M^{(a}\overline{M}^{b)} - \sum_i Q_i^a Q_i^b + d^a d^b + \sum_m q_m^a q_m^b - m^a m^b\right)(H_a \wedge H_b)|_S = 0\ . \tag{5.38}$$

This is precisely the condition for the cancellation of the gauge anomaly of the TLSM.

5.5.4 Ruling Out T^4

The case $S = T^4$ provides a revealing test case for our construction. Since T_{T^4} is (utterly) trivial, the Bianchi identity takes a particularly simple form—in fact, it is so simple that there are no non-trivial solutions [15]. This can be seen by integrating 5.1 over the base using the restricted forms of dH and $[ch(\mathcal{V}_X) - ch(T_X)]$ given in the previous section. Since F_S—the curvature of the bundle $\mathcal{V}_S$—is anti-Hermitian and anti-self-dual, and since $ch_2(T_{T^4}) = 0$, the right-hand side of 5.37 is non-positive for

$S = T^4$ while the left-hand side is manifestly non-negative for anti-self-dual ω (and only 0 when ω is exact). We would like to see this directly in the TLSM, at least in a specific example.

To this end, we build the base $S = T^4$ as the product of two $T^2 \subset \mathbb{P}^2$, but with H-flux lacing both factors. This ensures that any 4-form on the base must be proportional to $H_1 \wedge H_2|_S$, where H_1 and H_2 are the hyperplane classes of the two $\mathbb{P}^2$s (the restrictions of H_i^2 vanish trivially). Since the Hodge star on T^4 acts as

$$*_S H_1 = H_2 \qquad *_S H_2 = H_1 \ , \tag{5.39}$$

$(H_1 - H_2)$ is the only anti-self-dual 2-form on T^4 constructed from hyperplane classes. Since the Fu-Yau construction requires ω be anti-self-dual, we must have $\omega = M(H_1 - H_2)$. Two further conditions apply: (1) for our embedding of T^4, none of the coordinate fields are charged under both $U(1)$s, and so $d^1 d^2 = Q_i^1 Q_i^2 = 0$; and (2) the condition that $c_1(\mathcal{V}_S) = 0$ translates into $m^a = \sum_m q_m^a$. Plugging this into 5.38, only the $H_1 \wedge H_2$ cross-term does not vanish upon restriction to S and we find

$$\sum_{m \neq n} q_m^1 q_n^2 = -|M|^2 \ . \tag{5.40}$$

But for the gauge bundle to be stable, all charges must satisfy $q_m^a \geq 0$ [48], in which case the equation has no solution unless $M = 0$. We conclude that our TLSM does not allow us to build a non-trivial T^2-bundle over this T^4-base, in agreement with the the supergravity result.

5.5.5 Global Anomalies

Of course, vanishing of the gauge anomaly and satisfaction of the Bianchi identity are not sufficient to ensure that the TLSM flows to a consistent vacuum of the heterotic string. In order to couple to worldsheet supergravity, our theory must flow to a superconformal fixed point which admits a chiral GSO projection. This in turn requires [48, 47] the existence of a non-anomalous right-moving $U(1)$ R-current, J_R, and a non-anomalous left-moving flavor symmetry, J_L, leading to additional constraints on allowed charges beyond quantum gauge invariance. The relevant anomalies are thus the various mixed gauge-global and global-global anomalies; consistency of the gauge theory requires that they cancel.

Let us start with the R-current. R-invariance of the $\Upsilon\Theta$ terms in the superpotential require Θ to be an R-scalar, though it may carry a shift-charge under R-symmetry. This implies that the fermion χ in Θ carries R-charge $+1$. However, since χ is gauge neutral, it does not contribute to the mixed gauge-R anomaly. Since the chiral superfields Φ_i typically appear in quasi-homogeneous polynomials in the superpotential $\Gamma_0 G(\Phi_i)$, it is most natural to assign them R-charges proportional to their gauge charges rQ_i—this also fixes the R-charge of Γ_0 to $-rd-1$. Then one has the Fermi supermultiplets Γ_m appearing in the superpotential via $\Phi_0\Gamma_m J^m(\Phi_i)$, restricting charge assignments for Φ_0 and Γ_m to be $p-rm$ and rq_m-p-1, respectively. This additional shift of p is a freedom not available to us in $(2,2)$ models.

The anomaly in the left-moving flavor symmetry can be treated similarly. For example, by setting the flavor charge of each field proportional to its gauge charge, and assigning Θ an anomalous shift-charge under the flavor $U(1)$, vanishing of the gauge anomaly ensures the non-anomaly of the left-moving flavor symmetry. Note that the contribution of the torsion multiplet to the currents J_L, J_R, and J_{gauge}, is of the form $J_\Theta \sim \partial\theta$, so its contributions to the anomalies actually come from tree-diagrams rather than loops.

Two final anomaly relations are important. First, for J_R and J_L to be purely right- and left-moving, their mixed anomaly must also vanish, giving one integer constraint. Finally, the $J_R J_R$ OPE measures the conformal anomaly, which must be equal to 9, giving one last integer equation on the charges. In the typical model of interest, there are many more fields than equations, making it easy to satisfy these constraints.

5.5.6 Caveat Emptor: Spacetime vs. Wordsheet Constraints

One very important elision in the above is distinguishing which conditions on the charges are required on *a priori* 2D grounds, and which derive from spacetime arguments. For example, in a $(2,2)$ model the running of the D-term is equivalent to the R-anomaly, which in turn is equivalent to the vanishing first Chern class of the IR geometry, $c_1(T_S)$. However, in a $(0,2)$ model these three effects are decoupled.

The running of t is decoupled because we can always add a pair of massive spectators to the theory—a chiral and a Fermi superfield—whose contributions to all gauge and global anomalies vanish, but whose gauge charges can be chosen to limit the running of t to a finite shift [48, 47], something not possible in more familiar $(2,2)$ models. Meanwhile, the chiral content of the theory yields enough freedom in assigning R-charges that the R-anomaly is decoupled from $c_1(T_S) = 0$. Similarly, the conditions that $c_1(\mathcal{V}_S) = 0$, that ω be anti-self-dual, and that $\mathcal{V}_S$ be stable, are all required to ensure spacetime supersymmetry in the supergravity construction of the Fu-Yau compactification but do not appear as necessary constraints for the consistency of our 2D gauge theories.

