Topological string theory from D-brane bound states

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Abstract

We investigate several examples of BPS bound states of branes and their associated topological field theories, providing a window on the nonperturbative behavior of the topological string.

First, we demonstrate the existence of a large N phase transition with respect to the 't Hooft coupling in q-deformed Yang-Mills theory on S^2. The Ooguri-Strominger-Vafa [62] relation of this theory to topological strings on a local Calabi-Yau [7], motivates us to investigate the phase structure of the trivial chiral block at small Kahler moduli. Second, we develop means of computing exact degeneracies of BPS black holes on certain toric Calabi-Yau manifolds. We show that the gauge theory on the D4 branes wrapping ample divisors reduces to 2D q-deformed Yang-Mills theory on necklaces of $\mathbf{P}^1$'s. At large N the D-brane partition function factorizes as a sum over squares of chiral blocks, the leading one of which is the topological closed string amplitude on the Calabi-Yau, in agreement with the conjecture of [62].

Third, we complete the analysis of the index of BPS bound states of D4, D2 and D0 branes in IIA theory compactified on toric Calabi Yau, demonstrating that they are encoded in the combinatoric counting of restricted three dimensional partitions. We obtain a geometric realization of the torus invariant configurations as a crystal associated to the bound states of 0-branes at the singular points of a single D4 brane wrapping a high degree equivariant surface that carries the total D4 charge. The crystal picture provides a direct realization of the OSV relation to the square of the topological string partition function, which in toric Calabi Yau is also equivalent to a theory of three dimensional partitions. Finally, we apply the techniques of the quiver representation of the derived category of coherent sheaves to discover a topological matrix model for the index of BPS states of D2 and D0 branes bound to a 6-brane. This enables us to examine the quantum foam [37] description of the A-model by embedding it in the full string theory.

Contents

Citations to Previously Published Work

The arguments given in Chapter 2 were obtained in collaboration with Joseph Marsano. The text has appeared previously in

> D. Jafferis and J. Marsano. "A DK phase transition in q-deformed Yang-Mills on S^2 and topological strings," arXiv:hep-th/0509004.

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The results of Chapter 3 were obtained in collaboration with Mina Aganagic and Natalia Saulina. The text has appeared previously in

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Chapter 1

Introduction and Summary

One of the most effective windows on the theory of quantum gravity is the search for the microscopic origin of the entropy of black holes that is predicted from low energy field theory and thermodynamics. The matching of the microscopic statistics of D-brane bound states with the supergravity entropy of 1/4 the area of the event horizon by Strominger and Vafa has been referred to as the most important calculation in string theory.

A particularly rich extension of this original computation is to the case of black holes in $\mathcal{N} = 2$ four dimensional compactifications of type II strings on a Calabi Yau 3-fold which preserve half of the supersymmetries. The protection of the relevant indexed entropies from perturbative and nonperturbative g_s corrections, which allows one to directly identify the states of the weakly coupled worldvolume theory living on certain D-branes with those of the semiclassical black hole at strong coupling, appears to depend crucially on supersymmetry (although see [63] for a possible generalization). The case of half BPS objects in $\mathcal{N} = 2$ supergravity in four dimensions provides the most diverse and fascinating environment in which exact calculations have been done to date.

Almost as a side effect, this has led to deep mathematical insights in algebraic geometry, enumerative geometry and topological string theory. The curvature of the branes' worldvolume, wrapping cycles in the Calabi-Yau manifold, causes their worldvolume gauge theories to become twisted [14]. This twisting of the supersymmetries renders the resulting theories topological in the internal manifold. In this work, we will explore some of the wonderful relations among topological invariants that have been discovered via their embedding into string theory.

The fact that the dilaton lies in the universal hypermultiplet, together with a theorem that vectors and hypers cannot mix in $\mathcal{N} = 2$ supersymmetric theories, implies that BPS quantities involving the vector multiplets and thus Kahler geometry in IIA will be string tree level exact, to all orders in perturbation theory and even non-perturbatively. The Kahler structure of the moduli spaces of BPS branes can receive α' corrections, but index is also protected from these corrections that would depend on the background Kahler moduli away from walls in Calabi-Yau moduli space where it can jump.

On the other hand, the vector multiplets can gravitate. Hence one expects corrections to the classical geometry of moduli space when fields in the four dimensional gravity multiplet are turned on. Those which can influence BPS quantities turn out to have a particularly elegant form, coming from F-terms in the effective action,

$$\int d^4x F_g(X^I)(R_+^2 F_+^{2g-2}),$$

where R_+ and F_+ are the self-dual components of the Ricci tensor and graviphoton field strength, and X^I represent the geometric vector multiplets. This term aries at genus g, although no powers of the string coupling enter due to the magic of string perturbation theory, and the crucial function F_g is exactly the genus g topological string amplitude. Therefore the coupling of the topological string theory is naturally identified with the background field strength, F_+. Starting with the work of [62], it has become possible compare microscopic and supergravity calculations of these subleading corrections to the classical entropy to all orders in perturbation theory.

Their remarkable conjecture, which has been verified in numerous papers (see [35] for a list of references), is that the perturbative corrections to the indexed entropy of BPS black holes, in a mixed ensemble for the electric and magnetic brane charges, are exactly twice the real part of the topological string prepotential evaluated at the attractor values of the Calabi-Yau moduli. Hence the index of BPS states, which is obviously integral, can be expressed in the large charge limit as a formal Legendre transformation of the square of a holomorphic object, namely the topological string partition function.

The topological string behaves as a wavefunction in the quantization of the classical phase space of vector moduli, as was first derived from the holomorphic anomaly equations [13] [71]. In the near horizon limit of the BPS black holes we will be discussing, it appears to be related to quantization in the radial direction. A more complete understanding of the quantum evolution in this emergent dimension in the context of AdS/CFT might provide powerful new insights into the long standing problem of emergent time in string theory. The OSV relation provides a partial window into these ideas in an elegant and computable BPS regime of string theory, giving a tantalizing hint of the way the index of microstates, in the ordinary sense, is captured by the Wigner density associated to the wavefunction of a kind of radial quantization.

One of the main themes of this thesis is the search for a non-perturbative completion of topological string theory. The A-model partition function expressed in terms of the Gromov-Witten invariants as

$$\mathcal{F}_{top} = \sum_{g\geq 0} \sum_{n_A \geq 0 \in H_2} F_{g\ n_A}\ g_{top}^{2g-2} e^{-n\cdot\ t}, \tag{1.1}$$

is a perturbative expression both in the topological string coupling and about the large volume limit. Mirror symmetry relates this theory to the B-model, which can naturally be expanded about any basepoint in the moduli space of complex structures. The dependence of the partition function on the basepoint is expressed genus by genus in terms of the holomorphic anomaly equations.

By studying the B-model near singular points in the moduli of complex structures, corresponding to the formation of a conifold in the Calabi-Yau geometry, one can relate the genus amplitudes in an appropriate scaling limit to the perturbation series of the $c = 1$ string. This is known to be an asymptotic expansion, and moreover is not Borel resummable. Hence new objects must rescue the theory if it is to have a sensible non-perturbative completion.

Because the topological string computes corrections to the F-terms of the superstring, it is natural to assume that its non-perturbative completion is related to BPS objects in II theory compactified on a Calabi-Yau. In the following chapters, we will explore several distinct but interconnected ways of embedding the topological string in theories of BPS branes. The integer invariants constructed from these bound states shed light on the beautiful integral structures which emerge from topological string theory.

This thesis will be primarily focused on the description of BPS bound states of branes in type II string theory compactified on Calabi-Yau manifolds, and their relation to topological string theory. I will investigate both the qualitative aspects of these systems, including their phase structure on certain local Calabi-Yau manifolds, as well as their beautiful mathematics of modular forms, derived category of coherent sheaves, crystal partition functions, and asymptotic factorization. Much of this work has centered on various techniques of analyzing topological gauge theories in different dimensions, ranging from the physical methods utilizing the structure of the Higgs branch of the mass deformed theory, mathematical methods of counting equivariant sheaves, the quiver description of the derived category, and a geometric realization as branes wrapping toric nonreduced subschemes.

The worldvolume theory of D4 branes with fluxes carrying D2 and D0 charge in IIA theory compactified on a Calabi-Yau, X, has a whole web of dual descriptions that can naturally be thought of as expansions about particular regimes of the moduli space of the branes. The full quantum theory should explore the entire classical BPS moduli space, and the exact relations between the web of theories encoding these dynamics has yet to be completely elucidated. When the 4-branes are wrap very ample cycles with sufficient deformations in the Calabi-Yau, the most natural description at generic points in their moduli space is the MSW CFT obtained by lifting to M-theory. This theory encodes the effective dynamics of M5 branes wrapping a surface in X, and lives on the remaining T^2 of the M5 worldvolume. However, the exact theory is not yet known in any example.

The opposite limit is the branch of moduli space when the D4 branes become coincident, and are naturally thought of in terms of the $U(N)$ Vafa-Witten theory. Calculations here are much more doable, and in principle reproduce the complete answer when the compactification of the adjoint scalars is properly taken into account. In addition, branes wrapping rigid cycle can also be analyzed, although this has proved calculationally difficult in practise, often requiring one to resort to mathematical classifications of bundles, such as the work of [43] on equivariant $\mathbb{P}^2$. Moreover, because of the topological nature of the supersymmetric partition function, much of the index can be thought of as arising from the singularities in the classical D4 BPS moduli space, which is exactly when the enhancement to $U(N)$ occurs. Intermediate between these two limits is the crystal picture explained in the fourth chapter, where the limiting singular geometry of the wrapped surface is examined directly.

Tuning the moduli of the Calabi-Yau to the Gepner point mixes branes of various dimension, and leads to their description in terms of a quiver theory. The derived category of coherent sheaves, which is the mathematical object given by the moduli space of BPS bound states of D-branes in IIA theory, has a simplified representation in terms of fractional branes at this special point in the Kahler moduli space. We extrapolate the matrix models found here to the opposite, large volume, limit in the fifth chapter, finding a striking agreement with the worldvolume gauge theories typically used there. Moreover, these quiver theories can also be interpreted as the collective coordinates of the moduli space of instantons in the worldvolume theory of Donaldson-Thomas for 6-branes, and Vafa-Witten for 4-branes.

The D4 brane $U(N)$ gauge theory on a very ample divisor reduces to a theory on the canonical curve [70] [44] that is further highly constrained by modular invariance. In special cases when the Calabi-Yau has toric symmetries (including the S^1 symmetry of the fiber of local Calabi-Yau) there exist a significantly different two dimensional reduction to q-deformed Yang-Mills theory. The relation between these two ideas is not fully understand, and which description is more useful depends on the question. The canonical curve approach is the natural way to understand the modular invariance of the four dimensional Vafa-Witten theory , while q-deformed Yang-Mills provides a window into the large charge OSV factorization.

One amazing feature of this plethora of descriptions of the BPS sector of D-branes on a Calabi-Yau is that they all depend on the background Kahler moduli, rather than the attractor values. This is perhaps less surprising because of the topological nature of the index of BPS states, which is therefore locally independent of these asymptotic moduli. But incredibly these microscopic descriptions, which seem to naively live only inside the black hole horizon, are able to encode all the information about different attractor tree flows and multi-centered solutions.

This is particularly apparent in the fifth chapter, as we find that the topological quiver matrix model knows quite detailed information about the pattern of attractor flow. Perhaps most amusingly, the correct partition function can often be computed using the worldvolume theory of some branes that exist at the asymptotic values of the attractor moduli, but which have already decayed once we reach at the horizon! This is analogous to the fact that large pieces of the full entropy can be calculated even using the description valid in some small corner of the moduli space of the branes, as we see in many examples in the third and fourth chapters.

One of my primary motivations in this research has been to confirm, explore, and refine the OSV relation of the index of BPS bound states arising in four dimensional black holes in a mixed ensemble, to the square of the topological string wavefunction at the attractor values of the moduli. Of particular interest is the subtle interplay between the OSV relation and the AdS/CFT correspondence, which could shed further light on both. If the topological quiver models introduced in the fifth chapter become amenable to large N matrix models techniques, then there may be a sense in which they provide a zero dimensional "CFT" dual to the topological string, which is associated to the semi-classical Calabi-Yau geometry expected to emerge from the appropriate limit of the matrix model.

In the second chapter, I will investigate the instanton-induced large N phase transition with respect to the 't Hooft coupling in q-deformed Yang-Mills theory on a sphere. This theory is equivalent to the four dimensional topologically twisted $\mathcal{N} = 4$ Yang-Mills theory that describes the BPS bound states of D4 branes wrapping the divisor $\mathcal{O}(-p) \to \mathbb{P}^1$ in the local Calabi-Yau geometry $\mathcal{O}(-p) \oplus \mathcal{O}(p-2) \to \mathbb{P}^1$, with chemical potentials for D0 and D2 branes. Hence the third order phase transition we discover is relevant to the OSV conjecture, at least in its application to noncompact geometries. We study the phase diagram in terms of the 't Hooft coupling and instanton potential (θ angle), finding an intriguing interplay between the dominance of instanton contributions and the formation of a clumped saddle point distribution. Furthermore, this leads one to examine the related phase structure of the topological string partition function.

The q-deformed Yang Mills in two dimensions, which is a topological theory with no local degrees of freedom, can be obtained from the ordinary Yang Mills by compactifying the scalar dual to the field strength in two dimensions. This modification naturally arises in the reduction of the D4 worldvolume theory along the fiber direction, where this scalar field comes from a holonomy of the four dimensional gauge field. The precise choice of boundary conditions at infinity in the wrapped divisor must be those which correspond to physically having no D2 branes wrapping the fibers.

The partition function of this theory on a genus g Riemann surface can be solved exactly by doing the Gaussian integrals in the path integral, or by cutting and gluing rules, with the result

$$Z = \sum_{\mathcal{R}-U(N)} (\dim_q \mathcal{R})^{2-2g} \, q^{pC_2(\mathcal{R})/2} e^{i\theta C_1(\mathcal{R})}. \tag{1.2}$$

Expressing this in terms of the weights of the representation $\mathcal{R}$ in $\mathbb{Z}^N$, it looks like a matrix model with an attractive quadratic potential, competing with a repulsive force between the eigenvalues for genus 0. As in the pure Yang Mills theory studied by Gross-Taylor, Douglas-Kazakov, the strong coupling phase features a chiral factorization into a pair of Fermi seas on opposite sides of a clump with the maximal density imposed by integrality.

From the point of view of the D-brane theory, this corresponds to the regime where the attractor value of the size of the $\mathbb{P}^1$ is sufficiently large, and the OSV conjecture holds. The appearance of a sum over chiral blocks in the factorization into topological string and anti-topological string results from working from not working in the mixed ensemble for the noncompact D2 charges, but rather setting them to zero [6]. As the 't Hooft coupling, $\lambda = g_{YM}^2 N$, is reduced, a phase transition occurs, and there is no factorization when the attractor size becomes too small. This can be interpreted as a regime in which multi-centered black hole solutions are not suppressed.

The phase transition can be further illuminated by an analysis of the instanton which can contribute to the q-deformed Yang-Mills theory. The weak coupling phase is characterized by a single instanton sector dominating the path integral. This method enables one to investigate the structure of the phase diagram for nonzero θ, where the ordinary real saddle point techniques fail.

By now, there have been numerous checks and demonstrations of the OSV relation, which have shed further light on its meaning, and uncovered new subtleties. In the third chapter, we confirm the large charge chiral factorization of the indexed entropy into the square of the topological string wavefunction on various toric Calabi-Yau manifolds with a single compact 4-cycle. In addition we make contact between the methods used in [69] and [7] and the established calculations of [58] for topological gauge theory on resolutions of A_n singularities. We extend and develop the technology of solving Vafa-Witten theory in terms of q-deformed Yang-Mills to a much wider class of examples, and elucidate the role of very ampleness in the OSV relation from the perspective of the topological gauge theory on the D4 brane worldvolume.

The worldvolume theory of D4 branes wrapping ample divisors, with possible intersections in the fiber, is shown to reduce to a series of coupled q-deformed Yang-Mills on a necklace of $\mathbb{P}^1$'s. The explicit examples considered are local $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and A_k type ALE space times $\mathbb{C}$. We found that in general the structure of chiral blocks can be quite complicated, indicative of the structure of relevant noncompact moduli. Never the less, in all cases we were able to extract the leading term, and match it to the squared topological string amplitude on the Calabi-Yau at the attractor values of the moduli.

When D4 branes intersect, there will be bifundamental matter living on the intersection curve. If both wrap divisors of the form $\mathcal{O}(-p) \to \Sigma_g$, and intersect in the fiber over an intersection point of the base Riemann surfaces in the Calabi-Yau, then we show how this matter can be integrated out. The resulting q-deformed Yang-Mills theories on the base curve are thus coupled at the intersection point by the operator insertions we found. Thus we find that these two dimensional avatars of the intersecting Vafa-Witten theories are a useful tool in a wide range of geometries.

The modular properties of the BPS indexed partition functions were examined, and were found to be consistent with the s-duality of the four dimensional topological gauge theories. The exact results were often not very transparent, due to the mixing of different boundary conditions in the noncompact divisors under the electro-magnetic duality.

I found a beautiful connection between the statical mechanics of melting crystals in a truncated room and the torus invariant bound states of D4/D2/D0 branes in a large class of toric Calabi-Yau geometries, which is described in the fourth chapter. This involves the careful study of ideal sheaves on a nonreduced subscheme that makes its appearance in the "nilpotent Higgs branch" of the D4 brane moduli space, and the use of a transfer matrix approach to counting three dimensional partitions. In addition to providing an exciting new geometric realization of certain instanton configurations, this enables us to find an exceptionally elegant realization of the OSV factorization in terms of the asymptotically related topological string crystals.

The chiral limit of the 4-brane theory is obtained at large N by forgetting about the truncation, which immediately results in the correct crystal description of the topological string amplitude at the attractor value of the moduli. The anti-chiral block can be found by utilizing the *a priori* surprising invariance of the crystal partition function under conjugation of the representations along the toric legs. Subleading chiral blocks appear because of nontrivial configurations connecting the chiral and anti-chiral regions with order N boxes, which survive in the 't Hooft limit.

In this way, the curious relationship between three dimensional partitions of height less than N and unknot invariants of $U(N)$ Chern-Simons theory found in [61] is explained. The link invariants are ubiquitous, and the crystal is secretly a calculation involving 4-branes, rather than Donaldson-Thomas theory. Further connections between these points of view remain to be explored.

These crystals are a totally new method of counting the Euler character of the moduli space of equivariant instantons of the $U(N)$ Vafa-Witten theory. The three dimensional partitions we find literally correspond to the T^3 invariant BPS bound states, and are able to automatically encode the effect of topological bifundamental matter living on the intersections of D4 branes. One should think of the crystal theory as intermediate between the $U(N)$ Vafa-Witten theory which is a good description of the corner of moduli space where the D4 branes are coincident, and the MSW conformal field theory [49], which is suited to the generic branch of the moduli space when the D4 charge is carried by a smooth high degree surface in the Calabi-Yau.

It is natural to conjecture that even for non-toric Calabi-Yau, X, the partition function of twisted $\mathcal{N} = 4$ $U(N)$ Yang-Mills on a surface, $\mathcal{S}$, is equivalent to the generating function of the Euler characters of the moduli spaces of ideal sheaves on a "thickened" subscheme, $\hat{\mathcal{S}} \subset X$. This is the carefully defined version of the $U(1)$ theory living on the nonreduced subscheme which is the singular coincident limit of a smooth deformation of the N branes wrapping $\mathcal{S}$.

This chapter ends our foray among the BPS bound states of 4-branes in the OSV limit. It completes the work of chapter 3, [69], and [7] in developing techniques to compute these indexes for very amply wrapped 4-branes in toric Calabi-Yau by reduction to the two dimension skeleton of torus invariant $\mathbb{P}^1$'s. In particular, the general vertex for the triple intersection of D4 branes in a toric geometry is determined in terms of a combinatoric "crystal" partition function.

A topological matrix model whose partition function counts the index of BPS bound states of D6/D2/D0 branes in certain local geometries is constructed in the fifth chapter. This quiver matrix model has a cohomologically trivial action, and provides further insights into an ADHM-like description of instanton moduli spaces. One interesting corollary of this approach is the determination of the partition function of $U(M)$ Donaldson-Thomas theory in the vertex geometry. One main lesson is that the existence of a hypothetical superconformal quantum mechanics dual to a BPS black hole made from branes in a Calabi-Yau compactification means that the index can be computed in terms of its topological twisted analog, which is often easier to discover. We would like to regard these topological matrix models as a kind of holographic dual of the topological string. This makes future attempts to use large N matrix model techniques to analyze these partitions functions very attractive.

The quiver description of the twisted $\mathcal{N} = 2$ $U(1)$ gauge theory in six dimensions is a first step to understanding what the quantum foam picture of the topological A-model as a theory of fluctuating Kahler geometry means for the full type II string theory. We see that the blow up geometries relevant for quantum foam are exactly the physical geometry experienced by BPS 0-brane probes, deformed by the presence of the fixed D6 and D2 branes. In the local limit we focus on, the 2-brane fields are heavy, and their fluctuations about the frozen values can be integrated out trivially at 1-loop because of the topological nature of the matrix models.

In general, there are many smooth resolutions of the blown up geometry that are related by flop transitions. This is reflected in the effective quiver matrix model for the dynamical 0-brane degrees of freedom as different values for the FI parameters and the frozen fields. The theory is defined at the asymptotic values of the Kahler moduli, which we always take to have a large B-field so that the supersymmetric 6-brane bound states exist. The attractor tree pattern changes as one dials these background moduli, even away from walls of marginal stability [21]. This leads to the phenomenon that we find, in which the index is independent of the chosen blow up.

Chapter 2

Phase Transitions in q-Deformed Yang-Mills and Topological Strings

2.1 Introduction

One of the most exciting developments in the past few years has been the conjecture of Ooguri, Strominger, and Vafa [62] relating a suitably defined partition function of supersymmetric black holes to computations in topological strings. In particular, they argue that the perturbative corrections to the entropy of four dimensional BPS black holes, in a mixed ensemble, of type II theory compactified on a Calabi-Yau X are captured to all orders by the topological string partition function on X via a relation that takes the form

$$Z_{BH} \sim |Z_{top}|^2 \tag{2.1}$$

Moreover, since the left-hand side of (2.1) is nonperturbatively well-defined, it can also be viewed as providing a definition of the nonperturbative completeion of $|Z_{top}|^2$.

One of the first examples of this phenomenon is the case studied in [69, 7] of type II "compactified" on the noncompact Calabi-Yau $\mathcal{O}(-p) \oplus \mathcal{O}(p-2+2g) \to \Sigma_g$, with Σ_g a Riemann surface of genus g. The black holes considered there are formed from N D4 branes wrapping the 4-cycle $\mathcal{O}(-p) \to \Sigma_g$, with chemical potentials for D0 and D2 branes turned on. The mixed entropy of this system is given by the partition function of topologically twisted $\mathcal{N}=4$ $U(N)$ Yang Mills theory on the 4-cycle, which was shown to reduce to q-deformed $U(N)$ Yang Mills on Σ_g[1]. This theory is defined by the action

$$S = \frac{1}{g_{YM}^2}\int_{\Sigma_g} \mathrm{tr}\, \Phi \wedge F + \frac{\theta}{g_{YM}^2}\int_{\Sigma_g} \mathrm{tr}\, \Phi \wedge K - \frac{p}{2g_{YM}^2}\int_{\Sigma_g} \mathrm{tr}\, \Phi^2 \wedge K, \qquad (2.2)$$

where K is the Kahler form on Σ_g normalized so that Σ_g has area 1, Φ is defined to be periodic with period 2π, N denotes the large, fixed $D4$-brane charge, and the parameters g_{YM}^2, θ are related to chemical potentials for $D0$-brane and $D2-$brane charges by

$$\phi^{D0} = \frac{4\pi^2}{g_{YM}^2} \qquad \phi^{D2} = \frac{2\pi\theta}{g_{YM}^2}. \qquad (2.3)$$

The authors of [7] went on to verify that the partition function of this theory indeed admits a factorization of a form similar to (2.1)

$$Z = \sum_{\ell\in\mathbb{Z}}\sum_{P,P'} Z^{qYM,+}_{P,P'}(t+pg_s\ell) Z^{qYM,-}_{P,P'}(\bar{t}-pg_s\ell) \qquad (2.4)$$

[1]Before this connection was found, q-deformed Yang-Mills had previously been introduced, with different motivations, in [41].

where

$$
\begin{aligned}
Z^{qYM,+}_{P,P'}(t) &= q^{(\kappa_P+\kappa_{P'})/2} e^{-\frac{t(|P|+|P'|)}{p-2}} \\
&\qquad \times \sum_R q^{\frac{p-2}{2}\kappa_R} e^{-t|R|} W_{PR}(q) W_{P'^t R}(q) \qquad (2.5) \\
Z^{qYM,-}_{P,P'}(\bar{t}) &= (-1)^{|P|+|P'|} Z^{qYM,+}_{P^t,P'^t}(\bar{t})
\end{aligned}
$$

and

$$
g_s = g^2_{YM} \qquad t = \frac{p-2}{2} g^2_{YM} N - i\theta \qquad (2.6)
$$

where P, P', and R are $SU(\infty)$ Young tableaux that are summed over. The extra sums that seem to distinguish (4.11) from the conjectured form have been argued to be associated with noncompact moduli and are presumably absent for compact Calabi-Yau. The chiral blocks $Z^{qYM,+}_{P,P'}$ can be identified as perturbative topological string ampltiudes with 2 "ghost" branes inserted [7].

This factorization is analagous to a phenomenon that occurs in pure 2-dimensional $U(N)$ Yang-Mills, whose partition function can also be written at large N as a product of "chiral blocks" [30] [33] [32] that are coupled together by interactions analagous to the ghost brane insertions in (4.11). Indeed, because of the apparent similarity between the pure and q-deformed theories, it seems reasonable to draw upon the vast extent of knowledge about the former in order to gain insight into the behavior of the latter[2].

One particularly striking phenomenon of pure 2-dimensional Yang-Mills theory is a third order large N phase transition for $\Sigma_g = S^2$ that was first studied by Douglas

[2]A nice review of 2D Yang-Mills can be found, for instance in [16]

and Kazakov [27]. The general nature of this transition is easy to understand from a glance at the exact solution

$$Z_{YM,S^2} = \sum_{n_i \neq n_j} \left[\prod_{i<j} (n_i - n_j)^2 \right] \exp\left\{ -\frac{g_{YM}^2}{2} \sum_i n_i^2 \right\} \tag{2.7}$$

In particular, we see that the system is equivalent to a "discretized" Gaussian Hermitian matrix model with the n_i playing the role of "eigenvalues". Since the effective action for the "eigenvalues" is of order N^2, the partition function is well-approximated at large N by a minimal action configuration whose form is determined by a competition between the attractive quadratic potential and the repulsive Vandermonde term. This configuration is well-known to be the Wigner semi-circle distribution and accurately captures the physics of (2.7) at sufficiently small 't Hooft coupling. As the 't Hooft coupling is increased, however, the attractive term becomes stronger and the dominant distribution begins to cluster near zero with eigenvalue separation approaching the minimal one imposed by the "discrete" nature of the model. When these separations indeed become minimal, the system undergoes a transition to a phase in which the dominant configuration contains a fraction of eigenvalues clustered near zero at minimal separation. Sketches of the weak and strong coupling eigenvalue densities may be found in figure 2.1.

As mentioned before, this theory admits a factorization of the form $Z_{YM} = |Z_+|^2$ at infinite N and, moreover, the partition function can be reliably computed in the strong coupling phase as a perturbative expansion in $1/N$ [30] [33] [32]. In fact, roughly speaking the chiral blocks correspond to summing over configurations of eigenvalues to the right or left of the minimally spaced eigenvalues near zero. At the phase transition point, though, it is known that the perturbative expansion breaks down [67] so that the perturbative chiral blocks cease to capture the physics of the full theory, even when perturbative couplings between them are included.

A natural question to ask is whether the chiral factorization (2.1) exhibits similar behavior in any known examples. The goal of the present work is to lay the groundwork for studying this by first addressing the following fundamental questions. First, does a phase transition analagous to that of Douglas and Kazakov occur in the q-deformed theory on S^2? If so, is there a natural interpretation for the physics that drives it? Is there a correspondingly interesting phase structure of the perturbative chiral blocks that allows one to catch a glimpse of this physics? Can we extend our study of the phase structure to nonzero values of the θ angle?

We will find that the answer to the first question is affirmative for $p > 2$. Moreover, we will demonstrate that, as in the case of pure Yang-Mills on S^2 [31], the transition is triggered, from the weak coupling point of view, by instantons[3]. Using this observation, we will study the theory at nonzero θ angle and begin to uncover a potentially interesting phase structure there as well. To our knowledge, this has not yet been done even for pure Yang-Mills so the analysis we present for this case is also new. We then turn our attention to the trivial perturbative chiral block and find a phase transition at a value of the coupling which differs from the critical point of the full theory. In particular, for the case $p = 3$, which we analyze in greatest detail, the perturbative chiral block seems to pass through the transition point as the coupling is decreased without anything special occurring.