A natural guess is that ensuring spacetime supersymmetry of the massless modes of our theory requires the imposition of these constraints on the charges and fields in the TLSM. Checking this requires a more detailed discussion of the exact spectrum of our models than we have presented in this note; for now we will simply impose these conditions, as is often done in $(0, 2)$ models, because we can and because doing so matches us precisely onto the Fu-Yau construction. This question is under active investigation [4].

5.6 The Conformal Limit

So far, we have shown that our compact $(0, 2)$ TLSMs exist as non-anomalous, 2D $\mathcal{N} = 2$ quantum field theories which have Fu-Yau-type geometries as their one-loop classical moduli spaces. These are principal holomorphic T^2-bundles over Calabi-Yaus with torsionful G-structures which non-trivially satisfy the Green-Schwarz anomaly constraints. However, since Fu-Yau geometries are necessarily finite radius and generally contain small-volume cycles, the semi-classical geometric analysis is not obviously reliable. What we would like to argue is that the IR conformal fixed points to which these massive TLSMs flow should be taken to *define* the Fu-Yau CFT. For this to make sense, however, we must demonstrate that these TLSMs in fact flow to non-trivial CFTs in the IR.

This will take some work. The first step is to observe that the superpotential in a $(0,2)$ model is one-loop exact, so the vacuum is not destabilized at any order in perturbation theory; the concern is thus worldsheet instantons. It has long been understood that the perturbative moduli spaces of generic $(0,2)$ models are lifted by instanton effects [45, 46]. It has more recently been understood that $(0,2)$ GLSMs on Kähler targets with arbitrary (not necessarily linear) stable vector bundles are not lifted by instanton effects. This has been demonstrated in the class of "half-linear" models via an analysis of the analytic structure of the spacetime superpotential in a paper by Beasley & Witten [14] and, in the more limited case of GLSMs, via a generalized Konishi anomaly argument by Basu & Sethi [13]. Due to the gauge anomaly and gauge variance of the classical Lagrangian, neither of these analyses directly apply to our torsion models; however, the basic structure of the Beasley-Witten argument obtains, which suggests that the vacuum is indeed stable to worldsheet instanton corrections.

The basic ingredients in [14] were that the spacetime superpotential is a holomorphic section of a simple line bundle; that poles can appear only if the instanton moduli space has a non-compact dimension along which worldsheet correlators can diverge; that a simple residue theorem ensures that the sum over all poles is zero; and that the worldsheet theory respect a linearly realized $(0,2)$ with non-anomalous $U(1)$ R-symmetry. In the case of our TLSMs, the crucial step is verifying that the instanton moduli space is in fact compact; the rest appears to follow rather straightforwardly.

The instantons in our TLSM fall into two classes: those involving gauge fields coupled to torsion multiplets and those involving gauge fields coupled only to chiral multiplets. The latter class is identical to those studied in [14, 112] and have compact moduli spaces for the same reasons; these correspond to the homologically non-trivial lifts of holomorphic curves on the base Calabi-Yau. The former is more subtle. Recall that all that matters for the lifting of the massless vacuum are contributions to the chiral superpotential from BPS instantons. Significantly, BPS instantons in the torsion sector must satisfy an unusual BPS equation

$$\delta\psi = \partial_+\theta + M^a v_{+a} = 0 \ . \tag{5.41}$$

Since v_{+a} is singular for an instanton background, instantons aligned along M^a in G do not have finite action, so we appear to have *no* instantons along the curve associated to M^a. Actually, this makes a great deal of sense. The one-form on $K3$ associated to $M^a v_{+a}$ is α_M (see section 5.2); since α_M is not a globally-defined form, $\omega_M = d\alpha_M$—the 2-form curvature of the T^2-bundle—is non-trivial in $H^2(K3,\mathbb{Z})$. However, the connection 1-form on X, $d\theta + \pi^*\alpha_M$, *is* a globally defined 1-form on X, so $d(d\theta + \pi^*\alpha_M)$ is trivial in $H^2(X,\mathbb{Z})$. Thus, there is no 2-cycle in X associated with this gauge field.

Thus the BPS instantons of the TLSM are a refinement of the instantons of the base Calabi-Yau, and the moduli space is consequently compact. Elevating these heuristic arguments to a rigorous proof of the stability of the vacuum to instanton corrections does not appear impossible. We leave a more thorough discussion of instantons in torsion sigma models, and a formal proof of the stability of the vacuum, to future work.

5.7 Conclusions and Speculations

In this chapters, we have constructed gauged linear sigma models for non-Kähler compactifications of the heterotic string with non-trivial background NS-NS 3-form H satisfying the modified Bianchi identity, and we argued for the exact stability of their vacua to all orders and non-perturbatively in α'. This construction provides a microscopic definition of the Fu-Yau CFT and, via duality, for a related class of KST-like flux-vacua [80] involving non-trivial NS-NS and RR fluxes which stabilize various moduli in a fiducial Calabi-Yau orientifold compactification.

While motivated by the remarkable Fu-Yau construction, this construction is considerably more general, suggesting applications much richer than we have been able to cover explicitly. For example, while we have focused on $K3$ bases for simplicity, it is completely straightforward to construct more general compactifications over higher-dimensional Calabi-Yau bases, leading to 7 and 8 dimensional non-Kähler compactifications corresponding to torsionful G_2 and $Spin(7)$ structure manifolds. It is also natural to try to apply the technology of the torsion multiplet to non-CY bases—say, dP_8—by suitably adjusting the fibration structure. Perhaps the easiest cases to be studied are the type II examples in section 2; there is a rich story to be told there, including non-perturbative existence and a thorough study of the instanton structure of the theory. All of these points provide interesting directions for future research and are under active investigation [4].

One area where our construction should be of particular use is in the study of the moduli spaces—and hence low-energy phenomenology—of non-Kähler compactifications [17, 38, 19]. The necessary tools for analyzing the spectra of $(0, 2)$ GLSMs have long been know [48] and can presumably be applied with minor modifications. Relatedly, TLSMs should also provide a computationally effective tool to study the topological ring which was recently proved to exist for generic $(0,2)$-models [1, 2], as well as the action of mirror symmetry on these stringy geometries. In fact, the action of T-duality and mirror symmetry on these geometries is remarkably subtle—for example, it is easy to check that the T^2 fibre on the Fu-Yau geometry is, in fact, self-dual, corresponding to a pair of $SU(2)$ WZW models at level one. What is the relation between the self-dual circles and the NS-NS flux? Are these WZW models playing the anomaly-cancelling role of the WZW models in the $(0,2)$ Gepner model constructions of Berglund *et al* [20]? Clearly, a great deal remains to be learned form these torsion linear sigma models, and from the CFTs to which they flow.