The skeptical reader may wonder whether a detailed analysis is required to demon-

[3]The contribution of instanton sectors in pure Yang-Mills was first written in [54]

strate the existence of a DK type phase transition in q-deformed Yang-Mills as it may seem that any sensible "deformation" of pure Yang-Mills will retain it. However, we point out that the q-deformation is not an easy one to "turn off". In fact, from the exact expression for the partition function of the q-deformed theory

$$Z_{BH} = Z_{qYM,g} = \sum_{n_i \neq n_j} \prod_{i<j} \left(e^{-g_{YM}^2 (n_i - n_j)/2} - e^{-g_{YM}^2 (n_j - n_i)/2} \right)^{\chi(\Sigma)} \exp\left[-\frac{g_{YM}^2 p}{2} \sum n_i^2 - i\theta \sum n_i \right] \quad (2.8)$$

we see that, in order to make a connection with the 't Hooft limit of pure Yang-Mills it is necessary to take the limit $g_{YM} \to 0$, $N \to \infty$, and $p \to \infty$, while holding $g_{YM}^2 N p$, which becomes identified with the 't Hooft coupling of the pure theory, fixed. Because we are interested in finite values of p, we need to move quite far from this limit and whether the transition extends to this regime is indeed a nontrivial question. However, a glance at (2.8) gives hope for optimism as again the effective action for the n_i also exhibits competition between an attractive quadratic potential and repulsive term. We will later see that this optimism is indeed well-founded.

The outline of this paper is as follows. In section 2, we review the analysis of the Douglas-Kazakov phase transition in pure 2-dimensional Yang-Mills theory with $\theta = 0$ that was heuristically described above and initiate a program for studying the phase structure at nonzero θ. The primary purpose of this section is to establish the methods that will be used to analyze the q-deformed theory in a simpler and well-understood context though, as mentioned before, to the best of our knowledge the extension to nonzero θ is novel. In section 3, we study the q-deformed theory and demonstrate that it indeed undergoes a phase transition for $p > 2$ at $\theta = 0$ and

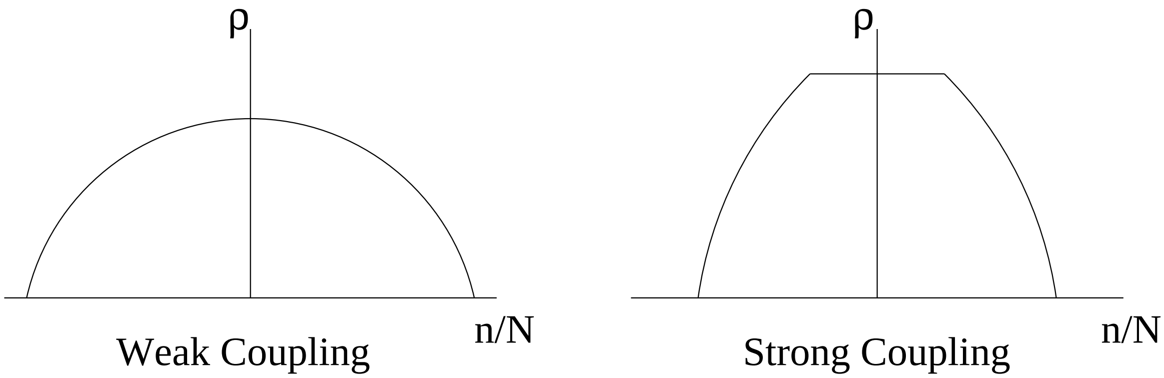

Figure 2.1: Sketch of the dominant distribution of n_i in the weak and strong coupling phases of 2-dimensional Yang-Mills on S^2

make steps toward understanding the phase structure at nonzero θ. In section 4, we turn our attention to the chiral factorization of the q-deformed theory as in (2.1) and study the phase structure of the trivial chiral block, namely the topological string partition function on $\mathcal{O}(-p) \oplus \mathcal{O}(p-2) \to \mathbb{P}^1$. We conjecture that, for $p > 2$, this quantity itself undergoes a phase transition but at a coupling smaller than the critical coupling of the full q-deformed theory. Finally, in section 5, we make some concluding remarks.

While this work was in progress, we learned that this subject was also under investigation by the authors of [8], whose work overlaps with ours. After the first preprint of this work was posted, a third paper studying similar issues appeared as well [15].

2.2 The Douglas-Kazakov Phase Transition in Pure 2-D Yang-Mills theory

In this section, we review several aspects concerning two-dimensional $U(N)$ Yang-Mills theory and the Douglas-Kazakov phase transition with an eye toward an analysis of the q-deformed theory in the next section. In addition to reviewing results for $\theta = 0$, we perform a preliminary analysis of the phase structure for nonzero θ using an expression for the partition function as a sum over instanton contributions.

2.2.1 Review of the Exact Solution

We begin by reviewing the exact solution of 2-dimensional Yang-Mills. The purpose for this review is to obtain an expression for the partition function as a sum over instanton sectors that will be useful in our later analysis. There are many equivalent ways of formulating the theory. For us, it will be convenient to start from the action

$$S = \frac{1}{g_{YM}^2}\int_{S^2} tr\Phi \wedge F + \frac{\theta}{g_{YM}^2}\int_{S^2} \mathrm{tr}\,\Phi \wedge K - \frac{1}{2g_{YM}^2}\int_{S^2} \mathrm{tr}\,\Phi^2 \wedge K \tag{2.9}$$

where Φ is a noncompact variable that can be integrated out to obtain the standard action of pure 2-dimensional Yang-Mills theory. To obtain the exact expression (2.7), we first use gauge freedom to diagonalize the matrix Φ, which introduces a Fadeev-Popov determinant over a complex scalar's worth of modes

$$\Delta_{FP} = \det([\phi, *]) \tag{2.10}$$

and reduces the action to

$$S_{\text{Diag }\Phi} = \frac{1}{g_{YM}^2}\int_\Sigma \phi_\alpha\left[dA_{\alpha\alpha} - iA_{\alpha\beta}\wedge A_{\beta\alpha}\right] + \frac{\theta}{g_{YM}^2}\int_\Sigma \phi_\alpha - \frac{1}{2g_{YM}^2}\int_\Sigma \phi_\alpha^2 \tag{2.11}$$

Integrating out the off-diagonal components of A in (2.11) yields an additional determinant over a Hermitian 1-form's worth of modes

$$\det\ ;_{1F}^{-1/2}([\phi,*]) \tag{2.12}$$

which nearly cancels the determinant (2.10). Combining (2.10) and (5.37), we are left with only the zero mode contributions

$$\frac{\Delta(\phi)^{2b_0}}{\Delta(\phi)^{b_1}} = \Delta(\phi)^2 \tag{2.13}$$

where $\Delta(\phi)$ is the usual Vandermonde determinant

$$\Delta(\phi) = \prod_{1\le i<j\le N}(\phi_i - \phi_j) \tag{2.14}$$

As a result, we obtain an Abelian theory in which each F_α is a separate $U(1)$ field strength [7]

$$Z = \int \Delta(\phi)^2 \exp\left\{\int_{S^2}\sum_i\left[\frac{1}{2g_{YM}^2}\phi_i^2 - \frac{\theta}{g_{YM}^2}\phi_i - \frac{1}{g_{YM}^2}F_i\phi_i\right]\right\} \tag{2.15}$$

To proceed beyond (2.15), let us focus on integration over the gauge field. The field strength F_i can be locally written as dA_i but this cannot be done globally unless F_i has trivial first Chern class. We thus organize the gauge part of the integral into a sum over Chern classes and integrations over connections of trivial gauge bundles.

In particular, we write $F_i = 2\pi r_i K + F_i'$ where $r_i \in \mathbb{Z}$ and F_i' can be written as dA_i for some A_i. In this manner, the third term of (2.15) becomes

$$-\frac{1}{g_{YM}^2}\int_\Sigma (2\pi r_i K\phi_\alpha + \phi_i dA_i) \tag{2.16}$$

Integrating by parts and performing the A_i integral yields a δ function that restricts the ϕ_i to constant modes on S^2. As a result, we obtain the following expression for the pure Yang-Mills partition function organized as a sum over sectors with non-trivial field strength

$$Z = \sum_{\vec{r}} Z_{\vec{r}} \tag{2.17}$$

where

$$Z_{\vec{r}} = \int \prod_{i=1}^{N} d\phi_i \, \Delta(\phi_i)^2 \exp\left\{-g_{YM}^2\phi_i^2 - i\phi_i\left(\theta + 2\pi r_i\right)\right\} \tag{2.18}$$

where we have Wick rotated (for convergence) and rescaled (for convenience) the ϕ_i. The sum over r_i can now be done to yield a δ-function that sets $\phi_i = n_i$ for integer n_i, leaving us with the known result for the exact 2D Yang-Mills partition function

$$Z = \sum_{n_i \neq n_j} \Delta(n_i)^{\chi(\Sigma)} \exp\left\{-\frac{g_{YM}^2 p}{2}\sum_i n_i^2 - i\theta\sum_i n_i\right\} \tag{2.19}$$

As noted in the introduction, this has precisely the form of a "discretized" Hermitian matrix model, with the ϕ_i playing the role of the eigenvalues. In particular, we note that the only effect of summing over r_i in (2.18) is to discretize the eigenvalues of this matrix model. Because it is precisely this discreteness that will eventually give rise to the phase transition, it is natural to expect that the transition can also be thought of as being triggered by instantons and the critical point determined by studying when their contributions become nonnegligible. We will come back to this point later.

Finally we note that the θ angle has a natural interpretation as a chemical potential for total instanton number. This is easily seen by shifting ϕ_i by $i\theta/g_{YM}^2$ in the contribution from a single instanton sector, (2.18), to obtain

$$Z_{\vec{r}} = \int \prod_{i=1}^{N} d\phi_i \, \Delta(\phi_i)^2 \exp\left\{ -\frac{g_{YM}^2}{2} \sum_j \phi_j^2 + 2\pi i \sum_j r_j \phi_j + \frac{2\pi\theta}{g_{YM}^2} \sum_j r_j - \frac{N\theta^2}{2g_{YM}^2} \right\} \tag{2.20}$$

2.2.2 The Phase Transition at $\theta = 0$

In this section, we review the derivation of the transition point in the theory with $\theta = 0$ using an analysis that we will eventually generalize to study the q-deformed theory. We can write the exact result (2.19) as

$$Z = \sum_{n_i \neq n_j} \exp\left\{ -N^2 S_{eff}(n_i) \right\} \tag{2.21}$$

where

$$S_{eff}(n_i) = -\frac{1}{2N^2} \sum_{i \neq j} \ln(n_i - n_j)^2 + \frac{\lambda}{2N} \sum_i \left(\frac{n_i}{N} \right)^2 \tag{2.22}$$

At large N, the sums can be computed in the saddle point approximation. Extremizing $S_{eff}(n_i)$ yields

$$\frac{1}{N}\sum_{j;j\neq i}\frac{N}{\phi_i-\phi_j}-\frac{\lambda}{2N}\phi_i=0 \tag{2.23}$$

Following standard methods, we define the density function ρ

$$\rho(u)=\frac{1}{N}\sum_i\delta\left(u-\frac{\phi_i}{N}\right) \tag{2.24}$$

and approximate $\rho(u)$ by a continuous function at large N for ϕ_i distributed in a region centered on zero. In terms of ρ, (2.23) becomes

$$P\int dx\,\frac{\rho(x)}{x-u}=-\frac{\lambda}{2}u \tag{2.25}$$

while the fixed number of eigenvalues leads to the normalization condition

$$\int du\,\rho(u)=1 \tag{2.26}$$

To solve the system (2.25),(2.26), it is sufficient to determine the resolvent

$$v(u)=\int_a^b dx\,\frac{\rho(x)}{x-u} \tag{2.27}$$

as $\rho(u)$ can be computed from $v(u)$ by

$$\rho(u)=-\lim_{\epsilon\to 0}\frac{1}{2\pi i}\left(v(u+i\epsilon)-v(u-i\epsilon)\right) \tag{2.28}$$

The resolvent for the system (2.25),(2.26) is of course well-known (see for instance [51])

$$v(u)=-\frac{\lambda}{2}\left(u-\sqrt{u^2-\frac{4}{\lambda}}\right) \tag{2.29}$$

and leads to the familiar Wigner semi-circle distribution

$$\rho(u) = \frac{\lambda}{2\pi}\sqrt{\frac{4}{\lambda} - u^2} \tag{2.30}$$

As pointed out by Douglas and Kazakov [27], the discreteness of the original sum imposes an additional constraint on the density function, ρ, namely

$$\rho(u) \leq 1 \tag{2.31}$$

which is satisfied for λ such that

$$\lambda \leq \pi^2 \tag{2.32}$$

For $\lambda > \pi^2$, (2.30) ceases to be an acceptable saddle point of the discrete model. Rather, the appropriate saddle point in this regime is one which saturates the bound (2.31) over a fixed interval. Douglas and Kazakov [27] performed this analysis and found an expression for the strong coupling saddle in terms of elliptic integrals. Since the weak and strong coupling saddles agree at $\lambda = \pi^2$, it is clear that the order of the phase transition which occurs at $\lambda = \pi^2$ must be at least second order. With the saddles in hand, it is also straightforward to compute $F'(\lambda)$, the derivative of the free energy with respect to λ, as it is simply proportional to the expectation value of n_i^2. On the saddle point, this becomes

$$F'(\lambda) \sim N^2 \int \rho(u) u^2 \tag{2.33}$$

From this, we immediately see that continuity of the saddle point distribution through the critical value also implies that $F'(\lambda)$ is continuous and that, in fact, the transition is at least of second order. To go further, it is necessary to obtain the strong coupling saddle and evaluate (2.33) in both phases. Douglas and Kazakov have done precisely this and demonstrated that the transition is actually of third order.

2.2.3 Instanton Sectors and $\theta \neq 0$

We now attempt to study the theory with $\theta \neq 0$. It is not clear to us how to impose the discreteness constraint on the saddle point distribution function ρ once the action becomes complex so we need to probe the phase structure with nonzero θ by another means. The key observation that permits us to proceed is that of [31], who demonstrated that the phase transition in the $\theta = 0$ theory is triggered by instantons in the sense that it occurs precisely when instanton contributions to (2.17) are no longer negligible at large N. Such a result is not surprising given that it is the discreteness of the model, which arises from the sum over instanton sectors, that drives the transition.

To study what happens with $\theta \neq 0$, we therefore turn to the instanton expansion (2.17) and ask at what value of the coupling does the trivial sector fail to dominate. Of course, we must be careful since taking $\theta \to -\theta$ is equivalent to taking $r_i \to -r_i$ and shifting $\theta \to \theta + 2\pi k$ is equivalent to shifting $r_i \to r_i + k$ for integer k so, while the trivial sector dominates near $\theta = 0$, the sector $(-n, -n, \ldots, -n)$ dominates at $\theta = 2\pi n$. We avoid any potential ambiguities by restricting ourselves to $0 \leq \theta \leq \pi$. There, we expect the trivial sector to dominate for λ below a critical point at which the first nontrivial instanton sectors, corresponding to $\vec{r} = (\pm 1, 0, \ldots, 0)$, become nonnegligible. It is the curve in the λ/θ plane at which this occurs that we now seek to determine.

We begin with the trivial sector, which is simply the continuum limit of the full discrete model (2.19). From (2.20), the partition function within this sector is given by

$$Z_{\vec{r}=(0,0,\ldots,0)} = \int \prod_{i=1}^{N} d\phi_i \, \exp\left\{ \sum_{i<j} \ln(\phi_i - \phi_j)^2 - \frac{g_{YM}^2}{2} \sum_i \phi_i^2 - \frac{N\theta^2}{2g_{YM}^2} \right\} \tag{2.34}$$

The saddle point equation for the ϕ_i integral is dominated by the Wigner semicircle distribution found in the previous section (2.30).

We can already see that the phase transition is effected by the value of θ, since the partition function in the weak coupling phase depends on θ according to $Z \sim e^{-N\theta^2/2g_{YM}^2}$, which does not respect the shift $\theta \to \theta + 2\pi$ of the full discrete theory.

We now turn to the family of instanton sectors with $\vec{r} = (r, 0, \ldots, 0)$ (we will eventually take $r = \pm 1$):

$$Z_{\vec{r}=(r,0,\ldots,0)} = \int \prod_{i=1}^{N} d\phi_i \, \exp\left\{ -N^2 S_{eff,(r,0,\ldots,0)} \right\} \tag{2.35}$$

where

$$S_{eff,(\pm 1,0,\ldots,0)} = -\frac{1}{N^2} \sum_{i<j} \ln\left(\phi_i - \phi_j\right)^2 + \frac{\lambda}{2N} \sum_i \left(\frac{\phi_i}{N}\right)^2 - \frac{2\pi i r}{N} \left(\frac{\phi_1}{N} - \frac{i\theta}{\lambda}\right) - \frac{\theta^2}{2\lambda} \tag{2.36}$$

To evaluate (2.35), we note that the saddle point configuration for $\phi_2, \ldots, \phi_N$ in (2.35) is precisely the same as that for the trivial sector since the only difference between the effective actions in these sectors lie in $\mathcal{O}(N)$ out of the $\mathcal{O}(N^2)$ terms. We may thus proceed by evaluating the $\phi_2, \ldots, \phi_N$ integrals using (2.30) and performing the ϕ_1 integral explicitly in the saddle point approximation[4]. The effective action for the ϕ_1 integral becomes

$$S_{eff}(\phi_1) = -\int dy\, \rho(y) \ln(\phi_1 - y)^2 + \frac{\lambda}{2}\phi_1^2 - 2\pi i r \left(\phi_1 - \frac{i\theta}{\lambda}\right) - \frac{\theta^2}{2\lambda} \tag{2.37}$$

and the saddle point value of ϕ_1 is determined by

$$\lambda\sqrt{\phi_1^2 - \frac{4}{\lambda}} - 2\pi i r = 0 \tag{2.38}$$

For $\lambda < \pi^2$, there is one saddle point, which lies along the imaginary axis at

$$(\phi_1)_{\lambda \leq \pi^2 r^2} = \frac{2i}{\lambda}\mathrm{sign}(r)\sqrt{r^2\pi^2 - \lambda} \tag{2.39}$$

As λ increases, this saddle point moves toward the real axis, eventually reaching it at $\lambda = \pi^2 r^2$ and splitting in two

$$(\phi_1)_{\lambda \geq \pi^2} = \pm\frac{2}{\lambda}\sqrt{\lambda - \pi^2 r^2} \tag{2.40}$$

We now wish to obtain the real part of the "free energy" in these nontrivial sectors by evaluating the effective action (2.37) on these saddle points. Comparing with the corresponding free energy of the trivial sector, we obtain

[4]Since we are only interested in the magnitude of the partition function of each sector, it will suffice to determine the real part of the free energy, which is obtained by simply evaluating the action at the potentially complex saddle point. In particular, we will not need to worry about the precise form of the relevant stationary phase contour.

$$
\begin{aligned}
N^{-1}\left(F_{\vec{r}=(\pm 1,0,\ldots,0)} - F_{\vec{r}=(0,0,\ldots,0)}\right) &= -\frac{2\pi\theta r}{\lambda} + \frac{2\pi^2 r^2}{\lambda}\gamma\left(\frac{\lambda}{\pi^2 r^2}\right) \qquad \lambda < r^2\pi^2 \\
&= -\frac{2\pi\theta r}{\lambda} \qquad \lambda > \pi^2 r^2
\end{aligned} \tag{2.41}
$$

where $\gamma(x)$ is defined as in Gross and Matytsin [31]

$$
\gamma(x) = \sqrt{1-x} - \frac{x}{2}\ln\left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right) = 2\sqrt{1-x}\sum_{s=1}^{\infty}\frac{(1-x)^{2s}}{4s^2-1} \tag{2.42}
$$

For $\theta = 0$, this result is in agreement with that of [31] and demonstrates that the partition function of the one-instanton sector is exponentially damped compared to that of the trivial sector at large N for $\lambda \leq \pi^2$. For $\lambda > \pi^2$, on the other hand, these sectors contribute with equal magnitudes at large N and hence instantons are no longer negligible.

In addition this this, we are now able to see at least part of the phase structure with nonzero θ as well. While we have only considered a restricted class of instantons here, it is natural to assume that, as λ is increased from zero, the most dominant of the nontrivial sectors will continue to be those of instanton number 1 and hence that they will continue to trigger the transition away from $\theta = 0$. We thus arrive at the following phase transition line in the λ/θ plane, which is plotted in figure 2.2

$$
\frac{\theta}{\pi} = \gamma\left(\frac{\lambda}{\pi^2}\right) \tag{2.43}
$$

Using the relation between shifts of θ and shifts of the r_i, we can now extend the phase transition line (2.43) outside of the region $0 \leq \theta \leq \pi$. Moreover, we can attempt to guess the form of the phase diagram beyond this line. While we know that there are no further transitions at $\theta = 0$, it seems likely to us that this is not the case for $\theta \neq 0$. The reason for this is that for $\theta \neq 0$, it is no longer true that several different sectors contribute equally beyond the transition as is the case for $\theta = 0$. Rather, because of the θ-dependent shift in the free energy, there will in general be one type of instanton sector which dominates all others for any particular value of λ and, as λ is increased, the instanton number of the dominant sector also increases. As a result it seems reasonable to conjecture that there are additional transitions that smooth out at $\theta = 0$.

We can now ask when sectors of higher instanton number begin to dominate the partition function. If we assume that the only relevant instantons for determining the full phase diagram are those of the sort considered here, then it is easy to proceed. However, we find it quite unlikely that the multi-instanton sectors that we have neglected in the present analysis remain unimportant throughout. To see this, consider following a trajectory at fixed small $\lambda > 0$ and along which θ is increased from 0. For θ less than a critical value θ_c given by (2.43), the dominant sector is the trivial one. Just beyond this value, the dominant sector is that with $\vec{r} = (-1, 0, \ldots, 0)$. Proceeding further, we expect to hit additional critical points at which a sectors with larger instanton number begin to dominate. Eventually, though, we will approach $\theta = 2\pi - \theta_c$, beyond which the $(-1, -1, \ldots, -1)$ sector becomes the dominant one. More generally, for each region in the range $0 \leq \theta \leq \pi$ in which an instanton sector $\vec{r}$ dominates, there is a region in the range $\pi \leq \theta \leq 2\pi$ in which an instanton sector $\vec{r}' = (1, 1, \ldots, 1) - \vec{r}$ dominates. A conjecture consistent with this is that a series of transitions occur along this trajectory in which the dominant sectors take the form $(-1, -1, \ldots, -1, 0, 0, \ldots, 0)$. We sketch a phase diagram based on this conjecture in

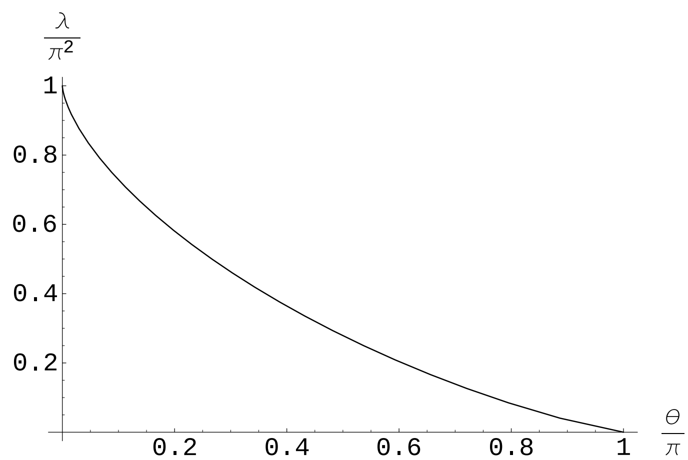

Figure 2.2: Transition line (2.43) in the λ/θ plane

figure 2.3. A deeper analysis which takes into account the multi-instanton sectors not considered here in order to confirm/reject a picture of this sort would be very interesting, but is beyond the scope of the present paper.

2.3 The q-deformed Theory

We now proceed to study the q-deformed theory in a manner analagous to the previous section[5]. Once again, we begin by reviewing the exact solution while obtaining an expression for the partition function as a sum over instanton sectors.

2.3.1 Review of the Exact Solution

We begin by recalling the action of the q-deformed theory

[5]In particular, we will once again use saddle points to study the large N behavior of various instanton sectors. In pure Yang-Mills, one can obtain more precise results by the technique of orthogonal polynomials [31]. This can in principle be generalized to the q-deformed case due to the recent identification of an appropriate set of orthogonal polynomials for such an analaysis [20]

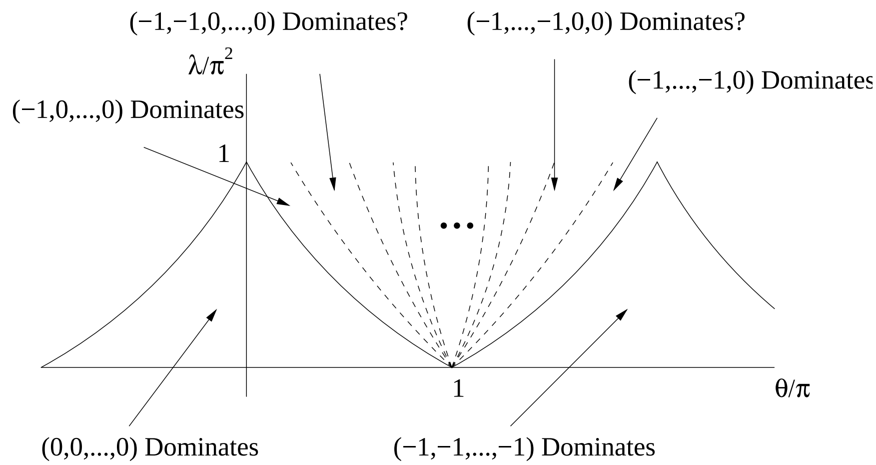

Figure 2.3: Conjecture for the 2D Yang-Mills phase diagram provided we take the curve (2.43) seriously over the entire range $0 \leq \theta \leq \pi^2$. The dotted lines represent the first of numerous conjectured transition lines that cluster near $\theta = \pi$.

$$S = \frac{1}{g_{YM}^2}\int_{S^2} \mathrm{tr}\,\Phi \wedge F + \frac{\theta}{g_{YM}^2}\int_{S^2} \mathrm{tr}\,\Phi \wedge K - \frac{p}{2g_{YM}^2}\int_{S^2} \mathrm{tr}\,\Phi^2 \wedge K \tag{2.44}$$

where Φ is periodic with period 2π. As demonstrated in [7], the analysis proceeds exactly as in section 2.1 with the periodicity of Φ giving rise to the replacement

$$\Delta(\phi_i) \to \tilde{\Delta}(\phi_i) = \prod_{1\leq i<j\leq N} \left[e^{i(\phi_i-\phi_j)/2} - e^{-i(\phi_i-\phi_j)/2}\right] \tag{2.45}$$

In particular, (2.17)-(2.18) become, after an identical Wick rotation and rescaling

$$Z_{qYM} = \sum_{\vec{r}} Z_{qYM,\vec{r}} \tag{2.46}$$

with

$$Z_{qYM,\vec{r}} = \int \prod_{i=1}^{N} d\phi_i \left[\prod_{1\le i<j\le N} \left(e^{-g_{YM}^2(\phi_i-\phi_j)/2} - e^{-g_{YM}^2(\phi_j-\phi_i)/2} \right)^2 \right] \times \exp\left\{ -\frac{g_{YM}^2 p}{2}\phi_i^2 - i(\theta + 2\pi r_i)\phi_i \right\} \quad (2.47)$$

Performing the sum over the r_i restricts ϕ_i to integer values, as in the case of pure Yang-Mills, and yields the known result

$$Z_{qYM} = \sum_{n_i \neq n_j} \left[\prod_{1\le i<j\le N} \left(e^{-g_{YM}^2(n_i-n_j)/2} - e^{-g_{YM}^2(n_j-n_i)/2} \right) \right]^2 \times \exp\left\{ -\frac{g_{YM}^2 p}{2} \sum_i n_i^2 - i\theta \sum_i n_i \right\} \quad (2.48)$$

As before the partition function resembles a matrix model, this time what appears to be a "discretized" Hermitian matrix model with unitary measure[6]. In addition, the sum in (2.49) can again be interpreted as a sum over instanton sectors with the trivial sector corresponding to the continuum limit and capturing all aspects of the full partition function except for the discreteness. Moreover, by shifting ϕ_i by $i\theta/g_{YM}^2 p$ in (2.49), we see that the θ angle continues to carry the interpretation of a chemical potential for total instanton number:

$$Z_{qYM,\vec{r}} = \int \prod_{i=1}^{N} d\phi_i \left[\prod_{1\le i<j\le N} \left(e^{-g_{YM}^2(\phi_i-\phi_j)/2} - e^{-g_{YM}^2(\phi_j-\phi_i)/2} \right)^2 \right] \times \exp\left\{ -\frac{g_{YM}^2 p}{2} \sum_j \phi_j^2 + 2\pi i \sum_j r_j\phi_j + \frac{2\pi\theta}{g_{YM}^2 p} \sum_j r_j - \frac{N\theta^2}{2g_{YM}^2 p} \right\} \quad (2.49)$$

[6]Matrix models of this sort have been studied before in the context of topological strings, first in [50] and later in [3]. Their "discretization" has also previously been studied in [20]

It seems reasonable to expect that the trivial sector will be dominated by a saddle point configuration analagous to the Wigner semicircle distribution for pure Yang-Mills below a critical coupling. Beyond this point, one might expect the attractive term in the potential to become sufficiently large that this distribution is too highly peaked to be consistent with discreteness, at which point we expect to find a phase transition that, due to its connection with discreteness, can again be thought of as arising from the effects of instantons.

2.3.2 The Phase Transition at $\theta = 0$

We now specialize to the case $\theta = 0$ for simplicity and determine the saddle point configuration for small values of the coupling. We will find a distribution analagous to the Wigner semicircle distribution which will, for sufficiently large coupling, violate the constraint arising from discreteness leading to a phase transition analagous to that of Douglas and Kazakov. To proceed, we write the exact result (2.48) as

$$Z = \sum_{n_i \neq n_j} \exp\left\{-N^2 S_{eff}(n_i)\right\} \tag{2.50}$$

where

$$S_{eff}(n_i) = -\frac{1}{2N^2}\sum_{i\neq j} \ln\left(2\sinh\left[\frac{\lambda(n_i - n_j)}{2N}\right]\right)^2 + \frac{\lambda p}{2N}\sum_i \left(\frac{n_i}{N}\right)^2 \tag{2.51}$$

We proceed to study this in the saddle point approximation. The saddle point condition is easily obtained

$$\frac{1}{N}\sum_{j;j\neq i}\coth\left[\frac{\lambda(n_i-n_j)}{2N}\right]=\frac{pn_i}{N} \tag{2.52}$$

We again follow standard techniques and introduce a density function $\rho(u)$ according to (2.24) in terms of which (2.52) can be written

$$P\int dy\,\rho(y)\coth\left[\frac{\lambda(x-y)}{2}\right]=px \tag{2.53}$$

and which must satisfy the normalization condition

$$\int dy\,\rho(y)=1 \tag{2.54}$$

To solve (2.53) and (2.54), it is convenient to make the change of variables $Y=e^{\lambda y}$. In this manner, we may rewrite the system (2.53), (2.54) as

$$\begin{aligned} P\int dY\,\frac{\tilde{\rho}(Y)}{Y-X} &= -\frac{p}{2\lambda}\ln\left[Xe^{-\lambda/p}\right] \\ \int dY\,\frac{\tilde{\rho}(Y)}{Y} &= 1 \end{aligned} \tag{2.55}$$

where

$$\tilde{\rho}(Y)=\tilde{\rho}\left(e^{\lambda y}\right)=\frac{1}{\lambda}\rho(y) \tag{2.56}$$

As usual, solving this system of integral equations is equivalent to determining the resolvent

$$v(X)=\int dY\frac{\tilde{\rho}(Y)}{Y-X} \tag{2.57}$$

Fortunately, this model has been studied before [3] and the resolvent found to be

$$v(X) = \frac{p}{\lambda} \ln \left[\frac{X + 1 + \sqrt{(X - a_+)(X - a_-)}}{2X} \right] \tag{2.58}$$

where

$$a_\pm = -1 + 2e^{\lambda/p} \left[1 \pm \sqrt{1 - e^{-\lambda/p}} \right] \tag{2.59}$$

From this, we find that $\rho(u)$ is nonzero only for $a_- \leq e^{\lambda y} \leq a_+$, where it takes the value

$$\rho(y) = \frac{p}{2\pi} \arccos \left[-1 + 2e^{-\lambda/p} \cosh^2 \left(\frac{\lambda u}{2} \right) \right] \tag{2.60}$$

The appropriate branch of the arccos function for this solution is that for which $0 \leq \arccos x < \pi$. As in the case of pure Yang-Mills, the discreteness of the model imposes an additional constraint

$$\rho(u) \leq 1 \tag{2.61}$$

which is satisfied provided

$$\lambda \leq \lambda_{crit} = -p \ln \cos^2 \frac{\pi}{p} \tag{2.62}$$

For $\lambda > \lambda_{crit}$, (2.60) ceases to be an acceptable saddle point of the discrete model and we expect a phase transition to a distribution analagous to the strong coupling saddle of Douglas and Kazakov at this point. Note that λ_{crit} is finite and nonzero only for $p > 2$ so we conclude that a phase transition of the Douglas-Kazakov type occurs only for these p in the q-deformed model.

To go beyond λ_{crit}, we must look for a new saddle which saturates the bound (2.61) over a fixed interval. As in the pure Yang-Mills case, this strong coupling saddle must agree with that at weak coupling at $\lambda = \lambda_{crit}$. Moreover, we note that the derivative of the free energy with respect to λ is given at large N by

$$\begin{aligned} F'(\lambda) &= N^2 \left[-\frac{1}{2} P \int dx\, dy\, \rho(x)\rho(y)(x-y)\coth\left(\frac{\lambda(x-y)}{2}\right) + \frac{\lambda p}{2} \int dy\, \rho(y) y^2 \right] \\ &= -\frac{N^2 p}{2} \int dy\, \rho(y) y^2 \end{aligned} \tag{2.63}$$

where we have used (2.53). Consequently, we see that, as in pure Yang-Mills theory, continuity of $\rho(y)$ through the transition point guarantees that $F'(\lambda)$ is continuous and hence the transition must be of at least second order. It seems very likely to us, however, that the transition will continue to be of third order as in the case of pure Yang-Mills. We include a brief discussion of the necessary tools to study the strong coupling saddle point in Appendix A. A more complete presentation, as well as further arguments for the third order nature of the transition, can be found in [8][15].

2.3.3 Instanton Sectors and $\theta \neq 0$

We now turn θ back on and attempt to study the phase structure. As in the pure Yang-Mills case, we seek to probe the phase structure by analyzing the instanton expansion (2.49). Again, the trivial sector dominates at small θ, λ and we expect the phase transition to occur when the first nontrivial instanton sector, corresponding to $\vec{r} = (\pm 1, 0, \ldots, 0)$, makes a nonnegligible contribution to the partition function at large N. We now work by analogy and seek to determine the curve in the λ/θ plane that bounds the region in which the trivial sector dominates.

We begin with the trivial sector, which is simply the continuum limit of the full discrete theory (2.48)

$$Z_{\vec{r}=(0,0,\ldots,0)} = \int \prod_{i=1}^{N} d\phi_i \, \exp\left\{ \frac{1}{2}\sum_{i\neq j} \ln\left(2\sinh\left[\frac{\lambda(\phi_i-\phi_j)}{2N}\right]\right)^2 - \frac{\lambda p}{2N}\sum_i \left(\frac{\phi_i}{N}\right)^2 - \frac{N\theta^2}{2g_{YM}^2 p} \right\} \tag{2.64}$$

The saddle point distribution for this integral is given by the function ρ found in the previous section (2.60).

We now turn to the family of nontrivial instanton sectors with $\vec{r} = (r, 0, \ldots, 0)$ (we will later take $r = \pm 1$) whose partition functions take the form

$$Z_{\vec{r}=(r,0,\ldots,0)} = \int \prod_{i=1}^{N} d\phi_i \, \exp\left\{-N^2 S_{eff,(r,0,\ldots,0)}\right\} \tag{2.65}$$

with

$$S_{eff,(r,0,\ldots,0)} = -\frac{1}{N^2}\sum_{i<j} \ln\sinh^2\left(\frac{\lambda(\phi_i-\phi_j)}{2}\right) + \frac{\lambda p}{2N}\sum_{i=1}^{N}\phi_i^2 - \frac{2\pi i r}{N}\left(\phi_1 - \frac{i\theta}{\lambda p}\right) - \frac{\theta^2}{2\lambda p} \tag{2.66}$$

The integrals over ϕ_i with $i > 1$ are dominated by the saddle point (2.60) leading to the following effective action for the ϕ_1 integral

$$S_{eff}(\phi_1) = -\int dy\,\rho(y)\ln\sinh^2\left(\frac{\lambda(\phi_1 - y)}{2}\right) + \frac{\lambda p}{2}\phi_1^2 - 2\pi i r\left(\phi_1 - \frac{i\theta}{\lambda p}\right) - \frac{N\theta^2}{2\lambda p} \tag{2.67}$$

It is easy to determine the saddle point equation for ϕ_1 using (2.53)

$$\frac{\lambda}{p}\left[1 - 2v\left(e^{\lambda\phi_1}\right)\right] = \lambda\phi_1 - \frac{2\pi i r}{p} \tag{2.68}$$

where v is the resolvent from (2.57). Solving this equation is straightforward, though requires a bit of care. If we write $r = np + m$ where n, m are integers and $m \in (-p/2, p/2]$, then the solution can be written as

$$\begin{aligned}\phi_1 &= \frac{2i}{\lambda}\mathrm{sign}(r)\arctan\left[\sqrt{e^{-\lambda/p}\sec^2\frac{\pi m}{p} - 1}\right] + \frac{2\pi i n}{\lambda} \qquad e^{-\lambda/p}\sec^2\frac{\pi m}{p} > 1\\ &= \frac{2}{\lambda}\ln\left[e^{\lambda/2p}\cos\frac{\pi m}{p}\left(1 \pm \sqrt{1 - e^{-\lambda/p}\sec^2\frac{\pi m}{p}}\right)\right] + \frac{2\pi i n}{\lambda} \qquad e^{-\lambda/p}\sec^2\frac{\pi m}{p} < 1\end{aligned} \tag{2.69}$$

where the appropriate branch of arctan in (2.69) is $-\pi/2 < \arctan x < \pi/2$ and that of ln is such that it has zero imaginary part for real argument.

As in the case of pure Yang-Mills, the saddle points lies along the imaginary axis for small λ and, as λ approaches the critical point $\lambda_{crit} = -p\ln\cos^2\frac{\pi}{p}$, the saddle points corresponding to $r = \pm 1$ hit the origin and splits in two along the real axis as λ is increased further.

To study when particular instanton sectors become nonnegligible, we must evaluate the (real part of) the free energy in each sector, namely

$$F_{\vec{r}} = \mathrm{Re}\left[-g(\phi_1) + \frac{\lambda p}{2}\phi_1^2 - 2\pi i r \phi_1 - \frac{2\pi r \theta}{\lambda p}\right] \tag{2.70}$$

where

$$g(\phi_1) = \int dy\, \rho(y) \ln \sinh^2 \left(\frac{\lambda(x-y)}{2}\right)^2 \tag{2.71}$$

In Appendix B, we obtain an exact expression for the function $g(\phi)$ and, in addition, simple integral formulae for the real part of $g(\phi)$ for ϕ along the real and imaginary axes. Using the results of Appendix B, we may write the difference between the real parts of free energies of the $\vec{r} = (r, 0, \ldots, 0)$ and $\vec{r} = (0, 0, \ldots, 0)$ sectors as

$$\begin{aligned} N^{-1}\left(F_{\vec{r}=(r,0,\ldots,0)} - F_{\vec{r}=(0,0,\ldots,0)}\right) = \\ \frac{4p}{\lambda}\int_0^{\alpha_0} d\alpha \left[\frac{\pi r}{p} - \mathrm{sign}(r) \arccos\left(e^{-\lambda/2p}|\cos\alpha|\right)\right] + \frac{2\pi r}{\lambda p}(2\pi n p - \theta) \\ \text{for } e^{\lambda/p}\cos^2\frac{\pi m}{p} < 1 \\ = \frac{2\pi r}{\lambda p}(2\pi n p - \theta) \qquad \text{for } e^{\lambda/p}\cos^2\frac{\pi m}{p} > 1 \end{aligned} \tag{2.72}$$

where

$$\alpha_0 = \mathrm{sign}(r) \arccos\left(e^{\lambda/2p}\left|\cos\frac{\pi m}{p}\right|\right) \tag{2.73}$$

and the appropriate branch of arccos is such that $0 \leq \arccos x \leq \pi$.

Let us first look at the case $\theta = 0$. The integrand in (2.72) is manifestly positive definite for α_0 nonzero and $e^{\lambda/p}\cos^2(\pi m/p) < 1$ so the integral is positive definite for

$\alpha_0 > 0$ and 0 for $\alpha_0 = 0$. As a result, we see that as the coupling is increased from zero, the trivial sector dominates until the critical point $\lambda = \lambda_{crit}$ (2.62) is reached, at which point α_0 becomes zero for $r = \pm 1$ and the difference in free energies vanishes for the one-instanton sectors $\vec{r} = (\pm 1, 0, \ldots, 0)$. Consequently we conclude that, as in the case of pure Yang-Mills, the phase transition at $\lambda = \lambda_{crit}$ (2.62) is triggered by nontrivial instanton sectors becoming nonnegligible and contributing with equal magnitude to the trivial one at large N.

Now, what happens when we turn θ back on? It is again reasonable to assume that, as λ is increased from zero, multi-instanton sectors of the type not considered here remain unimportant so that most dominant of the nontrivial sectors for $\lambda \leq \lambda_{crit}$ (2.62) will continue to be those of instanton number $\pm$ 1. This leads us to the following phase transition line in the λ/θ plane, which is plotted for $p = 3$ in figure 2.4 [7]

$$\frac{\theta}{\pi} = \frac{2p^2}{\pi^2} \int_0^{\alpha_0} d\alpha \left[\frac{\pi}{p} - \mathrm{sign}(r) \arccos\left(e^{-\lambda/2p} |\cos\alpha| \right) \right] \tag{2.74}$$

The qualitative behavior of this curve is identical to that of the pure Yang-Mills case in that the critical coupling decreases from λ_{crit} (2.62) at $\theta = 0$ to zero at $\theta = \pi$. Following the same reasoning applied to pure Yang-Mills, we also expect further transitions beyond the line (2.74) beyond which sectors with larger instanton number dominate the full partition function. The required symmetries under simultanoues shifts/flips of θ and $\vec{r}$ then lead us to conjecture a phase diagram of the form (2.3) in the case of the q-deformed theory as well. It would be interesting to pursue a further analysis in order to verify or reject this picture.

[7]We would like to thank M. Marino for correspondence regarding a mistake in the plotted curve in version 1, which was the result of a sign error.

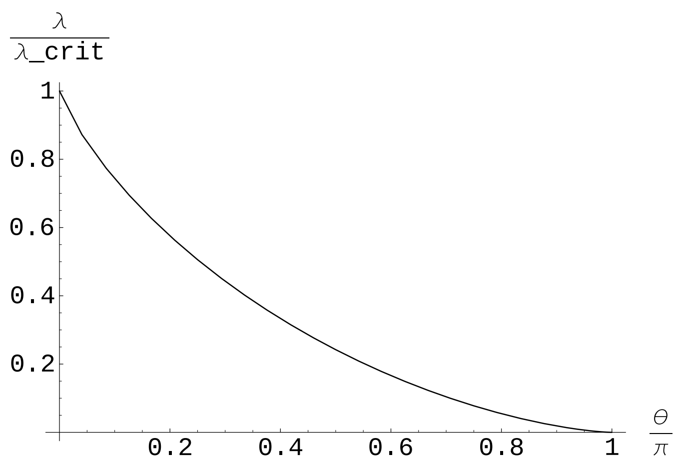

Figure 2.4: Plot of the curve (2.74) for $r = 1, p = 3$.

2.4 Chiral decomposition, and phase transition in topological string

One of the most interesting features of the q-deformed Yang-Mills theory we have been examining is its factorization into chiral blocks at large N. This occurs in an analogous manner to the chiral decomposition of pure Yang Mills, with the new ingredient being its identification by [7] with the topological string partition function. We will show that the chiral partition function has a natural interpretation in the strong coupling phase as a limiting description of the eigenvalues on a single side of the clumped region.

In contrast to the case of Yang-Mills on a torus, where a similar picture was developed [69] [22], here of course the Fermi surfaces do not decouple, even perturbatively, because of the nontrivial repulsion between the eigenvalues. We will see that this is encoded in the sum over "ghost" branes found in [7]. The presence of the weak coupling phase adds a new feature to this picture, since at some value of the 't Hooft coupling, the two Fermi surfaces simply vanish. We will investigate what effect this has on the trivial chiral block. Throughout this entire section, we consider only the case $\theta = 0$ for simplicity.

We now look a bit closer at the trivial chiral block, $Z^{qYM,+}_{\cdot\cdot}$, which is to be identified with the topological string partition function without any ghost brane insertions [7]. Using (2.5), we see that it takes a particularly simple form

$$\begin{aligned} Z_{top} &= \sum_R q^{\frac{p-2}{2}\kappa_R} e^{-t|R|} W_R(q)^2 \\ &= \sum_R q^{p\kappa_R/2} e^{-t|R|} W_R(q^{-1})^2 \end{aligned} \tag{2.75}$$

where the sum is over Young diagrams, R and the objects appearing in the sum are defined as follows:

$$\begin{aligned} \kappa_R &= \sum_i R_i(R_i - 2i + 1) \\ W_R(q^{-1}) &= s_R\left(q^{i-\frac{1}{2}}\right) \\ q &= e^{-g_s} = e^{-g_{YM}^2} \end{aligned} \tag{2.76}$$

where R_i denotes the number of boxes in the ith row of the diagram R and $s_R(x^i)$ is the $SU(\infty)$ Schur function associated to the diagram R.

It is clear that, if we include all Young diagrams, the series (2.75) diverges for any q for $p \geq 3$, since arbitrarily negative powers of q appear for diagrams with sufficiently large diagrams. This should not disturb us, however, because the topological string is merely an asymptotic expansion, and only the perturbation theory about $g_s = 0$ is well defined. Thus we truncate the sum over Young diagrams to chiral representations of $U(M)$, for some finite M, and take a 't Hooft limit keeping $\lambda = g_s M$ fixed. In fact, the chiral decomposition (2.5) indicates precisely what this cutoff should be as one can easily verify that (4.11) holds at finite N only when the Young diagrams R are taken to have at most $N/2$ rows. To be sure, the model that we study is the chiral block defined in this way. We then assume that, based on (4.11), this is the proper manner in which to define the perturbative topological string partition function.

2.4.1 Interpretation of the trivial chiral block

Heuristically, we expect the chiral partition function to describe a system in which only the eigenvalues on one side of the clump in the strong coupling phase are "dynamical" in the sense their locations are summed. We can isolate such contributions by placing an infinite clump of eigenvalues about the origin which drives the two fermi surfaces apart, effectively decoupling them. Of course, one must be careful when doing this to avoid introducing divergences in the partition function. A particularly natural way to proceed is to start with the q-deformed Yang-Mills partition function expressed as a sum over $U(N)$ Young diagrams. This is easily obtained from the exact expression (2.8) by the change of variables

$$\mathcal{R}_i = n_i - \rho_i \tag{2.77}$$

with ρ the Weyl vector of $U(N)$

$$\rho_i = \frac{1}{2}\left(N - 2i + 1\right) \tag{2.78}$$

Note that the $\mathcal{R}_i$ parametrize deformations from the maximally clumped rectangular distribution so organizing the partition function as a sum over the $\mathcal{R}_i$ is quite natural for the problem at hand. The resulting expression is

$$Z_{qYM} \sim \sum_{\mathcal{R}} \prod_{1\leq i<j\leq N} \left[\frac{[\mathcal{R}_i - \mathcal{R}_j + j - i]_q}{[j-i]_q}\right]^2 q^{\frac{p}{2}C_2(\mathcal{R})} e^{-i\theta|\mathcal{R}|} \tag{2.79}$$

where the sum is now over $U(N)$ Young diagrams $\mathcal{R}$. The quadratic Casimir and $|\mathcal{R}|$ are defined as

$$\begin{aligned} C_2(\mathcal{R}) &= \kappa_{\mathcal{R}} + N|\mathcal{R}| \\ |\mathcal{R}| &= \sum_i \mathcal{R}_i \end{aligned} \tag{2.80}$$

and $[x]_q$ is the q-analogue defined by

$$[x]_q = q^{x/2} - q^{-x/2} \tag{2.81}$$

To add an additional M eigenvalues, we simply pass from N to $M+N$ in the above expression, obtaining a sum over $U(M+N)$ representations that becomes

$$\begin{aligned} Z' &= \sum_{\mathcal{R}} \left[\prod_{1\leq i<j\leq M+N} \left(\frac{[\mathcal{R}_i - \mathcal{R}_j + j - i]_q}{[j-i]_q}\right)^2\right] q^{\frac{p}{2}([M+N]|\mathcal{R}|+\kappa_{\mathcal{R}})} e^{-i\theta|\mathcal{R}|} \\ &= \sum_{\mathcal{R}} \left[q^{-(N+M-1)|\mathcal{R}|} \prod_{1\leq i<j\leq M+N} \left(\frac{q^{\mathcal{R}_i - i} - q^{\mathcal{R}_j - j}}{q^{-i} - q^{-j}}\right)^2\right] q^{\frac{p}{2}([M+N]|\mathcal{R}|+\kappa_R} e^{-i\theta|\mathcal{R}|} \\ &= \sum_{\mathcal{R}} \left[s_{\mathcal{R}}\left(q^{1/2}, \ldots, q^{M+N-\frac{1}{2}}\right)\right]^2 q^{\frac{p-2}{2}(M+N)|\mathcal{R}|+\frac{p}{2}\kappa_{\mathcal{R}}} e^{-i\theta|\mathcal{R}|} \end{aligned} \tag{2.82}$$

If we now restrict to representations with at most $N/2$ rows, corresponding to treating only the largest $N/2$ eigenvalues as "dynamical", and take the limit $M \to \infty$ with $t = g_s(p-2)(M+N)/2$ fixed, we obtain the result

$$Z = \sum_R W_R \left(q^{-1}\right)^2 e^{-t|R|} q^{p\kappa_R/2} \tag{2.83}$$

which is our old friend (2.75). It is in precisely this sense that the chiral block describes the dynamics of eigenvalues on one side of the clump.

2.4.2 Phase Structure of the Trivial Chiral Block

We now perform a preliminary study of the phase structure of the trivial chiral block. As mentioned before, this quantity is defined with a cutoff on the sum over representations to those with at most $N/2$ rows so, to study this model as $g_s \to 0$, we must work in a 't Hooft limit where $g_{YM}^2 N \sim t/2$ is held fixed while taking $N \sim t/2g_s$ to ∞. It is convenient for this analysis to return to the n_i variables, of which there are now only $N/2$. The effective action in these variables becomes

$$\frac{S_{eff}}{N^2} = -\frac{1}{2N^2}\sum_{i,j=1}^{N/2} \ln\left(e^{-\lambda n_i/N} - e^{-\lambda n_j/N}\right)^2 + \frac{2}{\lambda N}\sum_{i=1}^{N/2} Li_2\left(e^{-\lambda n_i/N}\right) + \frac{p\lambda}{2N}\sum_{i=1}^{N/2}\left(\frac{n_i}{N}\right)^2 + \frac{i\theta}{N}\sum_{i=1}^{N/2}\frac{n_i}{N} \tag{2.84}$$

For simplicity, we now set $\theta = 0$. Proceeding via the usual saddle point method, we find that the configuration which extremizes the action satisfies

$$\frac{\lambda}{N}\sum_{j=1}^{N/2}\frac{1}{1-e^{\lambda(n_i-n_j)/N}} + \ln\left(1-e^{-\lambda n_i/N}\right) + \frac{p\lambda}{2}\left(\frac{n_i}{N}\right) = 0 \tag{2.85}$$

Introducing the eigenvalue distribution ρ as usual, we have

$$\int \frac{dy\,\rho(y)}{1-e^{\lambda(x-y)}} + \frac{1}{\lambda}\ln\left(1-e^{-\lambda x}\right) + \frac{px}{2} = 0 \tag{2.86}$$

where, because the cutoff is at $N/2$ rather than N, the normalization condition becomes

$$\int dy\,\rho(y) = \frac{1}{2} \tag{2.87}$$

To solve this, we make the change of variables

$$U = e^{\lambda u} \tag{2.88}$$

If we then define

$$\tilde{\rho}(U) = \frac{2}{\lambda}\rho\left(u = \frac{1}{\lambda}\ln U\right), \tag{2.89}$$

then the saddle point equation that we must solve becomes

$$P\int \frac{dV\,\tilde{\rho}(V)}{U-V} = \frac{p-2}{\lambda}\ln U + \frac{2}{\lambda}\ln\left(U-1\right) \tag{2.90}$$

with the constraint

$$\int \frac{\tilde{\rho}(U)\,dU}{U} = 1 \tag{2.91}$$

Because we expect clumping near zero, we look for a saddle point of the form

$$\rho(u) = \begin{matrix} 1 & 0 \le u \le a \\ \psi(u) & a \le u \le b \\ 0 & u > b \end{matrix} \tag{2.92}$$

We now define the function $\tilde{\psi}$ as

$$\tilde{\psi}(U) = \frac{2}{\lambda}\psi\left(u = \frac{1}{\lambda}\ln U\right) \tag{2.93}$$

and note that (2.90) and (2.91) may be written in terms of $\tilde{\psi}$ as

$$P\int_A^B \frac{dV\tilde{\psi}(V)}{U-V} = \frac{p-2}{\lambda}\ln U + \frac{2}{\lambda}\ln(S-A) \tag{2.94}$$

and

$$\int_A^B \frac{dV\tilde{\psi}(V)}{U-V} = -1 + \frac{2}{\lambda}\ln A \tag{2.95}$$

where

$$A = e^{\lambda a} \qquad B = e^{\lambda b} \tag{2.96}$$

As usual, we may determine $\tilde{\psi}(U)$ by first computing the resolvent

$$f(U) = \int dV\,\frac{\tilde{\rho}(V)}{U-V}, \tag{2.97}$$

The expression for $f(U)$ is given by the well-known "one-cut" solution (see, for instance, [51]):

$$f(U) = \frac{1}{2\pi i}\sqrt{(U-A)(U-B)}\oint_{[A,B]} dS\,\frac{\frac{2}{\lambda}\ln(S-A) + \frac{p-2}{\lambda}\ln S}{(U-S)\sqrt{(S-A)(S-B)}} \tag{2.98}$$

with integration contour going counterclockwise around the cut $[A, B]$. To evaluate this, we move the contour away from the cut, and pick up instead the branch cuts of the logarithms along $[-\infty, 0]$ and $[-\infty, A]$, as well as the pole at $S = U$. The resulting integrals can be easily evaluated, and we find that

$$f(U) = \frac{2}{\lambda}\ln\left(\frac{U-A}{U}\right) + \frac{p}{\lambda}\ln U + \frac{p-2}{\lambda}\ln\left(\frac{\sqrt{B(U-A)}+\sqrt{A(U-B)}}{\sqrt{B(U-A)}-\sqrt{A(U-B)}}\right) - \frac{p}{\lambda}\ln\left(\frac{\sqrt{(U-A)}+\sqrt{(U-B)}}{\sqrt{(U-A)}-\sqrt{(U-B)}}\right) \tag{2.99}$$

Using this, we may determine $\tilde{\psi}(U)$ and, from that, the density $\rho(u)$:

$$\rho(u) = \left\{ \begin{array}{ll} 1 & 0 \leq e^{\lambda u} < A \\ \frac{1}{\pi}\left[p \arctan\left(\sqrt{\frac{B-e^{\lambda u}}{e^{\lambda u}-A}}\right) - (p-2)\arctan\left(\sqrt{\frac{A(B-e^{\lambda u})}{B(e^{\lambda u}-A)}}\right)\right] & A \leq e^{\lambda u} \leq B \\ 0 & e^{\lambda u} > B \end{array} \right), \tag{2.100}$$

where we have returned to the original variables.