Appendix A

Non-Technical Summary

There are probably few subjects that are harder to explain to non-specialists than theoretical physics with its formidable combination of heavy mathematics and exotic concepts like quantum entanglement and curved spaces. But it is also an important and exciting topic that should be communicated beyond narrow academic circles and indeed ultimately derives its vigor and justification from its contribution to common human knowledge. This was eloquently put in the dedication of the textbook on general relativity by Misner, Thorne, and Wheeler [96]:

> *We dedicate this book*
> *To our fellow citizens*
> *Who, for love of truth,*
> *Take from their own wants*
> *By taxes and gifts,*
> *And now and then send forth*
> *One of themselves*
> *As dedicated servant,*
> *To forward the search*
> *Into the mysteries and marvellous simplicities*
> *Of this strange and beautiful Universe,*
> *Our home.*

This appendix is an attempt to honor that ethos by presenting a non-technical overview of the material in the preceding chapters. To write an even moderately readable introduction to all of string theory would require another thesis-length text, and it has already been done very skillfully in the popular books by Michael Greene (whose name is also well represented among the technical references) [65, 66]. Instead, an impressionistic rendition of the main issues and ideas in this thesis is presented, with the hope that it will be more than a blur.

A.1 The Use of Toy Models

In a sentence, string theory is the attempt to describe all matter and all forces in the universe as arising from tiny vibrating strings that can join together and split up, influencing each other's vibrations in the process. Rather than being fundamentally different entities, the many different types of particles that physicists have found so far would be merely different vibrational patterns of these strings, like different musical notes correspond to different vibrational patterns of a violin string. This hypothesis gives us a single source of all particles in a very elegant way, but unfortunately it is hard to test it directly. The problem is that these fundamental strings are, according to our best guesses, so tiny that they are far beyond the reach of our usual experimental techniques—so far beyond, in fact, that it is unlikely we will ever be able to probe them directly in a controlled experiment. This is not to say that they are entirely beyond the reach of experiment: a number of indirect ways of probing strings are possible (e.g. by observing high-energy astrophysical phenomena) and controlled experiment may reveal features that support the stringy view of physics.

However, this state of affairs *does* mean that string theory cannot make progress via the standard approach of back-and-forth between theory and experiment, which instructs us to build a theoretical model of the phenomenon we want to describe, go out and test it experimentally, and repeat until we have found a model that works. Many branches of physics, however, have found themselves in this predicament for at least some time, and one way to make progress nonetheless is to turn to the the exploration of *toy models*. 'Toy model' here means a simplified mathematical model which is known to not be a realistic depiction of the real-world phenomenon of interest, but which may capture some of the features expected of the full theory. Since the full theory in this case is a theory of all matter and force,[1] there are many features we expect to form part of it and so string theory has given rise to a correspondingly large set of toy models, spanning the range from semi-realistic attempts at reproducing the observed particles to very theoretical models designed mainly to investigate features of the underlying mathematics.

A.2 The Research in this Thesis

The research presented in this thesis is based on the approach expounded above and uses different toy models to probe string theory in the hope that it will lead us to a better understanding of some features of the final theory. It is based on three articles and the corresponding chapters are treated in turn below.

[1]The somewhat pompous term 'Theory of Everything' if often used, but it has the disadvantage that it is likely to be interpreted very differently by different people: to a physicist, it typically means a theory that explains the observed behavior of all matter and force, but to most other people it sounds like something that claims to explain why we fall in love or enjoy a Bach sonata. Physicists are as interested in these parts of life as everybody else, but they do not tend to think of it as part of their job!

A.2.1 Two-Dimensional String Universes

One standard way of building simplified toy models is to consider models that describe the phenomenon we are interested in, but in fewer dimensions. For instance, many models of how an electric current moves through a solid are much easier to deal with if we think of the current as only moving in two space dimensions (i.e. along a plane) rather than in the usual three. This technique is also well suited for string theory, particularly because most string theories lead us to believe that the real world has no less than nine spatial dimensions, as well as a time dimension (we will later discuss how this extraordinary claim can be reconciled with our everyday experience). One way to overcome this profusion of dimensions and the attendant mathematical complications is to consider instead a drastically reduced model in which there is only time and one spatial dimension. Clearly, we are going to lose many important features of the full theory, but in return we get a model which can in fact be solved exactly. The research in chapter 2 of this thesis is work on exactly such a model.

It may seem that there is rather little room, as it were, for a string in one space dimension—all we can do is stretch it taut. Indeed, it turns out that in this scenario, strings simply cannot vibrate and with only one vibrational pattern available (namely no vibration), the strings in this theory give rise to only one type of particle known as a *tachyon*. One further detail is that the single space dimension comes equipped with a 'wall': the tachyons can move as far as they want in one direction, but if we push them too far in the other direction, they will feel a repulsive force and bounce back.

Eliminating all the complicated vibration patterns is already a great simplification, leaving us with a toy universe with only one type of particle. But it gets better yet. By a mathematical sleight of hand which is surprising even by the standards of theoretical physics, we can show that the whole theory, with its wall and bouncing tachyons, can be described in terms of waves on a liquid surface. Imagine a small island sitting in the middle of a calm ocean. If we now send a small wave towards the island, it will wash up on the shore and then roll back, creating a wave going *away* from the island. This process is what describes tachyons bouncing off the wall. In fact, by describing how such waves would behave, we can calculate how a whole bunch of tachyons approaching the wall will interact and bounce back.

In chapter 2, we use the technical incarnation of the ocean-and-island analogy—which is named the *matrix model* after part of the underlying mathematics—to describe how tachyons might behave in a universe that is finite, i.e. in which they cannot move infinitely far in any direction. It turns out that the simplest way to depict this is to first imagine the sea drained dry, so that the island now sticks up like a mountain from the sea floor. We then hurl a huge ball of water—a *droplet*—towards the island. In our ideal island paradise, this droplet will bounce right back after hitting the island shore, with the same speed it came in with. If, at the same time, the droplet is large enough, we could also have waves moving on its surface meanwhile. These waves we would interpret as tachyons, but because the droplet has a definite size, they can only move so far. In other words, our tachyon now seem to move in a finite universe that corresponds to the finite surface of the droplet!