It remains to fix the position of the cut, A and B. This is done by noting that the normalization constraint implies $f(0) = -1 + \frac{2}{\lambda}\ln A$ while the definition of $f(U)$ implies that $f(U) \sim \mathcal{O}(U^{-1})$ as $U \to \infty$. Let's look at the first condition, namely $f(U) = 0$. In the limit $U \to 0$, $f(U)$ becomes

$$f(U \to 0) = \frac{2}{\lambda}\ln A - \frac{p}{\lambda}\ln\left(\frac{\sqrt{B}+\sqrt{A}}{\sqrt{B}-\sqrt{A}}\right) + \frac{p-2}{\lambda}\ln\left(\frac{4AB}{B-A}\right) = -1 + \frac{2}{\lambda}\ln A \tag{2.101}$$

On the other hand, in the limit $U \to \infty$, $f(U)$ becomes

$$f(U \to \infty) = \frac{p}{\lambda}\ln\left(\frac{B-A}{4}\right) + \frac{p-2}{\lambda}\ln\left(\frac{\sqrt{B}+\sqrt{A}}{\sqrt{B}-\sqrt{A}}\right) \tag{2.102}$$

Defining

$$X = \frac{\sqrt{B}-\sqrt{A}}{2} \qquad Y = \frac{\sqrt{B}+\sqrt{A}}{2}, \tag{2.103}$$

for convenience, we obtain the polynomial equations

$$X^{2(p-2)}(Y^2 - X^2) = e^{-\lambda/2} \tag{2.104}$$

$$XY^{p-1} = 1 \tag{2.105}$$

These can be solved easily in principle. As an example, let us consider the case $p = 3$. From (2.105) we find $Y = 1/\sqrt{X}$. We now plug into (2.104) and look for real solutions such that $A = (X^{-1/2} - X)^2 \geq 1$. There are four solutions, two of which are real, namely

$$X = \frac{\sqrt{\mu}}{2}\left[1 \pm \sqrt{\frac{2}{\mu^{3/2}} - 1}\right], \tag{2.106}$$

where

$$\mu = \frac{4}{\alpha}\left(\frac{2}{3}\right)^{1/3} e^{-\lambda/2} + \frac{\alpha}{2^{1/3}3^{2/3}} \tag{2.107}$$

$$\alpha = \left(9 + \sqrt{3}e^{-3\lambda/2}\sqrt{27e^{3\lambda} - 256e^{3\lambda/2}}\right)^{1/3} \tag{2.108}$$

These solutions are real provided

$$\lambda > (2/3)\ln\left(\frac{256}{27}\right) \approx 1.499 \tag{2.109}$$

but in this range the plus sign leads to $A \leq 1$ so it is only the minus solution that will interest us. Moreover, we find that at $A = 1$ at

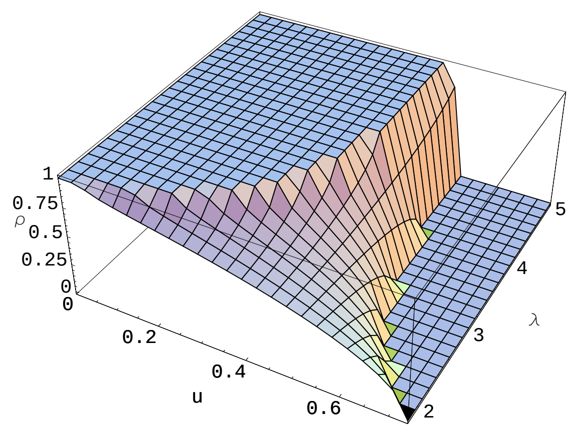

Figure 2.5: The saddle point density $\rho(u)$ as a function of λ in the case $p = 3$ for $\lambda >\approx 1.74174$.

$$\lambda \approx 1.74174 \tag{2.110}$$

and hence the density (2.100) ceases to be an acceptable saddle point of the model below this value of the coupling. This leads us to conjecture that the $p = 3$ chiral block, and presumably the chiral blocks for larger values of p as well, undergoes a phase transition at a critical value of λ. Note that, interestingly, this occurs far below the phase transition of the full q-deformed theory, which for $p = 3$ corresponds to $\lambda_{crit} = 3 \ln 4 \approx 4.159$. We plot $\rho(u)$ in this case for various values of λ in figure 2.5. Further study of the phase structure here would be very interesting. We postpone this to future work.

2.5 Concluding Remarks

We have found a phase transition in q-deformed Yang-Mills theory on S^2 in the 't Hooft limit, driven by instantons. Using saddle point techniques we evaluated the free energy in the weak coupling phase. We studied the structure of the contributions of various instanton sectors at strong coupling, and find evidence of a rich phase structure when a θ angle is turned on. These phase transitions have very interesting consequences for the chiral factorization into a sum over topological string expansions. We discover a phase transition in the trivial chiral block at a different value of the coupling, implying a nontrivial structure in the breakdown of the factorization as one moves into the weak coupling phase.

While in this work we have focused on $p > 2$, our analysis also clarifies certain aspects of the OSV relation for $p = 1$ and 2 as well. In those cases, the q-deformed Yang-Mills remains in the weak coupling phase for all λ, thus the chiral decomposition appears suscipious. In particular, we identify the absence of a clumped region of eigenvalues with the fact that the attractor value of the Kahler class of the S^2 is negative for $p = 1$. In [7], it is argued that one can perform a flop, which essentially turns on a θ angle of $-\pi$, in order to make sense of this case. This is yet another motivation to understand the phase structure at nonzero θ angle better. On the other hand, for $p = 2$, the attractor size is 0 which is consistent with the fact that only in the limit $\lambda \to \infty$ do we saturate the discreteness condition.

There are several intriguing directions which deserve further research. It would be very interesting to fully explore the phase structure of the q-deformed Yang-Mills for non-zero θ, extending our analysis of the instanton sectors deeper into the strong coupling regime where multi-instanton corrections may become important. This seems particularly promising since we have seen some indictations of a rich phase diagram. The manner in which the chiral decomposition breaks down in the weak coupling phase should also be further elucidated, since we found a phase transition in the topological string partition functions in the trivial sector only after the coupling had moved well inside the regime where we expect the decomposition into chiral blocks to break down. We believe that this phenomenon will be found to be driven by putatively nonperturbative corrections to the factorization becoming important.

Appendix A. Two cut solution in the strong coupling phase

We have seen that for $\lambda > \lambda_{crit}$, the instanton sectors are no longer suppressed, and we have a transition to a phase in which some of the eigenvalues are clumped together at the maximum density determined by the constraint (2.31). Following Douglas and Kazakov [27], we will substitute the ansatz,

$$\tilde{\rho}(u) = 1 \text{ for } -a < u < a, \text{ and } \rho(u) \text{ otherwise,} \tag{2.111}$$

for the eigenvalue density in the strong coupling regime.

Plugging this into the effective action for q-deformed Yang Mills (2.51), we find the new effective action

$$S_{eff} = \frac{p\lambda}{2}\int du\rho(u)\ u^2 + \frac{p\lambda}{2}\int_{-a}^{a} dy\ y^2 - \frac{1}{2}\int du\rho(u)\int dv\rho(v)\log\left(\sinh^2(\frac{\lambda}{2}(u-v))\right) \tag{2.112}$$

$$-\int du\rho(u)\int_{-a}^{a}dy\log\left(\sinh^2(\frac{\lambda}{2}(u-y))\right)-\frac{1}{2}\int_{-a}^{a}dx\int_{-a}^{a}dy\log\left(\sinh^2(\frac{\lambda}{2}(x-y))\right),$$

where we have set $\theta = 0$ at present for convenience. The saddle point equation is therefore

$$W'_{new}(u) = p\lambda u - \log\left(\frac{\sinh^2\left(\frac{\lambda}{2}(u+a)\right)}{\sinh^2\left(\frac{\lambda}{2}(u-a)\right)}\right) = \frac{\lambda}{2}\mathcal{P}\int dy\ \rho(y)\coth\left(\frac{\lambda}{2}(u-y)\right). \tag{2.113}$$

This is a system of repulsive eigenvalues in the effective potential, W_{new}, which has two wells, one on each side of the maximally clumped region, that the eigenvalues can fall into. Thus we will look for a symmetric two cut solution. It will be easier to first change variables to $U = e^{\lambda u}$ and $A = e^{\lambda a}$. This implies that

$$p\log U - \log\left(\left(\frac{U^{\frac{1}{2}}A^{\frac{1}{2}} - U^{-\frac{1}{2}}A^{-\frac{1}{2}}}{U^{\frac{1}{2}}A^{-\frac{1}{2}} - U^{-\frac{1}{2}}A^{\frac{1}{2}}}\right)^2\right) = \frac{1}{2}\mathcal{P}\int\frac{dY}{Y}\rho(Y)\frac{U^{\frac{1}{2}}Y^{-\frac{1}{2}} + U^{-\frac{1}{2}}Y^{\frac{1}{2}}}{U^{\frac{1}{2}}Y^{-\frac{1}{2}} - U^{-\frac{1}{2}}Y^{\frac{1}{2}}},$$

which can be simplified to

$$p\log U - 2\log\left(\frac{U-A^{-1}}{U-A}\right) + \lambda(\frac{1}{2}-2a) = \mathcal{P}\int\frac{\rho(Y)dY}{U-Y}. \tag{2.114}$$

Defining the resolvent by $v(U) = \int\frac{\rho(Y)dY}{U-Y}$, as in equation (2.57), we are left with a standard problem of two cut matrix model, see for instance [51]. The solution is determined by solving the Riemann-Hilbert problem for the resolvent, which is characterized by falling off as U^{-1} at infinity, having period $2\pi i$ about the two cuts, and taking the form $v(U+i\epsilon) - v(U-i\epsilon) = 2\pi i\rho(U)$ along the cuts. Using well known techniques in complex analysis, one finds that

$$v(U) = \oint_{\mathcal{C}}\frac{dz}{2\pi i}\frac{p\log z - 2\log\left(\frac{z-A^{-1}}{z-A}\right) + \lambda(\frac{1}{2}-2a)}{U-z}\sqrt{\frac{(A-U)(A^{-1}-U)(B-U)(B^{-1}-U)}{(A-z)(A^{-1}-z)(B-z)(B^{-1}-z)}}, \tag{2.115}$$

where the contour encircles the cuts from (B^{-1}, A^{-1}) and (A, B). Moving the contour toward infinity, we pick up instead the branch cut of the logarithm along (A^{-1}, A), the pole at $z = U$, and the branch cut of the $\log U$ along the negative real axis.

Putting all of this together, we obtain

$$v(U) = p\log U - 2\log\left(\frac{U-A^{-1}}{U-A}\right) + \lambda(\frac{1}{2} - 2a) + 2\int_{A^{-1}}^{A} ds \frac{\sqrt{(A-U)(A^{-1}-U)(B-U)(B^{-1}-U)}}{(U-s)\sqrt{(A-s)(A^{-1}-s)(B-s)(B^{-1}-s)}}$$
$$-p\sqrt{(A-U)(A^{-1}-U)(B-U)(B^{-1}-U)}\int_{-\infty}^{0} ds \frac{1}{(U-s)\sqrt{(A-s)(A^{-1}-s)(B-s)(B^{-1}-s)}}. \tag{2.116}$$

This last two integrals can be expressed as elliptic integrals of the third kind. The size of the cut can be determined by requiring that we reproduce the correct limit as $U \to \infty$, where we must have $v(U) \sim \mathcal{O}(\frac{1}{U})$, and as $U \to 0$, where $v(U) = \int \frac{dY}{Y}\rho(Y) = \lambda - 2a$. Hence we can now compute the free energy in the strong coupling phase.

Define the elliptic integral, f, as follows

$$f(s) = \int_{A^{-1}}^{s} \frac{dt}{U-t}\sqrt{\frac{(A-U)(A^{-1}-U)(B-U)(B^{-1}-U)}{(A-t)(A^{-1}-t)(B-t)(B^{-1}-t)}} =$$
$$\frac{2i}{B-A}\sqrt{\frac{(B-U)(1-BU)}{(A-U)(1-AU)}}\left\{(1-AU)F[z|m] + (A^2-1)\Pi[n, z|m]\right\}, \tag{2.117}$$

where

$$z = \sin^{-1}\sqrt{\frac{(B-A)(s-A^{-1})}{(B-A^{-1})(A-s)}},\ m = \left(\frac{AB-1}{B-A}\right)^2,\ \text{and } n = \frac{(B-A^{-1})(A-U)}{(B-A)(A^{-1}-U)}.$$

Imposing the limits derived above implies that

$$\int_{A^{-1}}^{A} \frac{ds}{\sqrt{(A-s)(A^{-1}-s)(B-s)(B^{-1}-s)}} = \frac{p}{2}\int_{-\infty}^{0} \frac{ds}{\sqrt{(A-s)(A^{-1}-s)(B-s)(B^{-1}-s)}}, \tag{2.118}$$

and

$$\lambda = p\int_{-\infty}^{-\epsilon} \frac{ds}{s\sqrt{(A-s)(A^{-1}-s)(B-s)(B^{-1}-s)}} - p\log\epsilon - 2\int_{A^{-1}}^{A} \frac{ds}{s\sqrt{(A-s)(A^{-1}-s)(B-s)(B^{-1}-s)}}, \quad (2.119)$$

which can be solved for A and B in terms of λ. Note that the divergence of the integral up to $-\epsilon$ exactly cancels the $\log\epsilon$.

Appendix B. The Function $g(\alpha)$

In this appendix, we study the function

$$g(\alpha) = \int dy\, \rho(y) \ln \sinh^2\left(\frac{\lambda(\alpha - y)}{2}\right) \quad (2.120)$$

where $\rho(y)$ is the density function (2.60) in the strong coupling phase of the q-deformed Yang-Mills theory at $\theta = 0$. To start, we note that

$$\begin{aligned} g'(\alpha) &= \lambda \int dy\, \rho(y) \coth\left(\frac{\lambda(\alpha - y)}{2}\right) \\ &= \lambda\left[1 - 2v\left(e^{\lambda\alpha}\right)\right] \end{aligned} \quad (2.121)$$

where v is the resolvent (2.57) first obtained in [3]. If we define $A = e^{\lambda\alpha}$, then we can express $g(\alpha)$ as a function of A. Moreover, we have that

$$\frac{dg}{dA}(A) = \frac{1}{A}\left(1 - 2v(A)\right) \quad (2.122)$$

which can be integrated in order to obtain $g(A)$:

$$g\left(A = e^{\lambda\alpha}\right) = \ln A + \frac{p}{\lambda}\left[\ln u(A)\right]^2 - \frac{2p}{\lambda}\left[\ln u(A)\right]\left[\ln\left(1 - e^{-\lambda/p}u(A)\right)\right] \\ - \frac{2p}{\lambda}Li_2\left(1 - u(A)\right) - \frac{2p}{\lambda}Li_2\left(e^{-\lambda/p}u(A)\right) + C \tag{2.123}$$

where

$$u(A) = \frac{1 + A + \sqrt{(A+1)^2 - 4e^{\lambda/p}A}}{2A} \tag{2.124}$$

and C is an α-independent constant. The result (2.123) is not very useful for learning about free energies. However, it is useful for noting that

$$g\left(\alpha + \frac{2\pi i}{\lambda}\right) = g(\alpha) + \frac{2\pi i}{\lambda} \tag{2.125}$$

and hence that shifting α in this manner has no effect on the real part of $g(\alpha)$. This fact is useful for the analysis described in the main text.

What we really need in order to pursue the analysis within the text is not necessarily $g(\alpha)$ itself, but rather simply $g(\alpha)$ evaluated for α purely real or purely imaginary. Writing g as $g(z = x + iy) = u(x, y) + iv(x, y)$ we note that

$$g'(z) = \partial_z g(z = x + iy) = \partial_x u(x, y) - i\partial_y u(x, y) \tag{2.126}$$

and hence that

$$u(x, 0) = \int \mathrm{Re} g'(x, y = 0) + D \qquad u(0, y) = -\int \mathrm{Im} g'(x = 0, y) + D' \tag{2.127}$$

where D, D' are constants independent of x and y that depend only on λ and p. Computing $u(x, 0)$ is particularly easy

$$u(x,0) = \frac{\lambda p x^2}{2} \tag{2.128}$$

For $u(0,y)$, we have the integral

$$u(0,\alpha) = -\lambda \int d\alpha \, \left[v\left(e^{i\lambda\alpha}\right) - v\left(e^{-i\lambda\alpha}\right) \right] \tag{2.129}$$

which simplifies to

$$u(0,\alpha) = -\frac{\lambda p \alpha^2}{2} + 2p \int d\alpha \, \arctan\left[\sqrt{e^{\lambda/p} \sec^2 \frac{\lambda\alpha}{2}} \right] \tag{2.130}$$

Chapter 3

Branes, Black Holes and Topological Strings on Toric Calabi-Yau

3.1 Introduction

Recently, Strominger, Ooguri and Vafa [62] made a remarkable conjecture relating four-dimensional BPS black holes in type II string theory compactified on a Calabi-Yau manifold X to the gas of topological strings on X. The conjecture states that the supersymmetric partition function Z_{brane} of the large number N of D-branes making up the black hole, is related to the topological string partition function Z^{top} as

$$Z_{brane} = |Z^{top}|^2,$$

to all orders in 't Hooft $1/N$ expansion. This provides an explicit proposal for what computes the corrections to the macroscopic Bekenstein-Hawking entropy of $d = 4$, $\mathcal{N} = 2$ black holes in type II string theory. Moreover, since the partition function Z_{brane} makes sense for any N, this is providing the non-perturbative completion of the topological string theory on X. A non-trivial test of the conjecture requires knowing topological string partition functions at higher genus on the one hand, and on the other explicit computation of D-brane partition functions. Since neither are known in general, some simplifying circumstances are needed.

Evidence that this conjecture holds was provided in [69] [7] in a special class of local Calabi-Yau manifolds which are a neighborhood of a Riemann surface Σ. The conjecture for black holes preserving 4 supercharges was also tested to leading order in [18] [19] [66] . The conjecture was found to have extensions to 1/2 BPS black holes in compactifications with $\mathcal{N} = 4$ supersymmetry [17] [65, 64] [18] [19] . In [6] the version of the conjecture for open topological strings was formulated.

In this paper we consider black holes on local Calabi-Yau manifolds with torus symmetries. The local geometry with the branes should be thought of as an appropriate decompactification limit of compact ones. While the Calabi-Yau manifold is non-compact, by considering D4-branes which are also non-compact as in [69] [7] , one can keep the entropy of the black hole finite. The non-compactness of the D4 branes turns out to also be the necessary condition to get a large black hole in four dimensions. Because the D-branes are noncompact, different choices of boundary conditions at infinity on the branes give rise to different theories. In particular, in the present setting, a given D4 brane theory cannot be dual to topological strings on all of X, but only to the topological string on the local neighborhood of the D-brane in X. This constrains the class of models that can have non-perturbative completion in terms of D4 branes and no D6 branes, but includes examples such as neighborhood of a shrinking $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$ in X.

The paper has the following organization. In section 2 we review the conjecture of [62] focusing in particular to certain subtleties that are specific to the non-compact Calabi-Yau manifolds. We describe brane configurations which should be dual to topological strings on the Calabi-Yau. In section 3 we explain how to compute the corresponding partition functions Z_{brane}. The D4 brane theory turns out to be described by qYM theory on necklaces and chains of $\mathbf{P}^1$. Where the different $\mathbf{P}^1$'s intersect, one gets insertions of certain observables corresponding to integrating out bifundamental matter from the intersecting D4 branes. The qYM theory is solvable, and corresponding amplitudes can be computed exactly. In section 4 we present our first example of local $\mathbf{P}^2$. We show that the 't Hooft large N expansion of the D-brane amplitude is related to the topological strings on the Calabi-Yau, and moreover and show that the version of the conjecture of [62] that is natural for non-compact Calabi-Yau manifolds [7] is upheld. In section 5 we consider an example of local $\mathbf{P}^1 \times \mathbf{P}^1$. In section 6 we consider N D-branes on (a neighborhood of an) A_k type ALE space. We show that at finite N our results coincide with that of H. Nakajima for Euler characteristics of moduli spaces of $U(N)$ instantons on ALE spaces, while in the large N limit we find precise agreement with the conjecture of [62].

3.2 Black holes on Calabi-Yau manifolds

Consider IIA string theory compactified on a Calabi-Yau manifold X. The effective $d = 4$, $\mathcal{N} = 2$ supersymmetric theory has BPS particles from D-branes wrapping holomorphic cycles in X. We will turn off the D6 brane charge, and consider arbitrary D0, D2 and D4 brane charges.

3.2.1 D-brane theory

Pick a basis of 2-cycles $[C^a] \in H_2(X, Z)$, and a dual basis of 4-cycles $[D_a] \in H_4(X, Z)$, $a = 1, \dots h^{1,1}(X)$,

$$\#(D_a \cap C^b) = \delta_a{}^b.$$

This determines a basis for $h^{1,1}$ $U(1)$ vector fields in four dimensions, obtained by integrating the RR 3-form C_3 on the 2-cycles C^a. Under these $U(1)'s$ D2 branes in class $[C] \in H_2(X, Z)$ and D4 branes in class $[D] \in H_4(X, Z)$ carry electric and magnetic charges $Q_{2\,a}$ and $Q_4{}^b$ respectively:

$$[C] = \sum_a Q_{2\,a}\,[C^a], \qquad [D] = \sum_a Q_4{}^a\,[D_a],$$

We also specify the D0 brane charge Q_0. This couples to the one extra $U(1)$ vector multiplet which originates from RR 1-form.

The indexed degeneracy

$$\Omega(Q_4{}^a, Q_{2\,a}, Q_0)$$

of BPS particles in spacetime with charges Q_0, $Q_{2,a}$, Q_4^a can be computed by counting BPS states in the Yang-Mills theory on the D4 brane [68]. This is computed by the supersymmetric path integral of the four dimensional theory on D in the topological sector with

$$Q_0 = \frac{1}{8\pi^2}\int_D trF\wedge F, \qquad Q_{2\,a} = \frac{1}{2\pi}\int_{C_2^a} trF.$$

Since D is curved, this theory is topologically twisted, in fact it is the Vafa-Witten twist of the maximally supersymetric $\mathcal{N}=4$ theory on D.

3.2.2 Gravity theory

When the corresponding supergravity solution exists, the massive BPS particles are black holes in 4 dimensions, with horizon area given in terms of the charges

$$A_{BH} = \sqrt{\frac{1}{3!}C_{abc}\,{Q_4}^a{Q_4}^b{Q_4}^c|Q_0'|}$$

where C_{abc} are the triple intersection numbers of X, and $Q_0' = Q_0 - \frac{1}{2}C^{ab}Q_{2a}Q_{2b}$.[1] The Bekenstein-Hawking formula relates this to the entropy of the black hole

$$S_{BH} = \frac{1}{4}A_{BH}.$$

For large charges, the macroscopic entropy defined by area, was shown to agree with the microscopic one [68] [49] . The corrections to the entropy-area relation should be suppressed by powers in $1/A_{BH}$ (measured in plank units).

Following [47, 46, 45], Ooguri, Strominger and Vafa conjectured that, just as the leading order microscopic entropy can be computed by the classical area of the horizon and genus zero free energy F_0 of A-model topological string on X, the string loop corrections to the macroscopic entropy can be computed from higher genus topological

[1] $C^{ab}C_{bd} = \delta^a_d, \quad C_{ab} = C_{abc}Q_4^c$

string on X:

$$Z_{YM}(Q_4^a, \varphi^a, \varphi^0) = |Z^{top}(t^a, g_s)|^2 \tag{3.1}$$

where

$$Z_{YM}({Q_4}^a, \varphi^a, \varphi^0) = \sum_{Q_{2\,a}, Q_0} \Omega(Q_4^a, Q_{2\,a}, Q_0)\;\; exp\big(-Q_0\varphi^0 - Q_{2\,a}\varphi^a\big).$$

is the partition function of the $\mathcal{N} = 4$ topological Yang-Mills with insertion of

$$exp\big(-\frac{\varphi^0}{8\pi^2}\int trF\wedge F - \sum_a \frac{\varphi^a}{2\pi}\int \omega_a \wedge trF\big) \tag{3.2}$$

where we sum over all topological sectors.[2] The Kahler moduli of Calabi-Yau,

$$t^a = \int_{C^a} k + iB$$

and the topological string coupling constant g_s are fixed by the attractor mechanism:

$$t^a = (\frac{1}{2}{Q_4}^a + i\varphi^a)\, g_s$$

$$g_s = 4\pi/\varphi^0$$

Moreover, since the loop corrections to the macroscopic entropy are suppressed by powers of $1/N^2$ where $N \sim (C_{abc}Q_4^aQ_4^bQ_4^c)^{1/3}$ [49] the duality in [62] should be a large N duality in the Yang Mills theory.

3.2.3 D-branes for large black holes

Evidence that the conjecture [62] holds was provided in [69] [7] for a very simple class of Calabi-Yau manifolds. We show in this paper that this extends to a broader

[2] Above, ω_a are dual to C^a, $\int_{C^a}\omega_b = {\delta^a}_b$.

class, provided that the classical area of the horizon is large. This imposes a constraint on the divisor D, which is what we turn to next.

Recall that for every divisor D on X there is a line bundle $\mathcal{L}$ on X and a choice of a section s_D such that D is the locus where this section vanishes,

$$s_D = 0.$$

Different choices of the section correspond to homologous divisors on X, so the choice of $[D] \in H_4(X, \mathbf{Z})$ is the choice of the first Chern-class of $\mathcal{L}$ (this is just Poincare duality but the present language will be somewhat more convenient for us) .

The classical entropy of the black hole is large when $[D]$ is deep inside the Kahler cone of X, [49] , i.e. $[D]$ is a "very ample divisor". Then, intersection of $[D]$ with any 2-cycle class on X is positive, which guarantees that

$$C_{abc} t^a t^b t^c \gg 0.$$

Moreover, the attractor values of the Kahler moduli are also large and positive

$$Re(t^a) \gg 0.$$

Interestingly, this coincides with the case when the corresponding twisted $\mathcal{N} = 4$ theory is simple. Namely, the condition that $[D]$ is very ample is equivalent to

$$h^{2,0}(D) > 0.$$

When this holds, [70] , the Vafa-Witten theory can be solved through mass deformation. In contrast, when this condition is violated, the twisted $\mathcal{N} = 4$ theory has lines of marginal stability, where BPS states jump, and background dependence.[3]

[3]We thank C. Vafa for discussions which led to the statements here.

In the next subsection, we will give an example of a toric Calabi-Yau manifold with configurations of D4 branes satisfying the above condition.

3.2.4 An Example

Take X to be

$$X = O(-3) \rightarrow \mathbf{P}^2.$$

This is a toric Calabi-Yau which has a $d = 2\, \mathcal{N} = (2,2)$ linear sigma model description in terms of one $U(1)$ vector multiplet and 4 chiral fields X_i, $i = 0, \ldots 3$ with charges $(-3, 1, 1, 1)$. The Calabi-Yau X is the Higgs branch of this theory obtained by setting the D-term potential to zero,

$$|X_1|^2 + |X_2|^2 + |X_3|^2 = 3|X_0|^2 + r_t$$

and modding out by the $U(1)$ gauge symmetry. The Calabi-Yau is fibered by T^3 tori, corresponding to phases of the four X's modulo $U(1)$. Above, $r_t > 0$ is the Kahler modulus of X, the real part of $t = \int_{C_t} k + iB$. The Kahler class $[k]$ is a multiple of the integral class $[D_t]$ which generates $H^2(X, Z)$, $[k] = r_t\, [D_t]$.

Consider now divisors on X. A divisor in class

$$[D] = Q\, [D_t]$$

is given by zero locus of a homogenous polynomial in X_i of charge Q in the linear sigma model:

$$D: \qquad s_D^Q(X_0, \ldots, X_3) = 0.$$

In fact s_D^Q is a section of a line bundle over X of degree $Q[D_t]$. A generic such divisor breaks the $U(1)^3$ symmetry of X which comes from rotating the T^3 fibers. There are

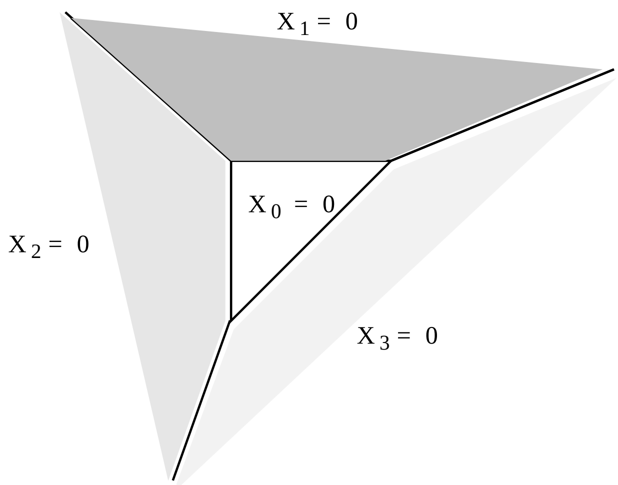

Figure 3.1: Local $\mathbf{P}^2$. We depicted the base of the T^3 fibration which is the interior of the convex polygon in $\mathbf{R}^3$. The shaded planes are its faces.

special divisors which preserve these symmetries, obtained by setting X_i to zero,

$$D_i : \qquad X_i = 0.$$

It follows that $[D_{1,2,3}] = [D_t]$, and that $[D_0] = -3[D_t]$. The divisor D_0 corresponds to the $\mathbf{P}^2$ itself, which is the only compact holomorphic cycle in X. As explained above, we are interested in D4 branes wrapping divisors whose class $[D]$ is positive, $Q = Q_4 > 0$. Since the compact divisors have negative classes, any divisor in this class is non-compact 4-cycle in X. The divisors have a moduli space $\mathcal{M}_Q$, the moduli space of charge Q polynomials, which is very large in this case since X is non-compact and the linear sigma model contains a field X_0 of negative degree. If D were compact, the theory on the D4 brane would involve a sigma model on $\mathcal{M}_Q$. Since D is not compact, in formulating the D4 brane theory we have to pick boundary conditions at infinity. This picks a point in the moduli space $\mathcal{M}_Q$, which is a particular divisor D.

Now, consider the theory on the D4 brane on D. Away from the boundaries of the moduli space $\mathcal{M}_Q$, the theory on the D4 brane should not depend on the choice of the divisor, but only on the topology of D. In the interior of the moduli space, D intersects the $\mathbf{P}^2$ along a curve Σ of degree Q, which is generically an irreducible and smooth curve of genus $g = (Q-1)(Q-2)/2$, and D is a line bundle over it. The theory on the brane is a Vafa-Witten twist of the maximally supersymmetric $\mathcal{N} = 4$ gauge theory with gauge group $G = U(1)$. At the boundaries of the moduli space, Σ and D can become reducible. For example, Σ can collapse to a genus zero, degree Q curve by having $s^Q = X_1^Q$, corresponding to having $D = Q \cdot D_1$. Then D is an $O(-3)$ bundle over $\mathbf{P}^1$, and the theory on the D4 brane wrapping D is the twisted $\mathcal{N} = 4$ theory with gauge group $G = U(Q)$ with scalars valued in the normal bundle to D.

Both of these theories were studied recently in [7] in precisely this context. In both cases, the theory on the D4 brane computes the numbers of BPS bound-states of D0 and D2 brane with the D4 brane. Correspondingly, the topological string which is dual to this in the $1/Q$ expansion describes *only* the maps to X which fall in the neighborhood of D. In other words, the D4 brane theory is computing the non-perturbative completion of the topological string on X_D where X_D is the total space of the normal bundle to D in X. It is not surprising that the YM theory on the (topologically) distinct divisors D gives rise to different topological string theories – because D is non-compact, different choices of the boundary conditions on D give rise to a-priori different QFTs.

It is natural to ask if there is a choice of the divisor D for which we can expect the YM theory theory to be dual to the topological string on $X = O(-3) \rightarrow \mathbf{P}^2$.

Consider a toric divisor in the class $[D] = Q[D_t]$ of the form

$$D = N_1 D_1 + N_2 D_2 + N_3 D_3, \tag{3.3}$$

where $Q = N_1 + N_2 + N_3$ for N_i positive integers. The D4 brane on D will form bound-states with D2 branes running around the edges of the toric base, and arbitrary number of the D0 branes. Recall furthermore that, because X has $U(1)$ symmetries, the topological string on X localizes to maps fixed under the torus actions, i.e. maps that in the base of the Calabi-Yau project to the edges. It is now clear that the D4 branes on D in (3.3) are the natural candidate to give the non-perturbative completion of the topological string on X. We will see in the next sections that this expectation is indeed fully realized.

The considerations of this section suggest that of all the toric Calabi-Yau manifolds, only a few are expected to have non-perturbative completions in terms of D4 branes. The necessary condition translates into having at most one compact 4-cycle in X, so that the topological string on the neighborhood X_D of an ample divisor can agree with the topological string on all of X. Even so, the available examples have highly non-trivial topological string amplitudes, providing a strong test of the conjecture.

3.3 The D-brane partition function

In the previous section we explained that D4-branes wrapping non-compact, toric divisors should be dual to topological strings on the toric Calabi-Yau threefold X. The divisor D in question are invariant under T^3 action on X, and moreover generically reducible, as the local $\mathbf{P}^2$ case exemplifies. In this section we want to understand what is the theory on the D4 brane wrapping D.

Consider the local $\mathbf{P}^2$ with divisor D as in (3.3) . Since D is reducible, the theory on the branes is a topological $\mathcal{N}=4$ Yang-Mills with quiver gauge group $G = U(N_1) \times U(N_2) \times U(N_3)$. The topology of each of the three irreducible components is

$$D_i : \qquad \mathcal{O}(-3) \to \mathbf{P}^1$$

In the presence of more than one divisor, there will be additional bifundamental hypermultiplets localized along the intersections. Here, D_1, D_2 and D_3 intersect pairwise along three copies of a complex plane at $X_i = 0 = X_j$, $i \neq j$.

As shown in [69] [7] , the four-dimensional twisted $\mathcal{N}=4$ gauge theory on

$$\mathcal{O}(-p) \to \mathbf{P}^1$$

with (3.2) inserted is equivalent to a cousin of two dimensional Yang-Mills theory on the base $\Sigma = \mathbf{P}^1$ with the action

$$S = \frac{1}{g_s} \int_\Sigma \operatorname{tr} \Phi \wedge F + \frac{\theta}{g_s} \int_\Sigma \operatorname{tr} \Phi \wedge \omega_\Sigma - \frac{p}{2g_s} \int_\Sigma \operatorname{tr} \Phi^2 \wedge \omega_\Sigma \tag{3.4}$$

where $\theta = \varphi^1/2\pi g_s$. The four dimensional theory localizes to constant configurations along the fiber. The field $\Phi(z)$ comes from the holonomy of the gauge field around the circle at infinity:

$$\int_{fiber} F(z) = \oint_{S^1_{z,\infty}} A(z) = \Phi(z). \tag{3.5}$$

Here the first integral is over the fiber above a point on the base Riemann surface with coordinate z. The (3.4) is the action, in the Hamiltonian form, of a $2d$ YM theory, where

$$\Phi(z) = g_s \frac{\nabla}{\nabla A(z)}$$

is the momentum conjugate to A. However, the theory is *not* the ordinary YM theory in two dimensions. This is because the the field Φ is periodic. It is periodic since it comes from the holonomy of the gauge field at infinity. This affects the measure of the path integral for Φ is such that not Φ but $exp(i\Phi)$ is a good variable. The effect of this is that the theory is a deformation of the ordinary YM theory, the "quantum" YM theory [7] .

Integrating out the bifundamental matter fields on the intersection should, from the two dimensional perspective, correspond to inserting point observables where the $\mathbf{P}^1$'s meet in the $\mathbf{P}^2$ base. We will argue in the following subsections, that the point observable corresponds to

$$\sum_{\mathcal{R}} \mathrm{Tr}_{\mathcal{R}}\, V_{(i)}^{-1}\, \mathrm{Tr}_{\mathcal{R}}\, V_{(i+1)} \tag{3.6}$$

where

$$V_{(i)} = e^{i\,\Phi^{(i)} - i \oint A^{(i)}}, \qquad V_{(i+1)} = e^{i\,\Phi^{(i+1)}}$$

The point observables $\Phi^{(i)}$ and $\Phi^{(i+1)}$ are inserted where the $\mathbf{P}^1$'s intersect, and the integral is around a small loop on $\mathbf{P}^1_i$ around the intersection point. The sum is over all representations $\mathcal{R}$ that exist as representations of the gauge groups on both $\mathbf{P}^1_i$ and $\mathbf{P}^1_{i+1}$. This means effectively one sums over the representations of the gauge group of smaller rank.

By topological invariance of the YM theory, the interaction (3.6) depends only on the geometry near the intersections of the divisors, and not on the global topology. For intersecting non-compact toric divisors, this is universal, independent of either D

or X. In the following subsection we will derive this result.

3.3.1 Intersecting D4 branes

In this subsection we will motivate the interaction (3.6) between D4-branes on intersecting divisors. The interaction between the D4 branes comes from the bi-fundamental matter at the intersection and, as explained above, since the matter is localized and the theory topological, integrating it out should correspond to universal contributions to path integral over D_L and D_R that are independent of the global geometry. Therefore, we might as well take D's, and X itself to be particularly simple, and the simplest choice is two copies of the complex 2-plane $\mathbf{C}^2$ in $X = \mathbf{C}^3$. We can think of the pair of divisors as line bundles fibered over disks C_a and C_b. One might worry that something is lost by replacing Σ by a non-compact Riemann surface, but this is not the case – as was explained in [7] because the theory is topological, we can reconstruct the theory on any X from simple basic pieces by gluing, and what we have at hand is precisely one of these building blocks.

The fields at the intersection $C_{a+b} = D_L \cap D_R$ transform in the bifundamental $(M, \bar{N})$ representations of the $U(M) \times U(N)$ gauge groups on the D-branes. We will first argue that the effect of integrating them out is insertion of

$$\sum_{\mathcal{R}} \mathrm{Tr}_{\mathcal{R}} \exp\left(i \oint_{S_b^1} A^{(L)}\right) \, \mathrm{Tr}_{\mathcal{R}} \exp\left(i \oint_{S_b^1} A^{(R)}\right) \tag{3.7}$$

where $\oint_{S_b^1} A^{(L)}$ and $\oint_{S_b^1} A^{(R)}$ are the holonomies of the gauge fields on D_L and D_R respectively around the circle at infinity on the cap C_{a+b}, i.e. $S_b^1 = \partial C_{a+b}$, see figure 2. (If this notation seems odd, it will stop being so shortly).

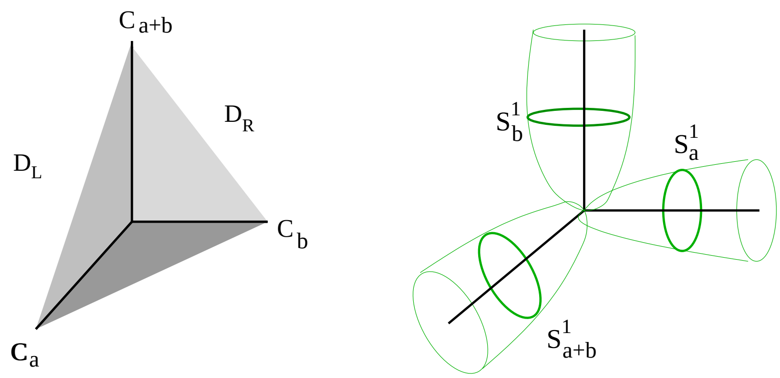

Figure 3.2: D4-branes are wrapped on the divisors $D_{L,R} = \mathbf{C}^2$. The three boldfaced lines in the figure on the left correspond to three disks C_a, C_b, C_{a+b} over which the a, b and $(a+b)$ 1-cycles of the lagrangian $T^2 \times \mathbf{R}$ fibration degenerate. The cycles of the T^2 which are finite are depicted in the figure on right.

We will argue this by consistency as follows.[4] First, note that there is correlation between turning of certain fluxes on D_L and D_R. To see this note that, if one adds D2 branes along C_{a+b}, the D2 branes have the effect of turning on flux on *both* D_L and D_R. Consider for simplicity the case where $M = 1 = N$. The fact that the corresponding fluxes are correlated is the statement that $\int F^{(L)} = \int F^{(R)}$ where integrals are taken over the fibers over a point on C_{a+b} in the divisors D_L and D_R respectively, where we view $D_{L,R}$ as fibrations over C_{a+b}. Since $S^1_b = \partial C_{a+b}$ this is equivalent to

$$\oint_{S^1_{a+b}} A^{(L)} = \oint_{S^1_{a+b}} A^{(R)} \tag{3.8}$$

where S^1_{a+b} is the one cycle in X that *vanishes* over C_{a+b} (this cycle is well defined in X as we will review shortly). This is consistent with insertion of

$$\sum_{n \in \mathbf{Z}} \exp\left(in \oint_{S^1_b} A^{(L)}\right) \exp\left(in \oint_{S^1_b} A^{(R)}\right). \tag{3.9}$$

because $\oint_{S^1_b} A^{(L,R)}$ and $\oint_{S^1_{a+b}} A^{(L,R)}$ are canonically conjugate, (one way to see this is to

[4]We thank C. Vafa for suggesting use of this approach.

consider the qYM theory one gets on C_{a+b}. Then insertion of (3.9) implies (3.8) as an identity inside correlation functions). For general M, N gauge and Weyl invariance imply precisely (3.7) .

We must still translate the operators that that appear in (3.7) , in terms of operators $\Phi^{(L,R)}$ and $A^{(L,R)}$ in the qYM theories on C_a and C_b. This requires understanding of certain aspects of T^3 fibrations. While any toric Calabi-Yau threefold is a lagrangian T^3 fibration, it is also a special lagrangian $T^2 \times \mathbf{R}$ fibration, where over each of the edges in the toric base a (p,q) cycle of the T^2 degenerates. The one-cycle which remains finite over the edge is ambiguous. In the case of $\mathbf{C}^3$, we will choose a fixed basis of finite cycles (up to $SL(2,\mathbf{Z})$ transformations of the T^2 fiber), that will make the gluing rules particularly simple.[5] This is described in figure 2. In the figure, the 1-cycles of the T^2 that vanish over C_a, C_b and C_{a+b} are S^1_a, S^1_b, S^1_{a+b}, respectively. These determine the point observables Φ's in the qYM theories on the corresponding disk. We have chosen a particular basis of the 1-cycles that remain finite. From the figure it is easy to read off that

$$\oint_{S^1_b} A^{(L)} = \oint_{S^1_{a+b}} A^{(L)} - \Phi^{(L)}, \qquad \oint_{S^1_b} A^{(R)} = \Phi^{(R)},$$

which justifies (3.6) . In the next subsection we will compute the qYM amplitudes with these observables inserted.

3.3.2 Partition functions of qYM

Like ordinary two dimensional YM theory, the qYM theory is solvable exactly [7] . In this subsection we will compute the YM partition functions with the insertions

[5] In the language on next subsection, this corresponds to inserting precisely $q^{pC_2(\mathcal{R})}$ to get $\mathcal{O}(-p)$ line bundle

of observables (3.6) . In [7] it was shown that qYM partition function $Z(\Sigma)$ on an arbitrary Riemann surface Σ can be computed by means of operatorial approach. Since the theory is invariant under area preserving diffeomorphisms, knowing the amplitudes for Σ an annulus A, a pant P and a cap C, completely solves the theory – amplitudes on any Σ can be obtained from this by gluing. In the present case, we will only need the cap and the annulus amplitudes, but with insertions of observables. Since the Riemann surfaces in question are embedded in a Calabi-Yau, we are effectively sewing Calabi-Yau manifolds, so one also has to keep track of the data of the fibration. The rules of gluing a Calabi-Yau manifold out of $\mathbf{C}^3$ patches are explained in [4] and we will only spell out their consequences in the language of 2d qYM.

In the previous subsection, the theory on divisors D_L and D_R in $\mathbf{C}^3$ was equivalent to qYM theories on disks C_a and C_b, with some observable insertions. These are Riemann surfaces with a boundary, so the corresponding path integrals define states in the Hilbert space of qYM theory on S^1. Keeping the holonomy $U = Pe^{i\oint A}$ fixed on the boundary, the corresponding wave function can be expressed in terms of characters of irreducible representations $\mathcal{R}$ of $U(N)$ as:

$$Z(U) = \sum_{\mathcal{R}} Z_{\mathcal{R}} \, \mathrm{Tr}_{\mathcal{R}} U$$

The first thing we will answer is how to compute the corresponding states, and then we will see how to glue them together. As we saw in the previous section, the choice of the coordinate $\oint_{S^1} A$ on the boundary is ambiguous, as the choice of the cycle which remains finite is ambiguous. This ambiguity is related to the choice of the Chern class of a line bundle over a non-compact Riemann surface, i.e. how the divisors $D_{L,R}$ are fibered over the corresponding disks. The simplest choice is the one that gives trivial fibration, and this is the one we made in figure 2 (this corresponds to picking the cycle that vanishes over C_{a+b}).

The partition function on a disk with trivial bundle over it and no insertions is

$$Z(C)(U) = \sum_{\mathcal{R} \in U(N)} S_{0\mathcal{R}} \ e^{i\theta C_1(\mathcal{R})} \, Tr_{\mathcal{R}} U. \tag{3.10}$$

Above, $C_1(\mathcal{R})$ is the first casimir of the representation $\mathcal{R}$, and $S_{\mathcal{R}\mathcal{P}}(N, g_s)$ is a relative of the S-matrix of the $U(N)$ WZW model

$$S_{\mathcal{R}\mathcal{Q}}(N, g_s) = \sum_{w \in S_N} \epsilon(w) q^{-(\mathcal{R}+\rho_N)\cdot w(\mathcal{Q}+\rho_N)}, \tag{3.11}$$

where

$$q = exp(-g_s)$$

and S_N is Weyl group of $U(N)$ and ρ_N is the Weyl vector.[6]

Sewing Σ_L and Σ_R is done by

$$Z(\Sigma_L \cup \Sigma_R) = \int dU \ Z(\Sigma_L)(U) \ Z(\Sigma_R)(U^{-1}) = \sum_{\mathcal{R}} Z_{\mathcal{R}}(\Sigma_L) Z_{\mathcal{R}}(\Sigma_R)$$

For example, the amplitude corresponding to $\Sigma = \mathbf{P}^1$ with $O(-p)$ bundle over it and no insertions can be obtained by gluing two disks and an annulus with $O(-p)$ bundle over it:

$$Z(A,p)(U_1, U_2) = \sum_{\mathcal{R} \in U(N)} q^{pC_2(\mathcal{R})/2} \ e^{i\theta C_1(\mathcal{R})} \ Tr_{\mathcal{R}} U_1 \ Tr_{\mathcal{R}} U_2 \tag{3.12}$$

This gives

$$Z(P^1, p) = \sum_{\mathcal{R}} (S_{0\mathcal{R}})^2 q^{pC_2(\mathcal{R})/2} e^{i\theta C_1(\mathcal{R})} \tag{3.13}$$

[6]The normalization of the path integral is ambiguous. In our examples in sections 4-6 we will choose it in such a way that the amplitudes agree with the topological string in the large N limit.

In addition we will need to know how to compute expectation values of observables in this theory. As we will show in the appendix B, the amplitude on a cap with a trivial line bundle and observable $\mathrm{Tr}_Q\, e^{i\Phi - in\oint_{S^1} A}$ inserted equals

$$Z(C,\, \mathrm{Tr}_{\mathcal{Q}}\, e^{i\Phi - in\oint_{S^1} A})(U) = \sum_{\mathcal{R}} q^{\frac{n}{2}C_2(\mathcal{Q})} S_{\bar{\mathcal{Q}}\mathcal{R}}(N, g_s) \mathrm{Tr}_{\mathcal{R}} U. \tag{3.14}$$

where U is the holonomy on the boundary.

It remains to compute the expectation value of the observables in (3.6) in the two-dimensional theory on C_a and C_b. The amplitude on the intersecting divisors D_L, D_R is

$$\begin{aligned} Z(V)(U^{(L)}, U^{(R)}) &= \sum_{\mathcal{Q}\in U(M), \mathcal{P}\in U(N)} V_{\mathcal{Q}\mathcal{P}}(M,N) Tr_{\mathcal{Q}} U^{(L)} Tr_{\mathcal{P}} U^{(R)} \\ V_{\mathcal{Q}\mathcal{P}}(M,N) &= \sum_{\mathcal{R}\in U(M)} S_{\mathcal{Q}\bar{\mathcal{R}}}(M, g_s)\, q^{\frac{1}{2}C_2^{(M)}(\mathcal{R})}\, S_{\mathcal{R}\mathcal{P}}(N, g_s). \end{aligned} \tag{3.15}$$

In the above, $U^{(L,R)}$ is the holonomy at the boundary of C_a and C_b.

When $M = N$, there is a simpler expression for the vertex amplitude in (3.15). Using the definition of $S_{\mathcal{P}\mathcal{R}}$ (3.11) and summing over $\mathcal{R}$ we have

$$V_{\mathcal{P}\mathcal{Q}} = \theta^N(q)\; q^{-\frac{1}{2}C_2(\mathcal{P})}\; S_{\overline{\mathcal{P}}\mathcal{Q}} q^{-\frac{1}{2}C_2(\mathcal{Q})} \tag{3.16}$$

and where $\theta(q) = \sum_{m\in\mathbf{Z}} q^{\frac{m^2}{2}}$. This is related to the familiar realization in WZW models of the relation

$$STS = (TST)^{-1}$$

between $SL(2,\mathbf{Z})$ generators S and T in WZW models where

$$T_{\mathcal{R}\mathcal{Q}} = q^{\frac{1}{2}C_2(\mathcal{R})}\delta_{\mathcal{R}\mathcal{Q}}, \quad S^{-1}_{\mathcal{R}\mathcal{P}}(g_s, N) = S_{\mathcal{R}\mathcal{P}}(-g_s, N) = S_{\bar{\mathcal{R}}\mathcal{P}}(g_s, N). \tag{3.17}$$

The difference is that there is no quantization of the level k here. Even at a non-integer level, this is more straightforward in the $SU(N)$ case, where the theta function in (3.16) would not have appeared.

3.3.3 Modular transformations

The partition functions of D4 branes on various divisors with chemical potentials

$$S_{4d} = \frac{1}{2g_s}\int \mathrm{tr}\, F\wedge F + \frac{\theta}{g_s}\int \mathrm{tr}\, F\wedge\omega,$$

turned on, are computing degeneracies of bound-states of Q_2 D2 branes and Q_0 D0 branes with the D4 branes, where

$$Q_0 = \frac{1}{8\pi^2}\int \mathrm{tr}\, F\wedge F, \quad Q_2 = \frac{1}{2\pi}\int \mathrm{tr}\, F\wedge\omega, \tag{3.18}$$

so the YM amplitudes should have an expansion of the form

$$Z^{q\mathrm{YM}} = \sum_{q_0,q_1}\Omega(Q_0,Q_2,Q_4)\exp\left[-\frac{4\pi^2}{g_s}Q_0 - \frac{2\pi\theta}{g_s}Q_2\right]. \tag{3.19}$$

The amplitudes we have given are not expansions in $exp(-1/g_s)$, but rather in $exp(-g_s)$, so the existence of the (3.19) expansion is not apparent at all. The underlying $\mathcal{N}=4$ theory however has S duality that relates strong and weak coupling expansions, so we should be able to make contact with (3.19) .

Since amplitudes on more complicated manifolds are obtained from the simpler ones by gluing, it will suffice for us to show this for the propagators, vertices and caps. Consider the annulus amplitude (3.12) Using the Weyl-denominator form of the $U(N)$ characters $Tr_{\mathcal{R}}U = \Delta_H(u)^{-1}\sum_{w\in S_N}(-)^{\omega}e^{\omega(iu)\cdot(\mathcal{R}+\rho_N)}$ we can rewrite $Z(A,p)$ as

$$Z(A,p)(U,V) = \Delta_H(u)^{-1}\Delta_H(v)^{-1}\sum_{n\in\mathbf{Z}^N}\sum_{w\in S_N} q^{\frac{p}{2}n^2}e^{n(iu-w(iv))}$$

which is manifestly a modular form,[7] which we can write

$$Z(A,p)(U,V) = \Delta_H(u)^{-1}\Delta_H(v)^{-1}\left(\frac{g_s p}{2\pi}\right)^{-\frac{N}{2}} \sum_{m\in\mathbf{Z}^N}\sum_{w\in S_N} \tilde{q}^{\frac{1}{2p}\left(m-\frac{u-w(v)}{2\pi}\right)^2} \tag{3.20}$$

where in terms of $\tilde{q} = e^{-4\pi^2/g_s}$. In the above, the eigenvalues U_i of U are written as $U_i = exp(iu_i)$, and $\Delta_H(u)$ enters the Haar measure:

$$\int dU = \int \prod_i du_i \Delta_H(u)^2$$

Note that, in gluing, the determinant $\Delta_H(u)^2$ factors cancel out, and simple degeneracies will be left over.

Similarly, the vertex amplitude (3.15) corresponding to intersection of N and M D4 branes can be written as (see appendix C for details):

$$Z(U,V) = \Delta_H(u)^{-1}\Delta_H(v)^{-1}\theta^M(q)\sum_{m\in\mathbf{Z}^M} q^{-\frac{1}{2}m^2}e^{m\cdot v}\sum_{w\in S_N}(-)^w\sum_{n\in\mathbf{Z}^N} e^{n(\cdot w(iu)+iv-g_s(\rho_N-\rho_M))} \tag{3.21}$$

where v, ρ_M are regarded as N dimensional vectors, the last $N-M$ of whose entries are zero. We see that $Z(U,V)$ is given in terms of theta functions, so it is modular form, its modular transform given by

$$Z(U,V) = \Delta_H(u)^{-1}\Delta_H(v)^{-1}\left(\frac{g_s}{2\pi}\right)^{-M/2}\theta^M(\tilde{q})\sum_{m\in\mathbf{Z}^M}\tilde{q}^{-\frac{1}{2}(m+iv/2\pi)^2} \sum_{w\in S_N}(-)^w\sum_{n\in\mathbf{Z}^N} e^{n(\cdot w(iu)+iv-g_s(\rho_N-\rho_M))} \tag{3.22}$$

In a given problem, it is often easier to compute the degeneracies of the BPS states from the amplitude as a whole, rather than from the gluing the S-dual amplitudes as in (3.22) . Nevertheless, modularity at the level of vertices, propagators and caps, demonstrates that the $1/g_s$ expansion of our amplitudes does exist in a general case.

[7] Recall, $\theta(\tau,u) = (-i\tau)^{-\frac{1}{2}}e^{-i\pi\frac{u^2}{\tau}}\theta(-\frac{1}{\tau},\frac{u}{\tau})$, where $\theta(\tau,u) = \sum_{n\in\mathbf{Z}} e^{i\tau n^2}e^{2\pi i u}$.

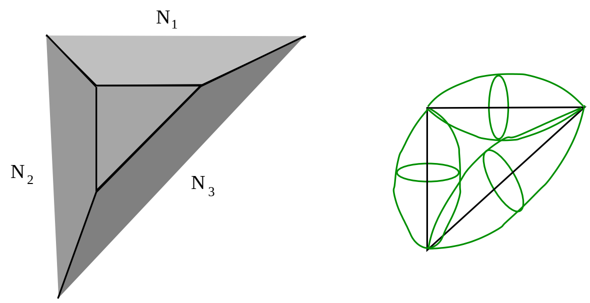

Figure 3.3: Local $\mathbf{P^2}$, depicted as a toric web diagram. The numbers of D4 branes wrapping the torus invariant non-compact 4-cycles are specified.

3.4 Branes and black holes on local $\mathbf{P}^2$.

We will now use the results of the previous section to study black holes on $X = O(-3) \to \mathbf{P}^2$. As explained in section 2, to get large black holes on $\mathbf{R}^{3,1}$ we need to consider D4 branes wrapping very-ample divisors on X, which are then necessarily non-compact. Moreover, the choice of divisor D that should give rise to a dual of topological strings on X corresponds to

$$D = N_1 D_1 + N_2 D_2 + N_3 D_3$$

where D_i, $i = 1, 2, 3$ are the toric divisors of section 2.

Using the results of section 3, it is easy to compute the amplitudes corresponding to the brane configuration. We have $N_1 \geq N_2 \geq N_3$ D4 branes on three divisors of topology $D_i = O(-3) \to \mathbf{P}^1$. From each, we get a copy of quantum Yang Mills theory on $\mathbf{P}^1$ with $p = 3$, as discussed in section 3. From the matter at the intersections, we get in addition, insertion of observables (3.6) at two points in each $\mathbf{P}^1$.

All together this gives:

$$Z_{qYM} = \alpha \sum_{\mathcal{R}_i \in U(N_i)} V_{\mathcal{R}_2\mathcal{R}_1}(N_2, N_1)\, V_{\mathcal{R}_3\mathcal{R}_2}(N_3, N_2)\, V_{\mathcal{R}_3\mathcal{R}_1}(N_3, N_1)\, \prod_{j=1}^{3} q^{\frac{3C_2(\mathcal{R}_i)}{2}} e^{i\theta_i C_1(\mathcal{R}_i)} \tag{3.23}$$

Note that in the physical theory there should be only one chemical potential for D2-branes, corresponding to the fact that $H_2(X, \mathbf{Z})$ is one dimensional. In the theory of the D4 brane we $H_2(D, \mathbf{Z})$ is three dimensional, generated by the 3 $\mathbf{P}^1$'s in D – the three chemical potentials θ_i above couple to the D2 branes wrapping these. While all of these D2 branes should correspond to BPS states in the Yang-Mills theory, not all of them should correspond to BPS states once the theory is embedded in the string theory. Because the three $\mathbf{P}^1$'s that the D2 brane wrap are all homologous in $H_2(X, \mathbf{Z})$,

$$[\mathbf{P}^1_1] - [\mathbf{P}^1_3] \sim 0, \qquad [\mathbf{P}^1_2] - [\mathbf{P}^1_3] \sim 0$$

there will be D2 brane instantons that can cause those BPS states that carry charges in $H_2(D, \mathbf{Z})$ to pair up into long multiplets. Decomposing $H_2(D, \mathbf{Z})$ into a $H_2(D, \mathbf{Z})^{||} = H_2(X, \mathbf{Z})$ and $H_2(D, \mathbf{Z})^{\perp}$, it is natural to turn off the the chemical potentials for states with charges in $H_2(D, \mathbf{Z})^{\perp}$. This corresponds to putting

$$\theta_i = \theta, \qquad i = 1, 2, 3.$$

For some part, we will keep the θ-angles different, but there is only one θ natural in the theory.

The normalization α of the path integral is chosen in such a way that Z_{qYM} has chiral/anti-chiral factorization in the large N_i limit (see 4.6 and 4.10 below).

$$\alpha = q^{-(\rho_{N_2}^2 + \frac{N_2}{24})} q^{-2(\rho_{N_3}^2 + \frac{N_3}{24})}\, e^{\frac{(N_1+N_2+N_3)\theta^2}{6g_s}}\, q^{\frac{(N_1+N_2+N_3)^3}{72}}$$

The partition function simplifies significantly if we take equal numbers of the D4 branes on each D_i,

$$N_i = N, \quad i = 1, 2, 3$$

since in this case, we can replace (3.15) form of the vertex amplitude with the simpler (3.16) , and the D-brane partition function becomes

$$Z_{qYM} = \alpha\, \theta^{3N}(q) \sum_{\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3 \in U(N)} S_{\mathcal{R}_1\bar{\mathcal{R}}_2}(g_s, N)\, S_{\mathcal{R}_2\bar{\mathcal{R}}_3}(g_s, N)\, S_{\mathcal{R}_3\bar{\mathcal{R}}_1}(g_s, N) \prod_{j=1}^{3} q^{\frac{C_2(\mathcal{R}_i)}{2}} e^{i\theta_i C_1(\mathcal{R}_i)} \tag{3.24}$$

In the following subsections we will first take the large N_i limit of Z_{qYM} to get the closed string dual of the system. We will then use modular properties of the partition function to compute the degeneracies of the BPS states of D0-D2-D4 branes.