The original work in this chapter consist in deriving the theory that governs the motion of waves on these droplets and describing the features of this theory. Perhaps

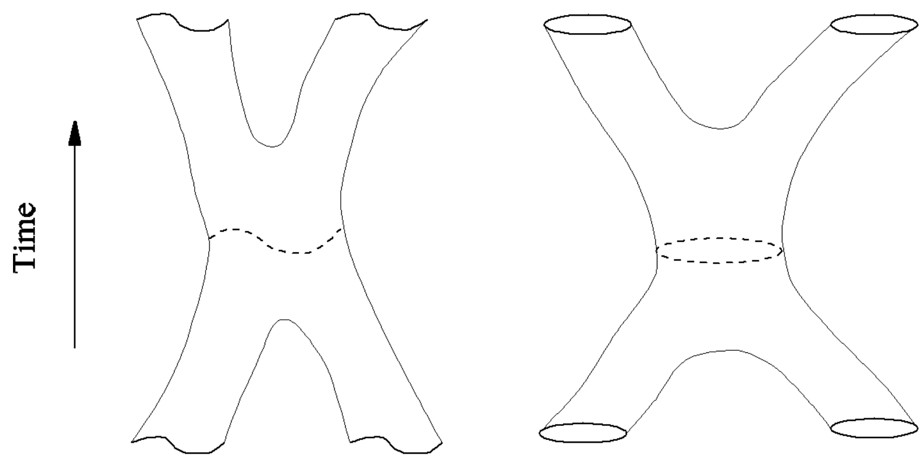

Figure A.1: Examples of worldsheets traced out by open strings (left) and closed strings (right). The diagrams here correspond to a process in which two strings collide and join into a single string which subsequently decays back into two strings. The initial state (with two separate strings) is indicated by fat outlines at the base of the figures, while the final state that the strings move into after the collision (also two separate strings) is indicated by fat outlines at the top of the figures. At any given time, we can find the arrangement of strings in space by making a horizontal slice through the figure. The dashed lines indicate one such slice, representing a point in time where the two strings have joined into one.

the most surprising discovery is that in this toy universe, *time* is also finite. That is, the universe comes into existence at some definite point and then exists for a finite period before vanishing again. However, it is not so obvious how we should interpret this behavior—in particular, we do not know what happens at the beginning and end of time. But on the other hand, it is quite extraordinary that we can even construct a model of an entire universe in such simple terms.

A.2.2 Simple Subsets of Complicated Theories

To understand the second part of this thesis, we have to introduce one further central concept from string theory, namely that of the *worldsheet*. Consider a string—which may either be an open piece or a closed loop—moving through space and time (we shall lump the two together as physicists often prefer to do, thinking of time as being just another direction). If we imagine the string had been dripped in soap water, it would trace out a sheet of soap-film as it moves, indicating its path (see fig. A.1). It is this surface—'the surface of past positions'—which we call the worldsheet.

It turns out that one can describe a string theory—also the ones in nine-plus-one space-and-time dimensions—by mathematical models that, as it were, live *on* the worldsheet. That is, they are confined to this two-dimensional surface, just like we are confined to our usual three space dimensions and time. In this case, going to two dimensions is not an approximation, but merely a change of perspective. However, similarly to the previous section, working in two dimensions has distinct advantages.

Typically, our worldsheet theory is still mathematically very complicated. But in the mid-eighties, the three string-theorist Wolfgang Lerche, Cumrun Vafa, and Nicholas Warner discovered that there is a small part of this theory that can be extracted and has a very simple structure. This part is known technically as the *chiral ring*. 'Ring' is intended in its mathematical meaning: roughly, a mathematical ring is a set of objects with the property that there exists two operations that allow us to combine two objects to yield a third one. The most common example of this is in fact very mundane: the set of all whole numbers forms a ring, with addition and multiplication providing the two ways of combining numbers. The rings that appear on the worldsheet are more complicated, but the basic idea is the same.

The reason such a nice structure could be found was that Lerche, Vafa, and Warner considered a very special class of worldsheet theories. In particular, they were *supersymmetric* theories. What this means is that the theories do not change if one performs a certain mathematical operation on them. This operation that is an abstract generalization of a rotation, more specifically one that allow us to "rotate" particles of one type (called *bosons*) into particles of another type (called *fermions*). In fact, the theories had a full four distinct rotations of this type. Though it may not be obvious straight away, having such symmetries forces the theory to take a very constrained form. As a further analogy, suppose you were given a piece of paper with a line drawn down the middle and one drawn across the page. You are now instructed to draw a picture that is symmetric around both these lines. In that case, you would effectively only have the freedom to draw whatever you wanted on one fourth of the paper. Once you had filled, say, the upper right square, the demand that the drawing be symmetric would fix what must go in the other three squares, namely the appropriate mirror image of what you just drew. Similarly, supersymmetry cuts down the mathematical freedom we have in writing down our worldsheet theory—so much so that we are forced to include the nice ring structure. The chiral ring is also more than a mathematical nicety: knowing what it is partly specifies how the particles that arise from the string theory will interact with each other.

If we eliminate half of these supersymmetry rotations, we are much less constrained, just as we would have the freedom to fill half the page rather than just a fourth if one of the lines on our paper were removed. The central new result in chapter 4 is a proof that in fact, the chiral ring persists (at least in many cases) even if we throw out half the supersymmetry. This is important because so-called *heterotic string theories* only have this reduced amount of supersymmetry. And the heterotic theories, in turn, are among the best candidates for a string theory that will capture the physics of our universe. The chiral ring itself would only fix a very modest part of this physics, but its existence gives hope that these theories can in fact be treated mathematically.

A.2.3 Flux Models

The final part of the original research is contained in chapter 5 and deals with more explicit models of the heterotic string theory. As mentioned above, string theories (including the heterotic) typically require a total of ten space and time dimensions. To overcome the apparent contradiction with everyday experience, the six extra space dimensions are usually 'hidden' by curling them up and making them so small that we do not notice them, even in sensitive experiments. 'Curling up' here means looping a dimension back on itself such that if we move along this direction, we will eventually come back to where we started. If the size of this loop is sufficiently small, we may be running round it all the time without even noticing.

It might seem that this is just a cheap trick to sweep the problem under the rug. But in fact, compactifying dimensions like this gains us a number of nice properties. The compactified theory, like all other string theories, has many different mathematical 'dials' that can be turned, and depending on how they are set, the resulting universe can end up looking quite different. In particular, there is a range of dials for the shape and size of the way the extra dimensions are curled up. Just by turning these dials we can, to a large extent, choose which particles appear in the remaining four large dimensions that would correspond to the ones we usually experience.

String theorists have been playing this game for over 20 years and have explored many settings of these dials. But there is one dial they usually just left at zero. This dial bears the label ‘torsion’ and turning it up corresponds to switching on what is known as an *H-flux*. Such a flux corresponds roughly to a constant stream of a certain kind of particles throughout space, much like sunshine may permeate the atmosphere on a sunny day.

The problem with turning this dial is that as soon as we move away from the zero-setting, the mathematics becomes even more complicated than it already was. In fact, it took a full 20 years to merely prove the existence of a single solution to the resulting equations, a proof that was published by Fu and Yau in 2006. But proving that a solution exists does not finish the job, for we also want to be able to actually calculate things. In the most direct approach, this would entail actually *solving* the equations to find the solutions we know exist, or at least extracting some particular details of it. Although this job is much easier after Fu and Yau have pointed us in the right direction, it is still a task of great mathematical intricacy.

In these cases, the natural instinct of a physicist is to look for an easy way out, something that is not a full-fledged solution but which can be a useful tool. The original result of chapter 5 is the construction of exactly such a tool which allows an explicit investigation of the Fu-Yau solution. We make use of a brilliant trick invented by Edward Witten more than a decade ago: rather than working directly with a complicated theory, we work with a much simpler theory *which resembles to the complicated one in certain respects*. More specifically, we can construct a simple theory which begins to resemble the complicated one if we ‘cool it down’ and look only in the regime where everything happens at very low energies.

The particular theories used by Witten were called *gauged linear sigma models*. The construction in chapter 5 generalizes these models to include non-zero torsion and hence we dub them *torsion linear sigma models*. This approach by no means gives us a free lunch since many details are lost in the mathematical equivalent of cooling down the theory. But it does give us some genuine new results. For instance, it allows us to find out which particles we would observe in our normal three-plus-one-dimensional world if the Fu-Yau solution was indeed the true theory of the universe (though that particular calculation is not done in this thesis). A further reward is that constructing such a model gives us confidence that the Fu-Yau solution actually does represent a proper string theory. This may sound peculiar, but what Fu and Yau actually did was to prove the existence of a solution to a set of *approximate* equations. The existence of a corresponding torsion linear sigma model, on the other hand, turns out to imply that this approximation can in fact be modified into a solution of the *exact* equations. Hopefully, this new tool will allow string theorists to push the study of theories with non-zero torsion further in other ways and thus open up a whole new class of possible universes.

Bibliography

[1] Allan Adams, Anirban Basu, and Savdeep Sethi. (0,2) duality. *Adv. Theor. Math. Phys.*, 7:865–950, 2004, hep-th/0309226.

[2] Allan Adams, Jacques Distler, and Morten Ernebjerg. Topological heterotic rings. *Adv. Theor. Math. Phys.*, 10:657, 2005, hep-th/0506263.

[3] Allan Adams, Morten Ernebjerg, and Joshua M. Lapan. Linear models for flux vacua. 2006, hep-th/0611084.

[4] Allan Adams and Joshua M. Lapan. Work in progress.

[5] Ofer Aharony, Steven S. Gubser, Juan M. Maldacena, Hirosi Ooguri, and Yaron Oz. Large N field theories, string theory and gravity. *Phys. Rept.*, 323:183–386, 2000, hep-th/9905111.

[6] Sergei Alexandrov. Backgrounds of 2d string theory from matrix model. 2003, hep-th/0303190.

[7] Sergei Alexandrov. *Matrix quantum mechanics and two-dimensional string theory in non-trivial backgrounds*. PhD thesis, Service de Physique Théorique – C.E.A.-Saclay and Université Paris XI, 2003, hep-th/0311273.

[8] Sergei Yu. Alexandrov, Vladimir A. Kazakov, and Ivan K. Kostov. Time-dependent backgrounds of 2d string theory. *Nucl. Phys.*, B640:119–144, 2002, hep-th/0205079.

[9] Jan Ambjørn, B. Durhuus, and T. Jonsson. *Quantum geometry. A statistical field theory approach*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1997.

[10] Paul S. Aspinwall. The moduli space of N=2 superconformal field theories. 1994, hep-th/9412115. Lectures given at Summer School in High Energy Physics and Cosmology, Trieste, Italy, 13 Jun - 29 Jul 1994. Published in *Trieste HEP Cosmology 1994*, 352-401.

[11] Paul S. Aspinwall and Brian R. Greene. On the geometric interpretation of N=2 superconformal theories. *Nucl. Phys.*, B437:205–230, 1995, hep-th/9409110.

[12] Paul S. Aspinwall, Brian R. Greene, and David R. Morrison. Calabi-Yau moduli space, mirror manifolds and spacetime topology change in string theory. *Nucl. Phys.*, B416:414–480, 1994, hep-th/9309097.

[13] Anirban Basu and Savdeep Sethi. World-sheet stability of (0,2) linear sigma models. *Phys. Rev.*, D68:025003, 2003, hep-th/0303066.

[14] Chris Beasley and Edward Witten. Residues and world-sheet instantons. *JHEP*, 10:065, 2003, hep-th/0304115.

[15] Katrin Becker, Melanie Becker, Ji-Xiang Fu, Li-Sheng Tseng, and Shing-Tung Yau. Anomaly cancellation and smooth non-Kaehler solutions in heterotic string theory. *Nucl. Phys.*, B751:108–128, 2006, hep-th/0604137.

[16] Katrin Becker and Keshav Dasgupta. Heterotic strings with torsion. *JHEP*, 11:006, 2002, hep-th/0209077.

[17] Katrin Becker and Li-Sheng Tseng. Heterotic flux compactifications and their moduli. *Nucl. Phys.*, B741:162–179, 2006, hep-th/0509131.