3.4.1 Black holes from local $\mathbf{P}^2$

According to the conjecture of [62] (or more precisely, its version for the non-compact Calabi-Yau manifolds proposed in [7]) the large N limit of the D-brane partition function Z_{brane}, which in our case equals Z_{qYM}, should be given by

$$Z_{qYM}(D, g_s, \theta) \approx \sum_{\alpha} |Z^{top}_{\alpha}(t, g_s)|^2$$

where

$$t = \frac{1}{2}(N_1 + N_2 + N_3)g_s - i\theta$$

since $[D] = (N_1+N_2+N_3)[D_t]$ where $[D_t]$ is dual to the class that generates $H_2(X, \mathbf{Z})$. In the above, the two expressions should equal up to terms of order $O(exp(-1/g_s))$, hence the “approximate” sign. The sum over α is the sum over chiral blocks which should correspond to the boundary conditions at infinity of X. More precisely, the leading chiral block should correspond to including only the normalizable modes of topological string on X, which count holomorphic maps to $\mathbf{P}^2$, the higher ones containing fluctuation in the normal direction [7]ANV . We will see below that this prediction is realized precisely.

The Hilbert space of the qYM theory, spanned by states labeled by representations $\mathcal{R}$ of $U(N)$, at large N splits into

$$\mathcal{H}^{qYM} \approx \oplus_\ell \, \mathcal{H}_\ell^+ \otimes \mathcal{H}_\ell^-$$

where $\mathcal{H}_l^+$ and $\mathcal{H}_l^-$ are spanned by representations R_+ and R_- with small numbers of boxes as compared to N, and ℓ is the $U(1)$ charge. Correspondingly, the qYM partition function also splits as

$$Z_{qYM} \approx \sum_\ell \, Z_\ell^+ \, Z_\ell^-,$$

where $Z_\ell^\pm$ are the chiral and anti-chiral partitions. We will now compute these, and show that they are given by topological string amplitudes.

i. The $N_i = N$ case.

We'll now compute the large N limit of the D-brane partition function (3.24) for $N_i = N$, $i = 1, 2, 3$. At large N, the $U(N)$ Casimirs in representation $\mathcal{R} = R_+ \bar{R}_-[\ell_R]$ are given by

$$\begin{aligned} C_2(\mathcal{R}) &= \kappa_{R+} + \kappa_{R-} + N(|R_+| + |R_-|) + N{\ell_R}^2 + 2\ell_R(|R_+| - |R_-|), \\ C_1(\mathcal{R}) &= N\ell_R + |R_+| - |R_-| \end{aligned} \tag{3.25}$$

where

$$\kappa_R = \sum_{i=1}^{N-1} R_i(R_i - 2i + 1)$$

and $|R|$ is the number of boxes in R.

The S-matrix $S_{\mathcal{RQ}}$ is at large N given in [6]

$$\begin{aligned} q^{-(\rho^2+\frac{N}{24})} S_{\mathcal{RQ}}(-g_s, N) =& M(q^{-1})\eta(q^{-1})^N (-)^{|R_+|+|R_-|+|Q_+|+|Q_-|} \\ & \times q^{N\ell_R\ell_Q} q^{\ell_Q(|R_+|-|R_-|)} q^{\ell_R(|Q_+|-|Q_-|)} q^{\frac{N(|R_+|+|R_-|+|Q_+|+|Q_-|)}{2}} \\ & \times q^{\frac{\kappa_{R_+}+\kappa_{R_-}}{2}} \sum_P q^{-N|P|}(-)^{|P|} \hat{C}_{Q_+^T R_+ P}(q) \hat{C}_{Q_-^T R_- P^T}(q). \end{aligned} \tag{3.26}$$

The amplitude $\hat{C}_{RPQ}(q)$ is the topological vertex amplitude of [4] .[8] In (3.26) $M(q)$ and $\eta(q)$ are MacMahon and Dedekind functions.

Putting this all together, let us now parameterize the integers ℓ_{R_i} as follows

$$3\ell = \ell_{R_1} + \ell_{R_2} + \ell_{R_3}, \quad 3n = \ell_{R_1} - \ell_{R_3}, \quad 3k = \ell_{R_2} - \ell_{R_3}.$$

It is easy to see that the sum over n and k gives delta functions: at large N

$$Z_{qYM}(\theta_i, g_s) \sim \delta\big(\, N(\theta_1 - \theta_3)\,\big)\ \delta\big(\, N(\theta_2 - \theta_3)\,\big) \times\ Z_{qYM}^{finite}(\theta, g_s) \tag{3.27}$$

where $\theta_i = \theta$ in the finite piece. As we will show in Sec. 4.2 there is the same δ-function singularity as in the partition function of the bound-states of N D4 branes. There it will be clear that it comes from summing over D2 branes with charges in $H_2(X, \mathbf{D})^\perp$, as mentioned at the beginning of this section. The finite piece in (3.27) is given by

$$Z_{qYM}^{finite}(N, \theta, g_s) = \sum_{m\in\mathbf{Z}} \sum_{P_1,P_2,P_3} (-)^{\sum_{i=1}^3 |P_i|} Z^+_{P_1,P_2,P_3}\big(t + mg_s\big) Z^+_{P_1^T,P_2^T,P_3^T}\big(\bar{t} - mg_s\big). \tag{3.28}$$

[8] The conventions of this paper and [4] differ, as here $q = e^{-g_s}$, but $q_{there} = e^{g_s}$, consequently the topological vertex amplitude C_{RPQ} of [4] is related to the present one by $\hat{C}_{RPQ}(q) = C_{RPQ}(q^{-1})$.

The chiral block in (3.28) is the topological string amplitude on $X = O(-3) \to \mathbf{P}^2$,

$$Z^{+}_{P_1,P_2,P_3}(t) = \hat{Z}_0(g_s,t)e^{-t_0\sum_i |P_i|} \sum_{R_1,R_2,R_3} e^{-t\sum_i |R_i|} q^{\sum_i \kappa_{R_i}} \hat{C}_{R_2^T R_1 P_1^T}(q)\, \hat{C}_{R_3^T R_2 P_2^T}(q)\, \hat{C}_{R_1^T R_3 P_3^T}(q) \tag{3.29}$$

where $t_0 = -\frac{1}{2}Ng_s$ and the Kahler modulus t is (we will return to the meaning of t_0 shortly):

$$t = \frac{3Ng_s}{2} - i\theta.$$

More precisely, the chiral block with trivial ghosts $P_i = 0$,

$$Z^{+}_{0,0,0}(t,g_s) = Z^{top}(t,g_s)$$

is exactly equal to the perturbative closed topological string partition function for $X = O(-3) \to \mathbf{P}^2$, as given in [4] . This exactly agrees with the prediction of [62] .

The prefactor $\hat{Z}_0(g_s,t)$ is given by

$$\hat{Z}_0(g_s,t) = e^{-\frac{t^3}{18g_s^2}}\; M^3(q^{-1})\eta^{\frac{t}{g_s}}(q^{-1})\theta^{\frac{t}{g_s}}(q)$$

As explained in [7] the factor $\eta^{\frac{t}{g_s}} \sim \eta^{\frac{3N}{2}}$ comes from bound states of D0 and D4 branes [70] without any D2 brane charge, and moreover, it has only genus zero contribution perturbatively.

$$\eta^{\frac{t}{g_s}} \sim exp\left(-\frac{\pi^2 t}{6g_s^2}\right) + (non-perturbative)$$

The factor $\theta^{\frac{t}{g_s}}$ comes from the bound states of D4 branes with D2 branes along each of three the non-compact toric legs in the normal direction to the $\mathbf{P}^2$, and without any D0 branes. This gives no perturbative contributions

$$\theta^{\frac{t}{g_s}} \sim 1 + (non-perturbative)$$

The subleading chiral blocks correspond to open topological string amplitudes in X with D-branes along the fiber direction to the $\mathbf{P}^2$, which can be computed using the topological vertex formalizm [4] . The appearance of D-branes was explained in [6] where they were interpreted as non-normalizable modes of the topological string amplitudes on X. The reinterpretation in terms of non-normalizable modes of the topological string theory is a consequence of the open-closed topological string duality on [1] . While this is a duality in the topological string theory, in the physical string theory the open and closed string theory are the same only provided we turn on Ramond-Ramond fluxes. We cannot do this here however, since this would break supersymmetry, and the only correct interpretation is the closed string one.

To make contact with this, define

$$Z^{+}(U_1, U_2, U_3) = \sum_{R_1,R_2,R_3} Z^{+}_{R_1,R_2,R_3} \, Tr_{R_1}U_1 \, Tr_{R_2}U_2 \, Tr_{R_3}U_3.$$

where U_i are unitary matrices. This could be viewed as an open topological string amplitude with D-branes, or more physically, as the topological string amplitude, with non-normalizable deformations turned on. These are not most general non-normalizable deformations on X, but only those that preserve torus symmetries – correspondingly they are localized along the non-compact toric legs, just like the topological D-branes that are dual to them are. The non-normalizable modes of the geometry can be identified with [1]

$$\tau_i^n = g_s tr(U_i^n)$$

where the trace is in the fundamental representation. We can then write (3.28) as

$$Z^{finite} \sim \int dU_1 \, dU_2 \, dU_3 \, |Z^{+}(U_1, U_2, U_3)|^2$$

where we integrate over *unitary* matrices *provided* we shift

$$U \to U e^{-t_0}$$

where $t_0 = -\frac{1}{2} N g_s$. This shift is the attractor mechanism for the non-normalizable modes of the geometry [6] . In terms of the natural variables ${t^n}_i$, related by $\tau_i^n = exp(-t_i^n)$ to τ's we have

$$t_i^n = n t_0 \qquad (3.30)$$

This comes about as follows [6] . First note that size of any 2-cycle C in the geometry should be fixed by the attractor mechanism to equal its intersection with the 4-cycle class $[D]$ of the D4 branes, in this case $[D] = 3N[D_t]$. The relevant 2-cycle in this case is a disk C_0 ending on the topological D-brane. The real part of t_i^n measures the size of an n-fold cover of this disk (there is no chemical potential, i.e. t_0 is real, since there is associated BPS state of finite mass). Then (3.30) follows because

$$\#(C_0 \cap D) = -N.$$

To see this note that in homology, the class $3N[D_t]$ could equally well be represented by $-N$ D-branes on the base $\mathbf{P}^2$ and the latter has intersection number 1 with C_0. The factor of n in (3.30) comes about since t^n corresponds to the size of the n-fold cover of the disk.

ii. The general N_i case.

The case $N_1 > N_2 > N_3$ is substantially more involved, and in particular, the large N limit of the amplitudes (3.15) (3.23) is not known. However, as we will explain in the appendix D, *turning off* the $U(1)$ factors of the gauge theory, the large N limit can be computed, and we find a remarkable agreement with the conjecture of [62] .

Let us focus on the leading chiral block of the amplitude. The large N, M limit of the interaction $V_{QR}(M,N)$ (more precisely, the modified version of it to turn off the $U(1)$ charges) is

$$V_{\mathcal{QR}} \sim \; \beta_M \; q^{\frac{(|Q_+|+|Q_-|)(N-M)}{2}} q^{\frac{(|R_+|+|R_-|)(M-N)}{2}} \\ q^{-\frac{(\kappa_{R_+}+\kappa_{R_-}+\kappa_{Q_+}+\kappa_{Q_-})}{2}} W_{Q_+R_+}(q)\; W_{Q_-R_-}(q) \tag{3.31}$$

where

$$\beta_M = q^{\left(\rho_M^2+\frac{M}{24}\right)} M(q^{-1})\eta^M(q^{-1})\theta^M(q)$$

In (3.59) the W_{PR} is related to the topological vertex amplitude as $W_{PR}(q) = (-)^{|P|+|R|}\hat{C}_{0P^TR}(q)q^{\kappa_R/2}$. It is easy to see that for $N = M$ this agrees with the large N limit of the simpler form of the $V_{\mathcal{RQ}}$ amplitude in (3.16). It is easy to see that that the leading chiral block of (3.23) is

$$Z_{qYM} \sim Z^+_{0,0,0}\big(t\big)\, Z^-_{0,0,0}\big(\bar{t}\big) \tag{3.32}$$

where $Z^+_{0,0,0}(t)$ is

$$Z^+_{0,0,0} = \hat{Z}_0 \sum_{R_+,Q_+,p_+} W_{R_+Q_+}(q)W_{Q_+P_+}(q)W_{P_+R_+}(q)e^{-t(|R_+|+|Q_+|+|P_+)}$$

which is the closed topological string amplitude on X. In particular, this agrees with the amplitude in (3.29) . In the present context, the Kahler modulus t is given by

$$t = \frac{1}{2}(N_1 + N_2 + N_3)g_s - i\theta.$$

This is exactly as dictated by the attractor mechanism corresponding to the divisor $[D] = (N_1 + N_2 + N_3)[D_t]$!

The higher chiral blocks will naturally be more involved in this case. Some of the intersection numbers fixing the attractor positions of ghost branes are ambiguous, and correspondingly, far more complicated configurations of non-normalizable modes are expected.

3.4.2 Branes on local $\mathbf{P}^2$

In this and subsequent section we will discuss the degeneracies of BPS states that follow from (3.23) . Using the results of (3.20) and (3.21) or by direct computation, it is easy to see that Z_{qYM} is a modular form. Its form however is the simplest in the case

$$N_1 = N_2 = N_3 = N,$$

so let us treat this first.

i. Degeneracies for $N_i = N$.

In this case, the form of the partition function written in (3.24) is more convenient. By trading the sum over representations and over the Weyl-group, as in (3.21) , for sums over the weight lattices, the partition function of BPS states is

$$Z_{qYM}(N,\theta_i,g_s) = \beta \sum_{w\in S_N} (-)^w \sum_{n_1,n_2,n_3\in Z^N} q^{\frac{1}{2}\sum_{i=1}^3 n_i^2}\ q^{w(n_1)\cdot n_2+n_2\cdot n_3+n_3\cdot n_1}\ e^{i\sum_{i=1}^3 \theta_i e(N)\cdot n_i} \tag{3.33}$$

where $e(N) = (1,\ldots,1)$ and $\beta = \alpha\theta^{3N}(q)$. The amplitudes depend on the permutations w only through their conjugacy classes, consequently we have:

$$Z_{qYM} = \beta \sum_{\vec{K}} d(\vec{K})\ Z_{K_1} \times \ldots \times Z_{K_r} \tag{3.34}$$

where $\vec{K}$ labels a partition of N into natural numbers $N = \sum_{a=1}^{r} K_a$, and $d(\vec{K})$ is the number of elements in the conjugacy class of S_N, the permutation group of N elements, corresponding to having r cycles of length K_a, $a = 1, \dots, r$, and

$$Z_K(\theta_i, g_s) = (-)^{w_K} \sum_{n_1,n_2,n_3 \in \mathbf{Z}^K} q^{\frac{1}{2}\sum_{i=1}^3 n_i^2}\; q^{w_K(n_1)\cdot n_2 + n_2\cdot n_3 + n_3 \cdot n_1}\; e^{i\sum_{i=1}^3 \theta_i e(K)\cdot n_i} \tag{3.35}$$

Here w_K stands for cyclic permutation of K elements. Note that the form of the partition function (3.34) suggests that Z_{qYM} is counting not only BPS bound states, but also contains contribution from marginally bound states corresponding to splitting of the $U(N)$ to

$$U(N) \to U(K_1) \times U(K_2) \times \ldots \times U(K_r)$$

In each of the sectors, the quadratic form is degenerate. The contribution of bound states of N branes Z_N diverges as

$$Z_N(\theta_i, g_s) \sim \sum_{m_1,m_2 \in \mathbf{Z}} e^{iNm_1(\theta_1-\theta_3)} e^{iNm_2(\theta_2-\theta_3)} = \delta\big(N(\theta_1-\theta_3)\big)\delta\big(N(\theta_2-\theta_3)\big)$$

This is exactly the type of the divergence we found at large N in the previous subsection. This divergence should be related to summing over D_2 branes with charges in $H_2(D, \mathbf{Z})^{\perp}$ – these apparently completely decouple from the rest of the theory.

More precisely, writing $U(N) = U(1) \times SU(N)/\mathbf{Z}_N$, this will have a sum over 't Hooft fluxes which are correlated with the fluxes of the $U(1)$. Then, Z_N is a sum over sectors of different N-ality,

$$Z_N(\theta_i, g_s) = (-)^{w_N} \sum_{L_i=0}^{N-1} \sum_{\ell_i \in \mathbf{Z}+\frac{L_i}{N}} q^{\frac{N}{2}(\ell_1+\ell_2+\ell_3)^2} e^{iN\sum_i \theta_i \ell_i} \sum_{m \in \mathbf{Z}^{3(N-1)}+\vec{\xi}(L_i)} q^{\frac{1}{2}m^T \mathcal{M}_N m}$$

where $\mathcal{M}_N$ is a non-degenerate $3(N-1) \times 3(N-1)$ matrix with integer entries and

$\vec{\xi}_i$ is a shift of the weight lattice corresponding to turning on 't Hooft flux. Explicitly,

$$\xi_i^a = \frac{N-a}{N} L_i, \quad i = 1,2,3 \quad a = 0, \ldots N-1$$

where $\mathcal{M}_N$ is $3(N-1) \times 3(N-1)$ matrix

$$\mathcal{M}_N = \begin{pmatrix} M_N & W_N & M_N \\ W_N^T & M_N & M_N \\ M_N & M_N & M_N \end{pmatrix} \tag{3.36}$$

whose entries are

$$M_N = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & -1 & 2 \end{pmatrix} \tag{3.37}$$

and

$$W_N = \begin{pmatrix} -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ 0 & 0 & -1 & \ldots & . & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ -1 & 0 & 0 & \ldots & 0 & -1 \end{pmatrix} \tag{3.38}$$

We can express Z_N in terms of Θ-functions

$$Z_N(\theta_i, g_s) = (-)^{w_N} \delta\big(N(\theta_1 - \theta_3)\big) \delta\big(N(\theta_2 - \theta_3)\big) \sum_{L_i=0}^{N-1} \Theta_1[a(L_i), b](\tau)\ \Theta_{3N-3}[\mathbf{a}(L_i), \mathbf{b}](\hat{\tau})$$

where

$$\Theta_k[a,b](\tau) = \sum_{n\in\mathbf{Z}^k} e^{\pi i\tau(n+a)^2}\ e^{2\pi inb}$$

and

$$\tau = \frac{ig_s}{2\pi}N, \qquad \hat{\tau} = \frac{ig_s}{2\pi}\mathcal{M}_N$$
$$a = \frac{L_1+L_2+L_3}{N}, \quad b = \frac{N}{2\pi}\theta, \qquad \mathbf{a}_L = \vec{\xi}(L), \quad \mathbf{b} = \mathbf{0},$$

The origin of the divergent factor we found is now clear: from the gauge theory perspective it simply corresponds to a partition function of a $U(1) \in U(N)$ gauge theory on a 4−manifold whose intersection matrix is degenerate: $\#(C_i \cap C_j) = 1$, $i,j = 1,2,3$. More precisely, to define the intersection form of the reducible four-cycle D, note that D is homologous to the (punctured) $\mathbf{P}^2$ in the base, with precisely the intersection form at hand. The contribution of marginally bound states with multiple $U(1)$ factors have at first sight a worse divergences, however these can be regularized by ζ-function regularization to zero.[9] This is a physical choice, since in these sectors we expect the partition function to vanish due to extra fermion zero modes [70] [53].

To extract the black hole degeneracies we use that the matrix $\mathcal{M}_N$ is non-degenerate and do modular S-transformation using

$$\Theta[a,b](\tau) = det(\tau)^{-\frac{1}{2}} e^{2\pi iab}\Theta[b,-a](-\tau^{-1})$$

This brings Z_N to the form

$$Z_N(\theta_i,g_s) = \delta\big(N(\theta_1-\theta_3)\big)\delta\big(N(\theta_2-\theta_3)\big)(-)^{w_N}\ \left(\frac{2\pi}{Ng_s}\right)^{\frac{1}{2}}\ \left(\frac{2\pi}{g_s}\right)^{\frac{3(N-1)}{2}}\ det^{-\frac{1}{2}}\mathcal{M}_N$$

[9]For example, $Z_{N-M}(\theta_i,g_s)Z_M(\theta_i,g_s) \sim \delta\big(k(\theta_1-\theta_3)\big) \times \sum_{n\in\mathbf{Z}} 1 \times \delta\big(k(\theta_2-\theta_3)\big) \times \sum_{n\in\mathbf{Z}} 1$. where k is the least common divisor of N, M. Using $\zeta(2s) = \sum_{n=1}^{\infty} 1/n^{2s}$, where $\zeta(0) = -\frac{1}{2}$, we can regularize $\sum_{n\in\mathbf{Z}} 1 = 0$.

$$\sum_{L_i=0}^{N-1}\sum_{\ell\in\mathbf{Z}} e^{-\frac{2\pi^2}{Ng_s}(\ell+\frac{N\theta}{2\pi})^2} e^{-\frac{2\pi i(L_1+L_2+L_3)}{N}\ell} \sum_{m\in\mathbf{Z}^{3(N-1)}} e^{-\frac{2\pi^2}{g_s}m^T\mathcal{M}_N^{-1}m} e^{-2\pi i m\cdot\xi(L_i)}$$

where $\mathcal{M}_N$ is the matrix in (3.36) .

ii. Degeneracies for $N_1 > N_2 > N_3$.

When the number of branes is not equal the partition sum Z_{qYM} is substantially more complicated. By manipulations similar to the ones in appendix B, Z^{qYM} can be written as:

$$Z_{qYM} = \alpha\theta^{N_2+2N_3}(q)\sum_{\nu\in S_{N_1}}(-)^\nu \sum_{n_1\in\mathbf{Z}^{N_1}}\sum_{n_2\in\mathbf{Z}^{N_2}}\sum_{n_3\in\mathbf{Z}^{N_3}} q^{\frac{1}{2}(n_1^2-n_3^2)} q^{n_2\cdot\nu(n_1)+n_3\cdot n_2+n_3\cdot n_1}$$

$$q^{-\frac{1}{2}n_1\left(\nu^{-1}\hat{P}_{N_1|N_2}\nu\right)n_1} q^{-\frac{1}{2}n_2\hat{P}_{N_2|N_3}n_2} q^{-\frac{1}{2}n_1\hat{P}_{N_1|N_3}n_1}$$

$$q^{-\nu(n_1)\cdot(\rho_{N_1}-\rho_{N_2})} q^{-n_2\cdot(\rho_{N_2}-\rho_{N_3})} q^{-n_1\cdot(\rho_{N_1}-\rho_{N_3})} e^{i\theta_1 e(N_1)\cdot n_1+i\theta_2 e(N_2)\cdot n_2+i\theta_3 e(N_3)\cdot n_3}$$

where operator $\hat{P}_{N|M}$ projects N-dimensional vector on its first M components.

For example, consider $N_1 = 3, N_2 = 2, N_3 = 1$. In this case there are six terms in the sum

$$\nu_1 = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \quad \nu_2 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \quad \nu_3 = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

$$\nu_4 = \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \quad \nu_5 = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \quad \nu_6 = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

In this simple case Z_{qYM} has the form

$$Z_{qYM} = \alpha\theta^4(q')\left(\frac{\pi}{g_s}\right)^2\Big(Z_1 - Z_2 - Z_3 - Z_4 + Z_5 + Z_6\Big) \quad q' = e^{-\frac{\pi^2}{g_s}}$$

where

$$Z_i = \left(\frac{2\pi}{g_s}\right)^3 det^{-\frac{1}{2}}\mathcal{M}_{(i)} \sum_{f\in\mathbf{Z}^6} e^{-\frac{2\pi^2}{g_s}(f+\Lambda_{(i)})^T\mathcal{M}_{(i)}^{-1}(f+\Lambda_{(i)})}$$

where non-degenerate matrices $\mathcal{M}_{(i)}$ for $i = 1, \ldots 6$ are given by

$$\mathcal{M}_{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \end{pmatrix}, \quad \mathcal{M}_{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\mathcal{M}_{(3)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \end{pmatrix}, \quad \mathcal{M}_{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\mathcal{M}_{(5)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \end{pmatrix}, \quad \mathcal{M}_{(6)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \end{pmatrix}$$

and vectors $\Lambda_{(i)}$ for $i = 1, \ldots, 6$ have components

$$\Lambda_{(1)} = \frac{1}{2\pi}(\theta_1, \theta_1, \theta_1, \theta_2, \theta_2, \theta_3) + \frac{ig_s}{2\pi}(2, -\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 0)$$

$$\Lambda^a_{(2)} = \Lambda^a_{(1)}, \quad a = 1, \ldots, 6$$

$$\Lambda^1_{(3)} = \frac{1}{2\pi}(\theta_1, \theta_1, \theta_1, \theta_2, \theta_2, \theta_3) + \frac{ig_s}{2\pi}(\frac{1}{2}, -\frac{3}{2}, 1, \frac{1}{2}, -\frac{1}{2}, 0)$$

$$\Lambda^a_{(6)} = \Lambda^a_{(3)}, \quad a = 1, \ldots, 6$$

$$\Lambda^1_{(4)} = \frac{1}{2\pi}(\theta_1, \theta_1, \theta_1, \theta_2, \theta_2, \theta_3) + \frac{ig_s}{2\pi}(\frac{1}{2}, 0, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0)$$

$$\Lambda^a_{(5)} = \Lambda^a_{(4)}, \quad a = 1, \ldots, 6$$

3.5 Branes and black holes on local $\mathbf{P}^1 \times \mathbf{P}^1$

For our second example, we will take a noncompact Calabi-Yau threefold X which is a total space of canonical line bundle K over the base $\mathcal{B} = \mathbf{P}^1_B \times \mathbf{P}^1_F$

$$X = K \to \mathbf{P}^1_B \times \mathbf{P}^1_F$$

where $K = O(-2, -2)$. The linear sigma model whose Higgs branch is X has chiral fields X_i, $i = 0, \ldots 4$ and two $U(1)$ gauge fields $U(1)_B$ and $U(1)_F$ under which the chiral fields have charges $(-2, 1, 0, 1, 0)$ and $(-2, 0, 1, 0, 1)$. The corresponding D-term potentials are

$$|X_1|^2 + |X_3|^2 = 2|X_0|^2 + r_B$$

$$|X_2|^2 + |X_4|^2 = 2|X_0|^2 + r_F$$

The $H^2(X, Z)$ is generated by two classes $[D_F]$ and $[D_B]$. Correspondingly, there are two complexified Kahler moduli t_B and t_F, $t_B = r_B - i\theta_B$ and $t_F = r_F - i\theta_F$. There are 4 ample divisors invariant under the T^3 torus actions corresponding to setting

$$D_i : X_i = 0, \quad i = 1, 2, 3, 4$$

We have that $[D_1] = [D_3] = [D_B]$ and $[D_2] = [D_4] = [D_F]$. We take N_1 and N_2 D4 branes on D_1 and D_3, and M_1 and M_2 D4 branes on D_2 and D_4 respectively, corresponding to a divisor

$$D = N_1 D_1 + M_1 D_2 + N_2 D_3 + M_2 D_4$$

Since the topology of each D_i is $O(-2) \to \mathbf{P}^1$ we will get four copies of qYM theory of P^1 with ranks $N_{1,2}$ and $M_{1,2}$. In addition, from the matter at intersection we get 4 sets of insertions of observables (3.6) . All together, and assuming $N_{1,2} \geq M_{1,2}$, we have

$$Z_{qYM} = \gamma \sum_{\mathcal{R}_1, \mathcal{R}_2, \mathcal{Q}_1, \mathcal{Q}_2} V_{\mathcal{Q}_1 \mathcal{R}_1} V_{\mathcal{Q}_2 \mathcal{R}_2} V_{\mathcal{R}_1 \mathcal{Q}_2} V_{\mathcal{R}_2 \mathcal{Q}_1} \; q^{\sum_{i=1}^{2} C_2(\mathcal{R}_i) + C_2(\mathcal{Q}_i)} \\ e^{i\theta_{B,1} C_1(\mathcal{R}_1) + i\theta_{B,2} C_1(\mathcal{R}_2)} e^{i\theta_{F,1} C_1(\mathcal{Q}_1) + i\theta_{F,2} C_1(\mathcal{Q}_2)}. \tag{3.39}$$

Above, $\mathcal{R}_1, \mathcal{R}_2$ are representations of $U(N_1)$ and $U(N_2)$ and $\mathcal{Q}_1, \mathcal{Q}_2$ are representations of $U(M_1)$ and $U(M_2)$, respectively.