[18] Melanie Becker, Katrin Becker, and John C. Schwarz. *String Theory and M-theory: A Modern Introduction.* Cambridge University Press, Cambridge, UK, 2006.

[19] Melanie Becker, Li-Sheng Tseng, and Shing-Tung Yau. Moduli space of torsional manifolds. 2006, hep-th/0612290.

[20] Per Berglund, Clifford V. Johnson, Shamit Kachru, and Philippe Zaugg. Heterotic coset models and (0,2) string vacua. *Nucl. Phys.*, B460:252–298, 1996, hep-th/9509170.

[21] Nathan Berkovits. Super-Poincaré covariant quantization of the superstring. *JHEP*, 04:018, 2000, hep-th/0001035.

[22] Nathan Berkovits. ICTP lectures on covariant quantization of the superstring. 2002, hep-th/0209059. Prepared for ICTP Spring School on Superstrings and Related Matters, Trieste, Italy, 18-26 Mar 2002. Published in *Trieste 2002, Superstrings and related matters*, 57-107.

[23] R. Blumenhagen, R. Schimmrigk, and A. Wisskirchen. The (0,2) exactly solvable structure of chiral rings, Landau-Ginzburg theories, and Calabi-Yau manifolds. *Nucl. Phys.*, B461:460–492, 1996, hep-th/9510055.

[24] Ralph Blumenhagen, Mirjam Cvetic, Paul Langacker, and Gary Shiu. Toward realistic intersecting D-brane models. *Ann. Rev. Nucl. Part. Sci.*, 55:71–139, 2005, hep-th/0502005.

[25] Ralph Blumenhagen, Rolf Schimmrigk, and Andreas Wisskirchen. (0,2) mirror symmetry. *Nucl. Phys.*, B486:598–628, 1997, hep-th/9609167.

[26] Ralph Blumenhagen and Savdeep Sethi. On orbifolds of (0,2) models. *Nucl. Phys.*, B491:263–278, 1997, hep-th/9611172.

[27] Vincent Bouchard and Ron Donagi. An SU(5) heterotic standard model. *Phys. Lett.*, B633:783, 2006, hep-th/0512149.

[28] E. Braaten, T. Curtright, G. Ghandour, and Charles B. Thorn. Nonperturbative weak coupling analysis of the Liouville quantum-field theory. *Phys. Rev. Lett.*, 51:19, 1983.

[29] E. Braaten, T. Curtright, G. Ghandour, and Charles B. Thorn. Nonperturbative weak coupling analysis of the quantum Liouville field-theory. *Ann. Phys.*, 153:147, 1984.

[30] Eric Braaten, Thomas Curtright, and Charles B. Thorn. Quantum Backlund tranformation for the Liouville theory. *Phys. Lett.*, B118:115, 1982.

[31] Eric Braaten, Thomas Curtright, and Charles B. Thorn. An exact operator solution of the quantum Liouville field theory. *Ann. Phys.*, 147:365, 1983.

[32] Volker Braun, Yang-Hui He, Burt A. Ovrut, and Tony Pantev. The exact MSSM spectrum from string theory. *JHEP*, 05:043, 2006, hep-th/0512177.

[33] E. Brezin and V. A. Kazakov. Exactly solvable field theories of closed strings. *Phys. Lett.*, B236:144–150, 1990.

[34] Jr. Callan, Curtis G., E. J. Martinec, M. J. Perry, and D. Friedan. String in background fields. *Nucl. Phys.*, B262:593, 1985.

[35] P. Candelas, Gary T. Horowitz, Andrew Strominger, and Edward Witten. Vacuum configurations for superstrings. *Nucl. Phys.*, B258:46–74, 1985.

[36] T. P. Cheng and L. F. Li. *Gauge theory of elementary particle physics.* Oxford University Press, Oxford, UK, 1984.

[37] Thomas L. Curtright and Charles B. Thorn. Conformally invariant quantization of the Liouville theory. *Phys. Rev. Lett.*, 48:1309, 1982.

[38] Michelle Cyrier and Joshua M. Lapan. Towards the massless spectrum of non-Kaehler heterotic compactifications. 2006, hep-th/0605131.

[39] Sumit R. Das. D-branes in 2d string theory and classical limits. 2004, hep-th/0401067. Based on talks given at 3rd International Symposium on Quantum Theory and Symmetries (QTS3), Cininnati, Ohio, 10-14 Sep 2003, and at Workshop on Branes and Generalized Dynamics, Argonne, Illinois, 20-24 Oct 2003. Published in *Cincinnati 2003: Quantum theory and symmetries*, 218-233.

[40] Sumit R. Das, Joshua L. Davis, Finn Larsen, and Partha Mukhopadhyay. Particle production in matrix cosmology. *Phys. Rev.*, D70:044017, 2004, hep-th/0403275.

[41] Sumit R. Das and Antal Jevicki. String field theory and the physical interpretation of d=1 strings. *Mod. Phys. Lett.*, A5:1639–1650, 1990.

[42] Sumit R. Das and Joanna L. Karczmarek. Spacelike boundaries from the c = 1 matrix model. *Phys. Rev.*, D71:086006, 2005, hep-th/0412093.

[43] P. Di Francesco and D. Kutasov. Correlation functions in 2-d string theory. *Phys. Lett.*, B261:385–390, 1991.

[44] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal field theory*. Graduate texts in contemporary physics. Springer, Princeton, NJ, 1997.

[45] Michael Dine, N. Seiberg, X. G. Wen, and Edward Witten. Nonperturbative effects on the string world sheet. *Nucl. Phys.*, B278:769, 1986.

[46] Michael Dine, N. Seiberg, X. G. Wen, and Edward Witten. Nonperturbative effects on the string world sheet. 2. *Nucl. Phys.*, B289:319, 1987.

[47] Jacques Distler. Notes on (0,2) superconformal field theories. 1994, hep-th/9502012. Published in Trieste HEP Cosmology, 322-351.

[48] Jacques Distler and Shamit Kachru. (0,2) Landau-Ginzburg theory. *Nucl. Phys.*, B413:213, 1994, hep-th/9309110.

[49] Michael R. Douglas and Shamit Kachru. Flux compactification. 2006, hep-th/0610102. Submitted to Rev. Mod. Phys.

[50] Michael R. Douglas and Stephen H. Shenker. Strings in less than one-dimension. *Nucl. Phys.*, B335:635, 1990.

[51] Tohru Eguchi and Sung-Kil Yang. N=2 superconformal models as topological field theories. *Mod. Phys. Lett.*, A5:1693–1701, 1990.

[52] Morten Ernebjerg, Joanna L. Karczmarek, and Joshua M. Lapan. Collective field description of matrix cosmologies. *JHEP*, 09:065, 2004, hep-th/0405187.

[53] V. Fateev, Alexander B. Zamolodchikov, and Alexei B. Zamolodchikov. Boundary Liouville field theory. I: Boundary state and boundary two-point function. 2000, hep-th/0001012.

[54] Ji-Xiang Fu and Shing-Tung Yau. The theory of superstring with flux on non-Kähler manifolds and the complex Monge-Ampère equation. 2006, hep-th/0604063.

[55] Kazuo Fujikawa. Path integral measure for gauge invariant fermion theories. *Phys. Rev. Lett.*, 42:1195, 1979.

[56] Kazuo Fujikawa. Path integral for gauge theories with fermions. *Phys. Rev.*, D21:2848, 1980.

[57] S. J. Gates Jr., C. M. Hull, and M. Roček. Twisted multiplets and new supersymmetric nonlinear sigma models. *Nucl. Phys.*, B248:157, 1984.

[58] Jerome P. Gauntlett, Dario Martelli, and Daniel Waldram. Superstrings with intrinsic torsion. *Phys. Rev.*, D69:086002, 2004, hep-th/0302158.

[59] Paul H. Ginsparg and Gregory W. Moore. Lectures on 2-d gravity and 2-d string theory. 1993, hep-th/9304011. Published in *Boulder TASI 92*, 277-470.

[60] Edward Goldstein and Sergey Prokushkin. Geometric model for complex non-Kahler manifolds with SU(3) structure. *Commun. Math. Phys.*, 251:65–78, 2004, hep-th/0212307.

[61] Mariana Graña. Flux compactifications in string theory: A comprehensive review. *Phys. Rept.*, 423:91–158, 2006, hep-th/0509003.

[62] Michael B. Green, J. H. Schwarz, and Edward Witten. *Superstring Theory. Vol. 1: Introduction.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, UK, 1987.

[63] Michael B. Green, J. H. Schwarz, and Edward Witten. *Superstring theory. Vol. 2: Loop amplitudes, anomalies and phenomology.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, UK, 1987.

[64] Michael B. Green and John H. Schwarz. Anomaly cancellation in supersymmetric D=10 gauge theory and superstring theory. *Phys. Lett.*, B149:117–122, 1984.

[65] Brian R. Greene. *The elegant universe: Superstrings, hidden dimensions, and the quest of the ultimate theory.* Vintage, re-issue edition, 2000.

[66] Brian R. Greene. *The Fabric of the Cosmos: Space, Time, and the Texture of Reality.* Vintage, reprint edition, 2005.

[67] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry.* Wiley Classical Library. Wiley Interscience, New York, 1994.

[68] David J. Gross and Igor R. Klebanov. S = 1 for c = 1. *Nucl. Phys.*, B359:3–20, 1991.

[69] David J. Gross and Alexander A. Migdal. Nonperturbative two-dimensional quantum gravity. *Phys. Rev. Lett.*, 64:127, 1990.

[70] David J. Gross and Edward Witten. Superstring modifications of Einstein's equations. *Nucl. Phys.*, B277:1, 1986.

[71] Marco Gualtieri. *Generalized complex geometry.* DPhil thesis, Oxford University, Nov 2003, math.dg/0401221.

[72] Sergei Gukov, Cumrun Vafa, and Edward Witten. CFTs from Calabi-Yau four-folds. *Nucl. Phys.*, B584:69–108, 2000, hep-th/9906070.

[73] Nigel Hitchin. Generalized Calabi-Yau manifolds. *Quart. J. Math. Oxford Ser.*, 54:281–308, 2003, math.dg/0209099.

[74] Kentaro Hori et al., editors. *Mirror symmetry*, volume 1 of *Clay Mathematics Monographs.* American Mathematical Society and Clay Mathematics Institute, 2003. (ISBN 0-8218-2955-6).

[75] Kentaro Hori and Anton Kapustin. Worldsheet descriptions of wrapped NS five-branes. *JHEP*, 11:038, 2002, hep-th/0203147.

[76] Kentaro Hori and Cumrun Vafa. Mirror symmetry. 2000, hep-th/0002222.

[77] C. M. Hull and Edward Witten. Supersymmetric sigma models and the heterotic string. *Phys. Lett.*, B160:398–402, 1985.

[78] Amer Iqbal, Nikita Nekrasov, Andrei Okounkov, and Cumrun Vafa. Quantum foam and topological strings. 2003, hep-th/0312022.

[79] Shamit Kachru, Renata Kallosh, Andrei Linde, and SandipP. Trivedi. De Sitter vacua in string theory. *Phys. Rev.*, D68:046005, 2003, hep-th/0301240.

[80] Shamit Kachru, Michael B. Schulz, and Sandip Trivedi. Moduli stabilization from fluxes in a simple IIB orientifold. *JHEP*, 10:007, 2003, hep-th/0201028.

[81] Shamit Kachru and Cumrun Vafa. Exact results for N=2 compactifications of heterotic strings. *Nucl. Phys.*, B450:69–89, 1995, hep-th/9505105.

[82] Shamit Kachru and Edward Witten. Computing the complete massless spectrum of a Landau-Ginzburg orbifold. *Nucl. Phys.*, B407:637–666, 1993, hep-th/9307038.

[83] Anton Kapustin. Chiral de Rham complex and the half-twisted sigma-model. 2005, hep-th/0504074.

[84] Joanna L. Karczmarek and Andrew Strominger. Closed string tachyon condensation at c = 1. *JHEP*, 05:062, 2004, hep-th/0403169.

[85] Joanna L. Karczmarek and Andrew Strominger. Matrix cosmology. *JHEP*, 04:055, 2004, hep-th/0309138.

[86] Sheldon H. Katz and Eric Sharpe. Notes on certain (0,2) correlation functions. *Commun. Math. Phys.*, 262:611–644, 2006, hep-th/0406226.

[87] Igor R. Klebanov. String theory in two-dimensions. 1991, hep-th/9108019. Lectures given at Spring School on String Theory and Quantum Gravity, Trieste, Italy, Apr 15-23, 1991. Published in *Trieste Spring School 1991*, 30-101.