In principle, because $dim(H^2(D, \mathbf{Z})) = 4$, there 4 different chemical potentials that we can turn on for the D2 branes, corresponding to $\theta_{B,i}$, $\theta_{F,i}$. In X however, there are only two independent classes, $dim(H^2(D, \mathbf{Z})) = 2$, in particular

$$[\mathbf{P}^1_{B,1}] - [\mathbf{P}^1_{B,2}] = 0, \qquad [\mathbf{P}^1_{F,1}] - [\mathbf{P}^1_{F,2}] = 0$$

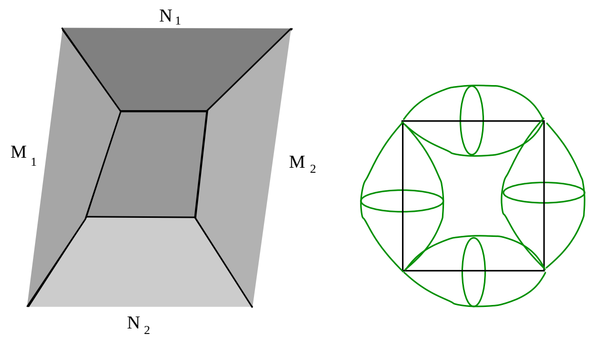

Figure 3.4: The base of the local $\mathbf{P}^1 \times \mathbf{P}^1$. The numbers of D4 branes wrapping the torus invariant non-compact 4-cycles are specified. This corresponds to qYM theory on the necklace of 4 $\mathbf{P}^1$'s with ranks M_1, N_1, M_2, and N_2.

We should turn off the chemical potentials for those states that can decay when the YM theory is embedded in string theory, by putting

$$\theta_{B,1} = \theta_{B,2}, \qquad \theta_{F,1} = \theta_{F,2}. \tag{3.40}$$

For the most part, we will keep the chemical potentials arbitrary, imposing (3.40) at the end. The prefactor γ is

$$\gamma = q^{-\left(2\rho^2_{M_1}+\frac{M_1}{12}\right)} q^{-\left(2\rho^2_{M_2}+\frac{M_2}{12}\right)} q^{-\frac{1}{96}\left((N_1+N_2)^3+(M_1+M_2)^3-3(N_1+N_2)^2(M_1+M_2)-3(M_1+M_2)^2(N_1+N_2)\right)}$$
$$\times\, e^{\frac{\theta_B\theta_F(N_1+N_2+M_1+M_2)}{4g_s}}$$

In the next subsections we will first take the large N limit of the qYM partition function, and then consider the modular properties of the exact amplitude to compute the degeneracies of the BPS bound states.

3.5.1 Black holes on local $\mathbf{P}^1 \times \mathbf{P}^1$.

We will now take the large N limit of Z_{qYM} in (3.39) and show that this is related to the topological string on X in accordance with the [62] conjecture.

i. The $N_1 = N_2 = N = M_1 = M_2$ case.

In this case, we can use the simpler form of the vertex amplitude in (3.15) to write the q-deformed Yang-Mills partition function as:

$$Z_{qYM} = \gamma' \sum_{\mathcal{R}_{1,2},\mathcal{Q}_{1,2}\in U(N)} S_{\mathcal{R}_1\bar{\mathcal{Q}}_1}(g_s,N)\, S_{\mathcal{Q}_1\bar{\mathcal{R}}_2}(g_s,N)\, S_{\mathcal{R}_2\bar{\mathcal{Q}}_2}(g_s,N)\, S_{\mathcal{Q}_2\bar{\mathcal{R}}_1}(g_s,N) \times e^{i\sum_i \theta_{B,i}C_1(\mathcal{R}_i)+i\theta_{F,i}C_1(\mathcal{Q}_i)}. \tag{3.41}$$

where $\gamma' = \gamma\theta^{4N}(q)$. Using the large N expansion for S-matrix (3.26) and parametrizing the $U(1)$ charges ℓ_{R_i} of the representations $\mathcal{R}_i$ as follows

$$2\ell_B = \ell_{R_1} + \ell_{R_2}, \quad 2\ell_F = \ell_{Q_1} + \ell_{Q_2}, \quad 2n_B = \ell_{R_1} - \ell_{R_2}, \quad 2n_F = \ell_{Q_1} - \ell_{Q_2}, \tag{3.42}$$

we find that the sum over $n_{B,F}$ gives delta functions

$$Z_{qYM}(N, g_s, \theta_{B,i}, \theta_{F,i}) \sim \delta\big(N(\theta_{B,1}-\theta_{B,2})\big)\ \delta\big(N(\theta_{F,1}-\theta_{F,2})\big)\ Z^{finite}_{qYM}(N, g_s, \theta_B, \theta_F)$$

where

$$Z^{finite}_{qYM} \sim \sum_{m_B,m_F\in\mathbf{Z}} \sum_{P_1,\dots,P_4} (-)^{\sum_{i=1}^4 |P_i|} Z^+_{P_1,\dots,P_4}\big(t_B + m_B g_s, t_F + m_F g_s\big)\; Z^+_{P_1^T,\dots,P_4^T}\big(\bar{t}_B - m_B g_s, \bar{t}_F - m_F g_s\big) \tag{3.43}$$

In (3.43) the chiral block $Z^+_{P_1,\dots,P_4}(t_B, t_F)$ is given by

$$Z^+_{P_1,\dots,P_4}(t_B,t_F) = \hat{Z}_0(g_s,t_B,t_F) e^{-t_0\sum_{i=1}^4 |P_i|} \sum_{R_1,R_2,Q_1,Q_2} e^{-t_B(|R_1|+|R_2|)} e^{-t_F(|Q_1|+|Q_2|)} \times q^{\frac{1}{2}\sum_{i=1,2}\kappa_{R_i}+\kappa_{Q_i}} \hat{C}_{Q_1^T R_1 P_1}(q)\, \hat{C}_{R_2^T Q_1 P_2}(q)\, \hat{C}_{Q_2^T R_2 P_3}(q)\, \hat{C}_{R_1^T Q_2 P_4}(q) \tag{3.44}$$

where Kahler moduli are

$$t_B = g_s N - i\theta_B, \qquad t_F = g_s N - i\theta_F.$$

The leading chiral block $Z^{+}_{0,\ldots,0}$ is the closed topological string amplitude on X. The Kahler moduli of the base $\mathbf{P}^1_B$ and the fiber $\mathbf{P}^1_F$ are exactly the right values fixed by the attractor mechanism: since the divisor D that the D4 brane wraps is in the class $[D] = 2N[D_F] + 2N[D_B]$. As we discussed in the previous section in detail, the other chiral blocks (3.44) correspond to having torus invariant non-normalizable modes excited along the four non-compact toric legs in the normal directions to the base $\mathcal{B}$. Moreover the associated Kahler parameters should also be fixed by the attractor mechanism – as discussed in the previous section, we can think of these as the open string moduli corresponding to the ghost branes. The open string moduli are complexified sizes of holomorphic disks ending on the ghost branes and these can be computed using the Kahler form on X. Since the net D4 brane charge is the same as that of $-N$ branes wrapping the base, and the intersection number of the disks C_0 ending on the topological D-branes with the base is $\#(C_0 \cap \mathcal{B}) = 1$, so the size of all the disks ending on the branes should be $t_0 = -\frac{1}{2}Ng_s$, which is in accord with (3.44) . The prefactor in (3.44) is

$$\hat{Z}_0(g_s, t_B, t_F) = e^{\frac{1}{24g_s^2}\left(t_F^3 + t_B^3 - 3t_F^2 t_B - 3t_B^2 t_F\right)}\, M^4(q^{-1})\eta^{\frac{t_B+t_F}{g_s}}(q^{-1})\theta^{\frac{t_B+t_F}{g_s}}(q)$$

As discussed before, the eta and theta function pieces contribute only to the genus zero amplitude, and to the non-perturbative terms.

ii. The general $N_{1,2}$, $M_{1,2}$ *case.*

We will assume here $N_i > M_j$, $i, j = 1, 2$. Using the large N, M limit of $V_{\mathcal{RQ}}(N, M)$ with $U(1)$ charges turned off (see Appendix D) we find that the leading chiral block of the YM partition function is

$$Z_{qYM} \sim Z^{+}_{0,\ldots,0}(t_B, t_F)\, Z^{-}_{0,\ldots,0}(\bar{t}_B, \bar{t}_F)$$

where $Z^{+}_{0,\ldots,0}(t_B, t_F)$ is precisely the topological closed string partition function on local $\mathbf{P}^1 \times \mathbf{P}^1$ [4] :

$$Z^{+}_{0,\ldots,0} = \hat{Z}_0 \sum_{Q_1^+, Q_2^+, R_1^+, R_2^+} W_{Q_1^+ R_1^+}(q) W_{Q_1^+ R_2^+}(q) W_{Q_2^+ R_1^+}(q) W_{Q_2^+ R_2^+}(q) e^{-t_F(|Q_1^+|+|Q_2^+|)} e^{-t_B(|R_1^+|+|R_2^+|)}$$

It is easy to see that this agrees with the amplitude given in (3.44) . Moreover, the Kahler parameters are exactly as predicted by the attractor mechanism corresponding to having branes on a divisor class

$$[D] = (N_1 + N_2)[D_B] + (M_1 + M_2)[D_F].$$

Namely,

$$t_B = \frac{1}{2}(M_1 + M_2) g_s - i\theta_B, \qquad t_F = \frac{1}{2}(N_1 + N_2) g_s - i\theta_F.$$

Note that the normal bundle to each of the divisor D_i is trivial, so the size of the corresponding $\mathbf{P}^1$ in $D_i = O(-2) \to \mathbf{P}^1$ is independent of the number of branes on D_i, but it does depend on the number of branes on the adjacent faces which have intersection number 1 with the $\mathbf{P}^1$.

It would be interesting to study the structure of the higher chiral blocks. In this case we expect the story to be more complicated, in particular because some of the intersection numbers that compute the attractor values of the brane moduli are now ambiguous.

3.5.2 Branes on local $\mathbf{P}^1 \times \mathbf{P}^1$

We will content ourselves with considering $N_{1,2} = M_{1,2} = N$ case, the more general case working in similar ways to the local $\mathbf{P}^2$ case. The partition function (3.41) may be written as

$$Z_{qYM}(N, \theta_i, g_s) = \gamma' \sum_{w \in S_N} (-)^w \sum_{n_1,\dots,n_4 \in Z^N} q^{w(n_1)\cdot n_2 + n_2\cdot n_3 + n_3\cdot n_4 + n_4\cdot n_1} \; e^{i\sum_{i=1}^4 \theta_i e(N)\cdot n_i} \tag{3.45}$$

where $e(N) = (1, \dots, 1)$. As before in the case of local $\mathbf{P}^2$, the bound states of N D4-branes are effectively counted by the Z_N term, i.e. the term with $w = w_N$. Like in that case, Z_N is again a sum over sectors of different N-ality,

$$Z_N(\theta_i, g_s) = \gamma' (-)^{w_N} \sum_{L_1,\dots,L_4=0}^{N-1} \sum_{\ell_i \in \mathbf{Z}+\frac{L_i}{N}} q^{N(\ell_1+\ell_3)(\ell_2+\ell_4)} e^{iN\sum_{i=1}^4 \theta_i \ell_i} \sum_{m \in \mathbf{Z}^{4(N-1)}+\vec{\xi}(L_i)} q^{\frac{1}{2}m^T \mathcal{M} m}$$

where $\mathcal{M}$ is a non-degenerate $4(N-1) \times 4(N-1)$ matrix with integer entries and $\vec{\xi_i}$ is a shift of the weight lattice corresponding to turning on 't Hooft flux.

More explicitly,

$$\xi_i^a = \frac{N-a}{N} L_i, \quad i = 1, \dots, 4 \quad a = 0, \dots N-1$$

$\mathcal{M}$ is $4(N-1) \times 4(N-1)$ matrix

$$\mathcal{M} = \begin{pmatrix} 0 & W_N & 0 & M_N \\ W_N^T & 0 & M_N & 0 \\ 0 & M_N & 0 & M_N \\ M_N & 0 & M_N & 0 \end{pmatrix} \tag{3.46}$$

whose entries are $(N-1)\times(N-1)$ matrices

$$M_N = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ . & . & . & . & \dots & . & . \\ . & . & . & . & \dots & . & . \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix} \tag{3.47}$$

and

$$W_N = \begin{pmatrix} -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & . & 0 \\ . & . & . & . & \dots & . & . \\ . & . & . & . & \dots & . & . \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \\ -1 & 0 & 0 & 0 & \dots & 0 & -1 \end{pmatrix} \tag{3.48}$$

We can express Z_N in terms of Θ-functions

$$\begin{aligned} Z_N(\theta_i, g_s) =& \gamma' \, (-)^{w_N} \delta\big(N(\theta_{B,1}-\theta_{B,2})\big)\delta\big(N(\theta_{F,1}-\theta_{F,2})\big) \\ & \sum_{L_1,\dots,L_4=0}^{N-1} \Theta_2[a(L_i), b](\tau) \;\; \Theta_{4N-4}[\mathbf{a}(L_i), \mathbf{b}](\hat{\tau}) \end{aligned}$$

where

$$\Theta_k[a,b](\tau) = \sum_{n\in\mathbf{Z}^k} e^{\pi i \tau (n+a)^2} \; e^{2\pi i n b}$$

and

$$\tau = \frac{i g_s}{2\pi} N \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\tau} = \frac{i g_s}{2\pi}\mathcal{M}$$

and

$$a = \big(\frac{L_1+L_3}{N}, \frac{L_2+L_4}{N}\big), \quad b = \big(\frac{N}{2\pi}\theta_B, \frac{N}{2\pi}\theta_F\big) \qquad \mathbf{a}_L = \vec{\xi}(L), \quad \mathbf{b} = \mathbf{0},$$

To extract black hole degeneracies we use that matrix $\mathcal{M}$ is non-degenerate and do modular S-transformation using

$$\Theta[a,b](\tau) = det(\tau)^{-\frac{1}{2}} e^{2\pi i ab}\Theta[b,-a](-\tau^{-1})$$

After modular S-transformation Z_N is brought to the form

$$\begin{aligned} Z_N(\theta, g_s) =& \gamma' \, \delta\big(N(\theta_{B,1}-\theta_{B,2})\big)\delta\big(N(\theta_{F,1}-\theta_{F,2})\big)(-)^{w_N} \left(\frac{2\pi}{Ng_s}\right)\left(\frac{2\pi}{g_s}\right)^{\frac{4(N-1)}{2}} det^{-\frac{1}{2}}\mathcal{M} \\ & \sum_{L_1,\dots,L_4=0}^{N-1} \sum_{\ell,\ell'\in\mathbf{Z}} e^{-\frac{\pi^2}{Ng_s}(\ell+\frac{N\theta_B}{2\pi})(\ell'+\frac{N\theta_F}{2\pi})} e^{-\frac{2\pi i(L_1+L_3)}{N}\ell} e^{-\frac{2\pi i(L_2+L_4)}{N}\ell'} \\ & \sum_{m\in\mathbf{Z}^{4(N-1)}} e^{-\frac{2\pi^2}{g_s}m^T\mathcal{M}^{-1}m} e^{-2\pi i m\cdot\xi(L_i)} \end{aligned}$$

3.6 Branes and black holes on A_k ALE space

Consider the local toric Calabi Yau X which is A_k ALE space times $\mathbf{C}$. This can be thought of as the limit of the usual ALE fibration over $\mathbf{P}^1$ as the size of the base $\mathbf{P}^1$ goes to ∞. In this section we will consider black holes obtained by wrapping N D4 branes on the ALE space.

This example will have a somewhat different flavor than the previous two, so we will discuss the D4 brane gauge theory on a bit more detail. On the one hand, the theory on the D4 brane is a topological $U(N)$ Yang-Mills theory on A_k ALE space which has been studied previously [58] [70] . On the other hand, the A_k ALE space

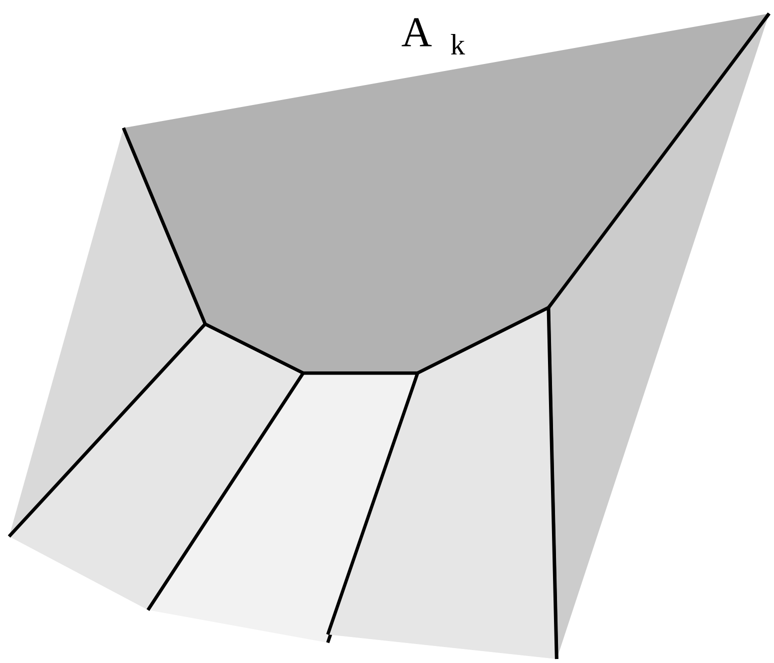

Figure 3.5: N D4-branes are wrapped on A_k type ALE space in $A_k \times \mathbf{C}$, for $k = 3$. The D-brane partition function is computed by $U(N)$ qYM theory on a chain of 3 $\mathbf{P}^1$'s.

has T^2 torus symmetries, so we should be able to obtain the corresponding partition function by an appropriate computation in the two dimensional qYM theory. We will start with the second perspective, and make contact with [58] [70] later.

As in [7] and in section 3, our strategy will be to cut the four manifold into pieces where the theory is simple to solve, and then glue the pieces back together. The A_k type ALE space can be obtained by gluing together $k+1$ copies of $\mathbf{C}^2$. Correspondingly, we should be able to obtain YM amplitudes on the ALE space by sewing together amplitudes on $\mathbf{C^2}$. Moreover, since the $\mathbf{C^2}$ and the ALE space have T^2 isometries, the 4d gauge theory computations should localize to fixed points of these isometries, and these are bundles with second Chern class localized at the vertices, and first Chern class along the edges.

Viewed as a manifold fibered by 2-tori T^2, $\mathbf{C}^2$ has contains two disks, say C_{base} and C_{fiber} that are fixed by torus action (see figure 2 by way of example). Viewed as a line bundle over a disk C_{base} as a base, the $U(1)$ isometry of the fiber allows us to do some gauge theory computations in the qYM theory on C_{base}. In particular, if the bundle is flat the qYM partition function on a disk (3.10) with holonomy $U = \exp\left(i \oint A\right)$ fixed on the boundary of the C_{base} fixed and no insertions is[10]

$$Z(C)(U) = \sum_{\mathcal{R}} e^{i\theta C_1(\mathcal{R})} S_{0\mathcal{R}}(N, g_s) Tr_{\mathcal{R}} U.$$

What is the four dimensional interpretation of this? The sum over $\mathcal{R}$ in the above

[10]More precisely, as we explained in section 3, the coordinate U is ambiguous since the choice of cycle which remains finite is ambiguous. This ambiguity relates to the choice of the normal bundle to the disk, and the present choice corresponds to picking this bundle to be trivial, which is implicit in the amplitude.

corresponds to summing over the four dimensional $U(N)$ gauge fields with

$$\int_{fiber} F_a = \mathcal{R}_a \, g_s, \qquad a = 1, \dots N, \tag{3.49}$$

where $\mathcal{R}_a$ are the lengths of the rows in the Young tableau of $\mathcal{R}$.[11] This is because on the one hand

$$S_{0\mathcal{R}}(N, g_s) = \langle \, Tr_{\mathcal{R}} \, e^{i \oint A} \, \rangle. \tag{3.50}$$

and on the other $\oint A_a = \int_{base} F_a$ is conjugate to $\Phi_a = \int_{fiber} F_a$, so inserting (3.50) shifts F as in (4.19) . The unusual normalization of F has to do with the fact that qYM directly computes the magnetic, rather than the electric partition function: In gluing two disks to get an $\mathbf{P}^1$ we sum over all $\mathcal{R}$'s labelling the bundles of the S-dual theory over the $\mathbf{P}^1$.

If we are to use 2d qYM theory to compute the $\mathcal{N} = 4$ partition function on ALE space, we must understand what in the 2d language is computing the partition function on $\mathbf{C}^2$ with

$$\int_{fiber} F_a = \mathcal{R}_a \, g_s, \qquad \int_{base} F_a = \mathcal{Q}_a \, g_s, \qquad a = 1, \dots N, \tag{3.51}$$

since clearly, what we call the "base" here versus the "fiber" is a matter of convention. Using once more the fact that Φ and $\oint A$ are conjugate, turning on $\int_{base} F_a = \mathcal{Q}_a g_s$ corresponds to inserting $Tr_{\mathcal{Q}} \, e^{-i\Phi}$ at the point on C_{base} where it intersects C_{fiber}. Thus, turning on (3.51) corresponds to computing $\langle Tr_Q e^{-i\Phi} \, Tr_{\mathcal{R}} \, e^{i \oint A} \rangle$. This is an amplitude we already know:

$$S_{Q\mathcal{R}}(N, g_s) = \langle \, Tr_Q e^{-i\Phi} \, Tr_R e^{i \oint A} \, \rangle. \tag{3.52}$$

[11] To be more precise, $\mathcal{R}_a$ in (4.19) is shifted by $\frac{1}{2}(N+1) - a$.

Alternatively, the amplitude on $\mathbf{C}^2$ with arbitrary boundary conditions (3.51) on the base and on the fiber is

$$\sum_{\mathcal{R},\mathcal{Q}} S_{\mathcal{R}\mathcal{Q}}(N,g_s)\, Tr_{\mathcal{R}}U\; Tr_{\mathcal{Q}}V \tag{3.53}$$

We then glue the pieces together using the usual local rules. The only thing we have to remember is that the normal bundle to each $\mathbf{P}^1$ is $O(-2)$, and that at the "ends" we should turn the fields off. In computing (3.52) we used the coordinates in which $\mathbf{C}^2$ is a trivial fibration over both C_{fiber} and C_{base}, and therefore to get the first Chern class of the normal bundle to come out to be -2, we must along each of them insert annuli with $O(-2)$ bundle over them. This gives:

$$Z = \sum_{\mathcal{R}_1\ldots\mathcal{R}_k} S_{0\mathcal{R}_{(1)}} S_{\mathcal{R}_{(1)}\mathcal{R}_{(2)}}\ldots S_{\mathcal{R}_{(k)}0}\; q^{\sum C_2(\mathcal{R}_{(j)})}\; e^{i\sum\theta_j|\mathcal{R}_{(j)}|}, \tag{3.54}$$

There is one independent θ angle for each $\mathbf{P}^1$ corresponding to the fact that they are all independent in homology. These θ angles will get related to chemical potentials for the D2 branes wrapping the corresponding 2-cycles.

3.6.1 Modularity

The S-duality of $\mathcal{N}=4$ Yang Mills acts on our partition function as $g_s \to \frac{4\pi^2}{g_s}$. By performing this modular transformation we will be able to read off the degeneracies of the BPS bound states contributing to the entropy. First, using the definition of the Chern Simons S-matrix, we find that

$$Z = \sum_{\omega\in\mathcal{W}}(-1)^{\omega} \sum_{n_1,\ldots n_k\in Z^N} q^{n_1^2+\ldots+n_k^2-n_1n_2-\ldots-n_{k-1}n_k}\; e^{i\theta_1|n_1|+\ldots\theta_k|n_k|} q^{\rho n_1+n_k\omega(\rho)} \tag{3.55}$$

Note the appearance of the intersection matrix of A_k ALE space. The fact that the Cartan matrix appears gives the k vectors $U(N)$ weight vectors n_i^a $i = 1, \dots k$, $a = 1, \dots N$ an alternative interpretation as N $SU(k)$ *root* vectors:

$$Z = \sum_{\omega \in \mathcal{W}} (-1)^\omega \prod_{a=1}^{N} \sum_{n_a \in \Lambda^{Root}_{SU(k)}} q^{\frac{1}{2} n_a n_a} \; e^{i\theta n_a} q^{(\rho + \omega(\rho))_a n_a}$$

where θ is a k-dimensional vector with entries θ_i. From the above, it is clear that Z is a product of N $SU(k)$ characters at level one. Recall that the level one characters are

$$\chi_\lambda^{(1)}(\tau, u) = \frac{\theta_\lambda^{(1)}(\tau, u)}{\eta^k(\tau)}$$

where

$$\theta_\lambda^{(1)}(\tau, u) = \sum_{n \in \Lambda^{Root}_{SU(k)}} e^{\pi i \tau (n+\lambda)^2 + 2\pi i (n+\lambda) u}$$

To be concrete, our amplitude is given as follows:

$$Z = \eta(q)^{Nk} \sum_{\omega \in \mathcal{W}} (-1)^\omega \prod_{a=1}^{N} \chi_0^{(1)}(\tau, u^a(\theta, \omega))$$

Here,

$$\tau = \frac{i g_s}{2\pi}, \qquad u_i^a(\theta, \omega) = \frac{\theta_i}{2\pi} + \frac{i g_s}{2\pi} (\rho + \omega(\rho))^a$$

Modular transformations act on the space of level one characters as:

$$\theta_\eta^{(1)}(-\frac{1}{\tau}, \frac{u}{\tau}) = e^{-\frac{uu}{2\tau}} \sum_{\omega \in \mathcal{W}_k} (-1)^\omega \sum_{\lambda} e^{\frac{2\pi i}{k+1} \omega(\eta+\rho)(\lambda+\rho)} \theta_\lambda^{(1)}(\tau, u),$$

consequently, the dual partition function also has an expansion in terms of N level one characters. The product of N level one characters can be expanded in terms of sums of level N characters, so this is consistent with the results of H. Nakajima. The fact that the partition function is a sum over level N characters, rather than a single one is natural given that we impose different boundary conditions at the infinity of ALE space from [58] .

3.6.2 The large N limit

In the 't Hooft large N expansion, using (3.26) , we find that the partition function (3.55) can be written as follows:

$$Z_{ALE} = \sum_{P_1,\ldots,P_{k+1}} (-)^{|P_1|+\ldots|P_{k+1}|} \sum_{m_1,\ldots,m_k \in \mathbf{Z}}$$

$$Z^{+}_{P_1,..,P_{k+1}}(t_1 + m_1 g_s, \ldots, t_k + m_k g_s)\; Z^{+}_{P_1^T,..,P_{k+1}^T}(\bar{t}_1 - m_1 g_s, \ldots, \bar{t}_k - m_k g_s),$$

where m's are related to the $U(1)$ charges of representations $\mathcal{R}_i$ as $m_i = 2\ell_i - \ell_{i-1} - \ell_{i+1}$, for $i = 1, \ldots, k$ (where $\ell_0 = \ell_{k+1} = 0$). The Kahler moduli are

$$t_j = -\, i\, \theta_j, \qquad j = 1, \ldots, k,$$

which is what attractor mechanism predicts: Since ALE space has vanishing first Chern class, the normal bundle of its embedding in a Calabi-Yau three-fold is trivial, and consequently $\#[D_{A_k} \cap C] = 0$ where D_{A_k} is (N times) the divisor corresponding to the ALE space and C is any curve class in X.

The normalization constant α_{ALE} in (3.54) was determined by requiring the large N limit factorizes in the appropriate way.

$$\alpha_{ALE} = q^{(k+1)(\rho^2 + \frac{N}{24})} e^{\frac{N}{2g_s}\theta^T A \theta}, \tag{3.56}$$

where A is the inverse of the intersection matrix of ALE.

The chiral block in the chiral(anti-chiral) decomposition of Z_{ALE} has the form

$$Z^{+}_{P_1,\ldots,P_{k+1}}(t_1, \ldots, t_k) = M(q)^{\frac{k+1}{2}} e^{-\frac{t_0\, t^T A\, t}{2g_s^2} + \frac{\pi^2 (k+1) t_0}{6 g_s^2}}\, e^{-t_0 \sum_{d=1}^{k+1} |P_d|} \times$$

$$\times \sum_{R_1 \ldots R_k} \hat{C}_{0 R_1{}^T P_1} q^{\kappa_{R_1}/2} e^{-t_1|R_1|} \hat{C}_{R_1 R_2{}^T P_2} q^{\kappa_{R_2}/2} e^{-t_2|R_2|} \ldots \hat{C}_{R_k 0 P_{k+1}}.$$

where

$$t_0 = \frac{1}{2} N g_s. \tag{3.57}$$

We see that the trivial chiral block $Z^{+}_{0,\ldots,0}(t_1, \ldots, t_k)$ is exactly the topological string partition function on ALE, in agreement with the conjecture of [62] . Moreover, the higher chiral blocks correspond to having $k+1$ sets of topological "ghost" branes in the $\mathbf{C}$ direction over the north and the south poles of the $\mathbf{P}^1$'s. The associated moduli, i.e. the size of the holomorphic disks ending on the topological ghost branes is also fixed by the attractor mechanism, to be $\#(D_{A_k} \cap C_{disk}) = N$. This is gives exactly (3.57) as the value of the corresponding Kahler moduli t_0, in agreement with the conjecture. As we discussed in section 4, in the closed string language, these are the non-normalizable modes in the topological string on X. The classical piece of the topological string amplitude

$$\frac{1}{2g_s^2}\, t_0\, t^T A t \tag{3.58}$$

deserves a comment. Because $X = A_k \times \mathbf{C}$, taking only the compact cohomology the triple intersection numbers would unambiguously vanish. The non-vanishing triple intersection numbers can be gotten only by a suitable regularization of the $\mathbf{C}$ factor. This *was* already regularized, in terms of the Kahler modulus t_0 of the non-normalizable modes – which exactly give the measure of the size of the disk, i.e. $\mathbf{C}$, making (3.58) a natural answer.[12]

[12] What is less natural is the appearance of the *inverse* intersection matrix of ALE. However, one has to remember that this is a non-compact Calabi-Yau, where intersection numbers are inherently ambiguous.