[88] Igor R. Klebanov, Juan M. Maldacena, and Nathan Seiberg. D-brane decay in two-dimensional string theory. *JHEP*, 07:045, 2003, hep-th/0305159.

[89] Wolfgang Lerche, Cumrun Vafa, and Nicholas P. Warner. Chiral rings in N=2 superconformal theories. *Nucl. Phys.*, B324:427, 1989.

[90] Ulf Lindstrom, Martin Rocek, Rikard von Unge, and Maxim Zabzine. Generalized Kaehler manifolds and off-shell supersymmetry. *Commun. Math. Phys.*, 269:833–849, 2007, hep-th/0512164.

[91] Juan M. Maldacena. The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2:231–252, 1998, hep-th/9711200.

[92] John McGreevy, Joerg Teschner, and Herman L. Verlinde. Classical and quantum D-branes in 2D string theory. *JHEP*, 01:039, 2004, hep-th/0305194.

[93] John McGreevy and Herman L. Verlinde. Strings from tachyons: The c = 1 matrix reloaded. *JHEP*, 12:054, 2003, hep-th/0304224.

[94] Jeremy Michelson. Compactifications of type IIB strings to four dimensions with non-trivial classical potential. *Nucl. Phys.*, B495:127–148, 1997, hep-th/9610151.

[95] Djordje Minic, Joseph Polchinski, and Zhu Yang. Translation invariant backgrounds in (1+1)-dimensional string theory. *Nucl. Phys.*, B369:324–350, 1992.

[96] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*. W. H. Freeman, San Francisco, 1973.

[97] David R. Morrison and M. Ronen Plesser. Towards mirror symmetry as duality for two dimensional abelian gauge theories. *Nucl. Phys. Proc. Suppl.*, 46:177–186, 1996, hep-th/9508107.

[98] David R. Morrison and M. Ronen Plesser. Summing the instantons: Quantum cohomology and mirror symmetry in toric varieties. *Nucl. Phys.*, B440:279–354, 1995, hep-th/9412236.

[99] M. Nakahara. *Geometry, Topology and Physics*. Institute of Physics Publishing, Bristol and Philadelphia, 2nd edition, 2003.

[100] Dennis Nemeschansky and Ashoke Sen. Conformal invariance of supersymmetric sigma models on Calabi-Yau manifolds. *Phys. Lett.*, B178:365, 1986.

[101] Michael E. Peskin and D. V. Schroeder. *An introduction to quantum field theory*. Graduate texts in contemporary physics. Perseus Books, 1995.

[102] J. Polchinski. *String theory. Vol. 1: An introductions to the bosnic string*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge and New York, 1998.

[103] J. Polchinski. *String theory. Vol. 2: Superstring theory and beyond*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge and New York, 1998.

[104] Joseph Polchinski. What is string theory? 1994, hep-th/9411028. Lectures given at NATO Advanced Study Institute: Les Houches Summer School, Session 62: Fluctuating Geometries in Statistical Mechanics and Field Theory, Les Houches, France, 2 Aug - 9 Sep 1994.

[105] Alexander M. Polyakov. Selftuning fields and resonant correlations in 2-d gravity. *Mod. Phys. Lett.*, A6:635–644, 1991.

[106] Ashoke Sen. Equations of motion for the heterotic string theory from the conformal invariance of the sigma model. *Phys. Rev. Lett.*, 55:1846, 1985.

[107] Ashoke Sen. (2,0) supersymmetry and space-time supersymmetry in the heterotic string theory. *Nucl. Phys.*, B278:289, 1986.

[108] Ashoke Sen. Non-BPS states and branes in string theory. 1999, hep-th/9904207. Published in *Cargese 1999: Progress in string theory and M-theory*, 187-234.

[109] E. Sharpe. Notes on certain other (0,2) correlation functions. 2006, hep-th/0605005.

[110] Eric Sharpe. Notes on correlation functions in (0,2) theories. 2005, hep-th/0502064.

[111] Eric R. Sharpe. Conjectures on (0,2) mirror symmetry. 1998, hep-th/9804066.

[112] Eva Silverstein and Edward Witten. Criteria for conformal invariance of (0,2) models. *Nucl. Phys.*, B444:161–190, 1995, hep-th/9503212.

[113] Andrew Strominger. Superstrings with torsion. *Nucl. Phys.*, B274:253, 1986.

[114] Andrew Strominger, Shing-Tung Yau, and Eric Zaslow. Mirror symmetry is T-duality. *Nucl. Phys.*, B479:243, 1996, hep-th/9606040.

[115] Gerard 't Hooft. A planar diagram theory for strong interactions. *Nucl. Phys.*, B72:461, 1974.

[116] J. Teschner. Remarks on Liouville theory with boundary. 2000, hep-th/0009138. Presented at the 4th Annual European TMR Conference on Integrability, Non-Perturbative Effects and Symmetry in Quantum Field Theory, Paris, France, 7-13 Sep 2000.

[117] David Tong. NS5-branes, T-duality and worldsheet instantons. *JHEP*, 07:013, 2002, hep-th/0204186.

[118] Cumrun Vafa. Topological mirrors and quantum rings. In Shing-Tung Yau, editor, *Mirror symmetry I*, AMS/IP studies in advanced mathematics. American Mathematical Society, Providence, RI, 1998, hep-th/9111017.

[119] Herman Verlinde and Martijn Wijnholt. Building the standard model on a D3-brane. 2005, hep-th/0508089.

[120] Steven Weinberg. *The quantum theory of fields. Vol. 2: Modern applications.* Cambridge University Press, Cambridge and New York, 1996.

[121] Edward Witten. Topological sigma models. *Commun. Math. Phys.*, 118:411, 1988.

[122] Edward Witten. On the structure of the topological phase of two-dimensional gravity. *Nucl. Phys.*, B340:281–332, 1990.

[123] Edward Witten. Phases of N=2 theories in two dimensions. *Nucl. Phys.*, B403:159–222, 1993, hep-th/9301042.

[124] Edward Witten. Two-dimensional models with (0,2) supersymmetry: Perturbative aspects. 2005, hep-th/0504078.

[125] Shing-Tung Yau. Calabi's conjecture and some new results in algebraic geometry. *Proc. Nat. Acad. Sci.*, 74:1798–1799, 1977.