Appendix A. Conventions and useful formulas

The S matrix is given by

$$S_{\mathcal{R}\mathcal{Q}}(N, g_s) = \sum_{w \in S_N} (-)^w q^{-w(\mathcal{R}+\rho_N)\cdot(\mathcal{Q}+\rho_N)}$$

where $q = exp(-g_s)$, and $\rho_N^a = \frac{N-2a+1}{2}$, for $a = 1, \ldots, N$. Note that while the expression for $S_{\mathcal{R}\mathcal{Q}}$ looks like that for the S-matrix of the $U(N)$ WZW model, unlike in WZW case, g_s is not quantized. Using Weyl denominator formula $Tr_{\mathcal{R}} x = \prod_{i<j}(x_i - x_j) \sum_{w \in S_N} (-)^w x^{w(\mathcal{R}+\rho_N)}$, the S-matrix can also be written in terms of Schur functions $s_{\mathcal{R}}(x_1, \ldots, x_N) = Tr_{\mathcal{R}} x$ of N variables.

$$S_{\mathcal{R}\mathcal{Q}}/S_{00}(g_s, N) = s_{\mathcal{R}}(q^{-\rho_N - \mathcal{Q}}) s_{\mathcal{Q}}(q^{-\rho_N}).$$

Above, x is an N by N matrix with eigenvalues x_i, $i = 1, \ldots N$, as

The S matrix has following important properties:

$$S_{\bar{\mathcal{R}}\mathcal{Q}}(N, g_s) = S_{\mathcal{R}\mathcal{Q}}(N, -g_s) = S^{-1}_{\mathcal{R}\mathcal{Q}}(N, g_s)$$

The first follows since (up to a sign that is $+1$ if N is odd and -1 if N is even), $\bar{\mathcal{Q}} + \rho_N = -\omega_N(\mathcal{Q} + \rho_N)$ where ω_N is the permutation that maps $a \to N - a + 1$ for $a = 1, \ldots, N$. The second is easily seen by computing

$$\begin{aligned}\sum_{\mathcal{P}} S_{\mathcal{R}\mathcal{P}}(N, -g_s) S_{\mathcal{P}\mathcal{Q}}(N, g_s) &= \sum_{w \in S_N} (-)^w \sum_{n \in \mathbf{Z^N}} q^{w(\rho_N+\mathcal{R})\cdot n} q^{-n\cdot(\rho_N+\mathcal{Q})} \\ &= \sum_{w \in S_N} (-)^w \delta^{(N)}\left(w(\rho_N + \mathcal{R}) - (\rho_N + \mathcal{Q})\right) = \delta_{\mathcal{R}\mathcal{Q}}.\end{aligned}$$

where we absorbed one sum over the Weyl group into the unordered vector, n. Note that $(\rho_N + \mathcal{R})^a$ and $(\rho_N + \mathcal{Q})^a$ are decreasing in a, so the delta function can only be satisfied when $w = 1$.

The large N limit of the S matrix for coupled representations $\mathcal{R} = R_+\bar{R}_-[\ell_R]$, $\mathcal{Q} = Q_+\bar{Q}_-[\ell_Q]$ is given in (3.26) in terms of the topological vertex amplitude

$$\hat{C}_{RQP}(q) = C_{RQP}(q^{-1}), \quad C_{RQP}(q) = q^{\frac{\kappa_R}{2}} s_P(q^\rho) \sum_\eta s_{R^t/\eta}(q^{P+\rho}) s_{Q/\eta}(q^{P^t+\rho})$$

This has cyclic symmetry $\hat{C}_{PQR} = \hat{C}_{QRP}$, and using the properties of the Schur functions under $q \to q^{-1}$: $s_R(q^{Q+\rho}) = (-1)^{|R|} s_{R^T}(q^{-Q^T-\rho})$ also a symmetry under inversion: $\hat{C}_{RQP}(q^{-1}) = (-)^{|R|+|Q|+|P|}\hat{C}_{R^tQ^tP^t}(q)$. The leading piece of S in the large N limit is significantly simpler than (3.26) . Since $\hat{C}_{0RQ}(q) = (-)^{|R|+|Q|} W_{R^TQ}(q) q^{-\frac{1}{2}\kappa_Q}$ we have:

$$\begin{aligned} S_{\mathcal{R}\mathcal{Q}}(g_s, N) =& (-)^{|R_+|+|Q_+|+|R_-|+|Q_-|} q^{-N\ell_R\ell_Q} q^{-\ell_R(|Q_+|-|Q_-|)} q^{-\ell_Q(|R_+|-|R_-|)} \\ & W_{R_+Q_+}(q) W_{R_-Q_-}(q) q^{-\frac{N}{2}(|R_+|+|R_-|+|Q_+|+|Q_-|)} \end{aligned}$$

where

$$W_{RQ}(q) = s_R(q^{\rho+Q}) s_Q(q^\rho)$$

where $\rho = -a + \frac{1}{2}$, for $a = 1, \ldots, \infty$.

Appendix B. Quantum Yang-Mills amplitudes with observable insertions

Consider the $U(N)$ q-deformed YM path integral on the cap. As shown in [7] this is given by

$$Z_{\mathrm{qYM}}(C)(U) = \sum_{\mathcal{R}} S_{0\mathcal{R}} Tr_{\mathcal{R}} U.$$

The Fourier transform to the Φ basis is given by the following path integral over the boundary of the disk,

$$Z_{\text{qYM}}(C)(U) = \int d_H\Phi \; e^{\frac{1}{g_s} Tr\Phi \oint A} \; Z_{\text{qYM}}(C)(\Phi).$$

Since the qYM path integral localizes to configurations where Φ is covariantly constant,so in particular Φ and A commute, integrating over the angles gives[13]

$$Z_{\text{qYM}}(C)(\vec{u}) = \int \prod_i d\phi_i \, \frac{\Delta_H(\phi)}{\Delta_H(u)} \; e^{\frac{1}{g_s}\sum_i \vec{\phi}\cdot\vec{u}} \; Z_{\text{qYM}}(C)(\vec{\phi}),$$

where we defined a hermitian matrix u by $U = e^{iu}$, and

$$\Delta_H(\phi) = \prod_{1\le i<j\le N} 2sin[(\phi_i - \phi_j)/2] = \prod_{\alpha>0} 2\sin(\vec{\alpha}\cdot\vec{\phi}).$$

comes from the hermitian matrix measure over $\vec{\phi}$ by adding images under $\vec{\phi} \to \vec{\phi} + 2\pi\vec{n}$, to take into account the periodicity of Φ.

Now, in the Φ basis, the path integral on the disk with insertion of $Tr_{\mathcal{Q}} e^{i\Phi}$ is simply given by:

$$Z(C, Tr_{\mathcal{Q}} e^{i\Phi})(\Phi) = Tr_{\mathcal{Q}} e^{i\Phi}$$

since Φ is a multiplication operator in this basis. Transforming this to U-basis, we use

$$Tr_{\mathcal{Q}} e^{i\Phi} := \chi_{\mathcal{Q}}(\vec{\phi}) = \frac{\sum_{\omega\in S_N} (-1)^{\omega} e^{i\omega(\vec{\mathcal{Q}}+\vec{\rho})\cdot\vec{\phi}}}{\sum_{\omega\in S_N} (-1)^{\omega} e^{i\omega(\vec{\rho})\cdot\vec{\phi}}},$$

where S_N is the Weyl group and $\vec{\rho}$ is the Weyl vector. We also use the Weyl denominator formula

$$\prod_{\alpha>0} \sin(\vec{\alpha}\cdot\vec{\phi}) = \sum_{\omega\in S_N} (-1)^{\omega} e^{i\omega(\vec{\rho})\cdot\vec{\phi}}.$$

[13]There was an error in [7] where the denominator $1/\Delta_H(u)$ was dropped. In that case this only affected the definition of the wave function (whether one absorbs the determinant $\Delta_H(\phi)$ into the wave function of ϕ or not), but here we need the correct expression. This normalization follows from [50] where the matrix model for a pair of commuting matrices with haar measure was first discussed.

Plugging this into the integral, and performing a sum over the weight lattice we get

$$Z(C, Tr_{\mathcal{Q}} e^{i\Phi})(U) = \frac{1}{\Delta_H(u)} \sum_{\omega \in W} (-1)^{\omega} \delta(\vec{u} + i g_s \omega(\vec{\rho} + \vec{\mathcal{Q}}))$$

We can extract the coefficient of this in front of $Tr_{\mathcal{R}} U$ by computing an integral

$$\int dU Z(C, Tr_{\mathcal{Q}} e^{i\Phi})(U) Tr_{\mathcal{R}} U^{-1}$$

which easily gives

$$Z(C, Tr_{\mathcal{Q}} e^{i\Phi})(U) = \sum_{\mathcal{R}} S_{\mathcal{R}\bar{\mathcal{Q}}}(g_s, N) Tr_{\mathcal{R}} U.$$

where

$$S_{\mathcal{R}\bar{\mathcal{Q}}}(g_s, N) = \sum_{\omega} q^{\omega(\mathcal{Q}+\rho)\cdot(\mathcal{R}+\rho)}$$

in terms of $q = e^{-g_s}$.

Another expectation value we need is of

$$Z(C, Tr_{\mathcal{Q}} e^{i\Phi - in\oint A})(U)$$

We can compute this by replacing Φ by $\Phi' = \Phi - n\oint A$ everywhere. The only difference is that we must now transform from $\Phi - n\oint A$ basis (with $\oint A$ as a momentum) where the computations are simple to $\oint A$ basis with Φ as a momentum, and this is done by

$$Z_{\rm 2dYM}(C)(U) = \int d\Phi' \, e^{\frac{1}{g_s}\,{\rm Tr}\,\Phi' u + \frac{n}{2g_s} Tr u^2} \, Z_{\rm 2dYM}(C)(\Phi').$$

This gives

$$Z(C, Tr_{\mathcal{Q}} e^{i\Phi - in\oint A})(U) = \sum_{\mathcal{R}} q^{\frac{n}{2}C_2(\mathcal{Q})} S_{\bar{\mathcal{Q}}\mathcal{R}} Tr_{\mathcal{R}} U$$

Appendix C. Modular transformations

Consider the vertex amplitude corresponding to intersecting D4 branes:

$$Z(U,V) = \sum_{\mathcal{R}\in U(N),\, \mathcal{Q}\in U(M)} V_{\mathcal{R}\mathcal{Q}}(N,M) Tr_{\mathcal{R}}U Tr_{\mathcal{Q}}V$$

where

$$V_{\mathcal{R}\mathcal{Q}} = \sum_{\mathcal{P}\in U(M)} q^{\frac{C_2^{(M)}(\mathcal{P})}{2}} S_{\mathcal{R}\mathcal{P}}(g_s,N)\, S_{\mathcal{P}\mathcal{Q}}(-g_s,M)$$

Using the definition (3.11) of $S_{\mathcal{R}\mathcal{Q}}$ and the Weyl-denominator form of the $U(N)$ characters $Z(U,V)$ becomes:

$$Z(U,V) = \frac{1}{\Delta_H(u)\Delta_H(v)} \sum_{w_1,w_3\in S_N} \sum_{w_3',w_2\in S_M} (-)^{w_1+w_2+w_3+w_3'} q^{\frac{||\mathcal{P}+\rho_M||^2}{2}}$$
$$q^{(\mathcal{P}+\rho_M)\cdot w_3'(\mathcal{Q}+\rho_M)} q^{-(\mathcal{P}+\rho_N)\cdot \omega_3(\mathcal{R}+\rho_N)} e^{iw_1(\mathcal{R}+\rho_N)\cdot u} e^{(\mathcal{Q}+\rho_M)\cdot w_2(iv)}$$

We can trade the sums over the Weyl groups, for sums over the full weight lattices: Put

$$w_2 = w_Q^{-1}, \quad w_3' = w_P^{-1} w_Q,$$

this defines elements $w_P, w_Q \in S_M$ uniquely given w_2, w_3'. Then, we can always find an element $w_R \in S_N$ such that

$$w_3 = w_P^{-1} w_R,$$

for a given w_3, by simply viewing w_P as an element of S_N acting on first M entries of any N dimensional vector, leaving the others fixed. Finally, find an $w \in S_N$ such that

$$w_1 = w^{-1} w_R,$$

Note now that

$$\omega_P(P+\rho_N) = \omega_P(P+\rho_M) + \rho_N - \rho_M$$

since ω_P acts only on first M entries of a vector and the first M entries of $\rho_N - \rho_M$ are all equal, hence invariant under ω_P. Using this and the fact that now only permutations w are counted with alternating signs, we can combine the sums over the Weyl-groups with the sums over the lattices to write:

$$Z(U,V) = \Delta_H(u)^{-1}\Delta_H(v)^{-1}\sum_{w\in S_N}(-)^w \sum_{m,p\in\mathbf{Z}^M;\, n\in\mathbf{Z}^N} q^{\frac{p^2}{2}} q^{p\cdot m} q^{-(p+\rho_N-\rho_M)\cdot n} e^{in\cdot w(u)} e^{im\cdot v}$$

Now split $n = (n', n'')$ where n' is the first M entries in n, n'' the remaining $N - M$, and similarly put $\rho_N - \rho_M = (\rho', \rho'')$, where we have treated ρ_M as N dimensional vector first M entries of which is the standard Weyl vector of $U(M)$, the remaining being zero, and $u = (u', u'')$. If one in addition defines $m' = m - n'$ above becomes

$$Z(U,V) = \theta^M(q)\Delta_H(u)^{-1}\Delta_H(v)^{-1}\sum_{w\in S_N}(-)^w \sum_{m'\in\mathbf{Z}^M} q^{\frac{-(m')^2}{2}} e^{im'\cdot v}$$
$$\sum_{n'\in\mathbf{Z}^M}\sum_{n''\in\mathbf{Z}^{N-M}} q^{-\rho'\cdot n'-\rho''\cdot n''} e^{n'(\cdot w(iu')+iv)+n''\cdot\omega(iu'')}$$

where $\theta(q) = \sum_{n\in\mathbf{Z}} q^{\frac{n^2}{2}}$ is the usual theta function. We write n again as an $N-$dimensional vector $(n', n') = n$ to get our final expression

$$Z(U,V) = \theta(q)^M\Delta_H(u)^{-1}\Delta_H(v)^{-1}\sum_{m'\in\mathbf{Z}^M} q^{-\frac{(m')^2}{2}} e^{im'\cdot v}\sum_{w\in S_N}(-)^w\delta(iv+w(iu)+(\rho_N-\rho_M)g_s)$$

where v, ρ_M are regarded as N dimensional vectors $(v, 0^{N-M})$, $(\rho_M, 0^{N-M})$.

Appendix D. Large N limit of the vertex amplitude

Here we find the large N, M limit of the interaction

$$V_{\mathcal{RQ}} = \sum_{\mathcal{P}} S_{\mathcal{RP}}(N, g_s)\, q^{\frac{C_2^{(M)}(\mathcal{P})}{2}}\, S_{\bar{\mathcal{P}}\mathcal{Q}}(M, g_s)$$

(we've dropped an overall factor). Using $TS^{-1} = \theta(q)^M S^{-1}T^{-1}S^{-1}T^{-1}$ in the $U(M)$ factor, this can be done by computing first the large N, M limit of

$$\sum_{\mathcal{P}} S_{\mathcal{R}\mathcal{P}}(N, g_s)\, S_{\bar{\mathcal{P}}\mathcal{A}}(M, g_s)$$

and then using large M limit of $(TST)^{-1}_{\mathcal{A}\mathcal{Q}}$ to get the full amplitude. In general, either version of the problem is very difficult and at present unsolved. Things simplify significantly if we *turn off* the $U(1)$ charges all together. This means we will effectively compute the $SU(N)$ rather than $U(N)$ version of interaction. It will turn out that the crucial features that one expects from the amplitudes *assuming* the conjecture holds, are unaffected by this. In this case, the representations $\mathcal{R}$ are effectively labelled by Young tableaux's.

From the free fermion description of the YM amplitudes it follows easily [48] that:

$$\begin{aligned}\sum_{\mathcal{P}\in U(M)} S_{\mathcal{R}\mathcal{P}}(g_s, N) S_{\bar{\mathcal{P}}\mathcal{A}}(g_s, M) \to\ & \alpha_N^{-1}(q)\alpha_M^{-1}(q)\ S_{0(R_+\bar{R}_-)}(g_s, N) S_{0(A_+\bar{A}_-)}(g_s, M) \\ & \times \prod_{i,j=1}^{\infty} \frac{[\frac{1}{2}N - \frac{1}{2}M + j - i]}{[R_i^+ - A_j^+ + \frac{1}{2}N - \frac{1}{2}M + j - i]} \frac{[\frac{1}{2}N - \frac{1}{2}M + j - i]}{[R_i^- - A_j^- + \frac{1}{2}N - \frac{1}{2}M + j - i]} \\ & \times \prod_{i,j=1}^{\infty} \frac{[\frac{1}{2}N + \frac{1}{2}M - j - i + 1]}{[R_i^+ + A_j^- + \frac{1}{2}N + \frac{1}{2}M - j - i + 1]} \frac{[\frac{1}{2}N + \frac{1}{2}M - j - i + 1]}{[R_i^- + A_j^+ + \frac{1}{2}N + \frac{1}{2}M - j - i + 1]}\end{aligned} \tag{3.59}$$

where the arrow indicates taking large $N,\ M$ limit and where $\alpha_N(q) = q^{-(\rho_N^2 + \frac{N}{24})} M(q)\eta^N(q)$, and similarly for α_M with M, N exchanged.

For simplicity, we will be are interested only in the leading chiral block of the amplitude which determines the Calabi-Yau manifold that the YM theory describes in the large N limit, and neglects the excitations of non-normalizable modes. In this limit, the piece $S_{0(R_+\bar{R}_-)}(g_s)S_{0(A_+\bar{A}_-)}(g_s)$ gives

$$\alpha_M(q)\alpha_N(q)W_{A_+^T0}(q)W_{A_-^T0}(q)W_{R_+^T0}(q)W_{R_-^T0}(q)q^{-\frac{M(|A_+|+|A_-|)}{2}}q^{-\frac{N(|R_+|+|R_-|)}{2}}$$

where $W_{RP}(q) = s_R(q^\rho)s_P(q^{R+\rho})$, and moreover $W_{R0}(q) = (-)^{|R|}q^{k_R/2}W_{R^T0}(q)$. Of the infinite product terms, in the leading chiral block limit only the second row in (3.59) contributes. This is because the interactions between the chiral and anti-chiral part of the amplitude are suppressed in this limit. Using

$$\prod_{i,j}\frac{1}{x_i - y_j} = \prod_i x_i^{-1}\sum_R s_R(x^{-1})s_R(y)$$

we get

$$const. \times \sum_{P_+,P_-} s_{P_+}(q^{R_++\rho})s_{P_+}(q^{-(A_++\rho)})s_{P_-}(q^{R_-+\rho})s_{P_-}(q^{-(A_-+\rho)})q^{(|P_+|+|P_-|)\frac{N-M}{2}}$$

The constant comes from regularizing the infinite products (see [48] for details) and can be determined by computing the leading large N, M scaling

$$\prod_{(i,j)\in A_+}\frac{[\frac{1}{2}N-\frac{1}{2}M-j+i]}{[-\frac{1}{2}N-\frac{1}{2}M-j+i]}\prod_{(i,j)\in A_-}\frac{[\frac{1}{2}N-\frac{1}{2}M-j+i]}{[-\frac{1}{2}N-\frac{1}{2}M-j+i]}$$
$$\prod_{(i,j)\in R_+}\frac{[\frac{1}{2}N-\frac{1}{2}M+j-i]}{[\frac{1}{2}N+\frac{1}{2}M+j-i]}\prod_{(i,j)\in R_-}\frac{[\frac{1}{2}N-\frac{1}{2}M+j-i]}{[\frac{1}{2}N+\frac{1}{2}M+j-i]} \sim q^{\frac{\kappa_{A_+}+\kappa_{A_-}}{2}}\; q^{\frac{M(|R_+|+|R_-|+|A_+|+|A_-|)}{2}}$$

where i goes over the rows and j over the columns. All together, this gives

$$\sum_{\mathcal{P}\in U(M)} S_{\mathcal{RP}}(g_s,N)S_{\bar{\mathcal{P}}\mathcal{A}}(g_s,M) \rightarrow (-)^{|R_+|+|R_-|}q^{-\frac{N-M}{2}(|R_+|+|R_-|)}q^{-\frac{\kappa_{R_+}+\kappa_{R_-}}{2}}q^{\frac{\kappa_{A_+}+\kappa_{A_-}}{2}}$$
$$\sum_{P_+}(-)^{|P_+|}W_{R_+P_+}(q)W_{P_+^TA_+^T}(q)q^{\frac{N-M}{2}|P_+|}$$
$$\sum_{P_-}(-)^{|P_-|}W_{R_-P_-}(q)W_{P_-^TA_-^T}(q)q^{\frac{N-M}{2}|P_-|}$$

Next, recall that (see appendix A) the large M limit (more precisely, the leading chiral block) of $(TST)^{-1}$ is

$$(T^{-1}S^{-1}T^{-1})_{AQ} = \alpha_M(q^{-1})W_{A_+Q_+}(q)W_{A_-Q_-}(q)q^{-\frac{\kappa_{A_+}+\kappa_{A_-}+\kappa_{Q_+}+\kappa_{Q_-}}{2}}$$

To compute our final expression, we need to sum:

$$\sum_{\mathcal{P}} S_{\mathcal{RP}}(N,g_s)\, q^{\frac{C_2^{(M)}(\mathcal{P})}{2}}\, S_{\bar{\mathcal{P}}\mathcal{Q}}(M,g_s) \to \alpha_M(q^{-1})(-)^{|R_+|+|R_-|}q^{-\frac{N-M}{2}(|R_+|+|R_-|)}$$
$$q^{-\frac{\kappa_{R_+}+\kappa_{Q_+}}{2}} \sum_{P_+,A_+} (-)^{|P_+|}W_{R_+P_+}(q)W_{P_+^TA_+^T}(q)W_{A_+Q_+}(q)q^{\frac{N-M}{2}|P_+|}$$
$$q^{-\frac{\kappa_{R_-}+\kappa_{Q_-}}{2}} \sum_{P_-,A_-} (-)^{|P_-|}W_{R_-P_-}(q)W_{P_-^TA_-^T}(q)W_{A_-Q_-}(q)q^{\frac{N-M}{2}|P_-|}.$$

Note that this contains an ill-defined expression

$$\sum_{A_+} W_{P_+^TA_+^T}(q)\, W_{Q_+A_+}(q)\; \sum_{A_-} W_{P_-^TA_-^T}(q)\, W_{Q_-A_-}(q) \tag{3.60}$$

The physical interpretation of a finite version of this amplitude, with insertions of $e^{-t|A_+|}$ and $e^{-\bar{t}|A_-|}$ also suggests how to define (3.60) . Namely, the finite amplitude is the topological string amplitude (more precisely, two copies of it) on $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbf{P}^1$ with D-branes as in the figure 6, where the size of the $\mathbf{P}^1$ is t. In the limit $t\to 0$ the $\mathbf{P}^1$ shrinks to zero size, and one can undergo a conifold transition, to a small S^3 of size ϵ. In this case, the only holomorphic maps correspond to those with $P_+ = Q_+$, so that

$$\sum_{A_+} W_{P_+^TA_+^T}(q)\, W_{Q_+A_+}(q) \;= \delta(P_+ - Q_+),$$

and similarly for the anti-chiral piece, which is independent of ϵ, as this is a complex structure parameter.

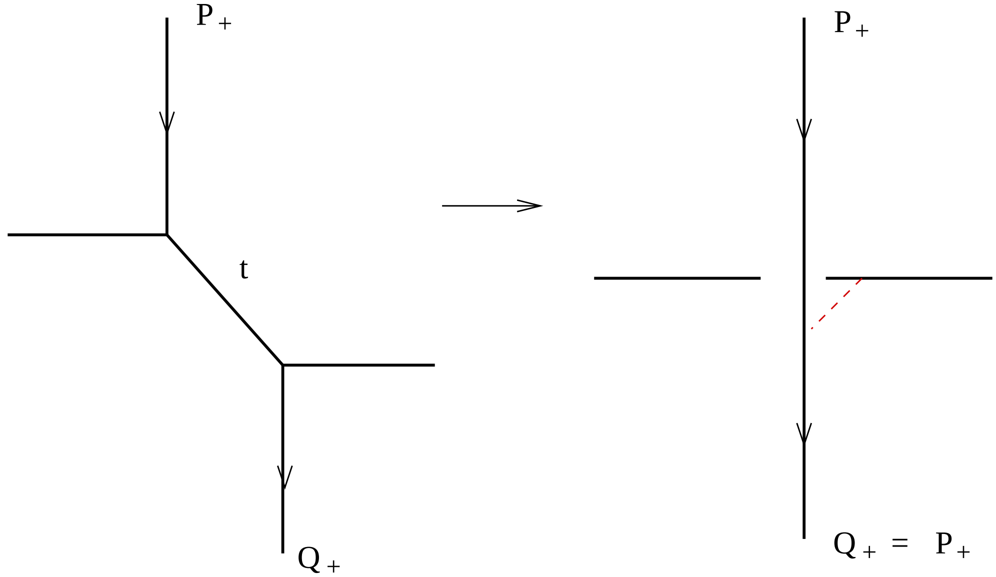

Figure 3.6: The figure on the left corresponds to $O(-1)\oplus O(-1) \to \mathbf{P}^1$ with $\mathbf{P}^1$ of size t with two stacks of lagrangian D-branes. The representations P_+ and Q_+ label the boundary conditions on open string maps. When $t = 0$ the Calabi-Yau is singular, but can be desingularized by growing a small S^3. The singular topological string amplitudes can be regulated correspondingly, and with this regulator, they vanish unless $P_+ = Q_+$. See [5] for more details.

Our final result is:

$$\sum_{\mathcal{P}} S_{\mathcal{R}\mathcal{P}}(N, g_s)\, q^{\frac{C_2^{(M)}(\mathcal{P})}{2}}\, S_{\bar{\mathcal{P}}\mathcal{Q}}(M, g_s) \to \alpha_M(q^{-1})\, \theta^M(q)\ (-)^{|R_+|+|R_-|+|Q_+|+|Q_-|}$$
$$q^{-\frac{N-M}{2}(|R_+|+|R_-|)} q^{\frac{N-M}{2}(|Q_+|+|Q_-|)} q^{-\frac{\kappa_{R_+}+\kappa_{R_-}}{2}} q^{-\frac{\kappa_{Q_+}+\kappa_{Q_-}}{2}} W_{R_+Q_+}(q) W_{R_-Q_-}(q) \tag{3.61}$$

Chapter 4

Crystals and intersecting branes

4.1 Introduction

The recent conjecture of Ooguri, Strominger, and Vafa [62] that the indexed entropy of $\mathcal{N} = 2$ black holes, in a mixed ensemble, factorizes, perturbatively to all orders for large charges, into the square of the topological string, together with subsequent checks for the case of IIA compactified on local Calabi Yau in [69] [7] [2] has generated much renewed interest in the structure of the partition functions of twisted $\mathcal{N} = 4$ Yang-Mills living on toric D4 branes. In this work, we will discover a situation in which this theory localizes to a discrete sum over equivariant configurations, associated to a new type of melting crystals, related to those of [60] which describe the topological A-model in the same geometry. Taking the limit of large D4 brane charge automatically reproduces the square of the A-model crystals, at the attractor values of the Kahler moduli.

Consider D4 branes wrapping various 4-cycles of a toric Calabi-Yau, with chemical potentials for D2 and D0 branes turned on. The partition function, studied in [70], is the generating function of the number (really index) of BPS bound states, which is given by the Euler characteristic of their moduli space. This can be computed in the $\mathcal{N}=4$ worldvolume $U(N_i)$ Yang Mills on the 4-cycles wrapped by N_i D4 branes, with some bifundamental couplings along the intersections. We will find that, at least locally in a single vertex geometry, the number of bound states is captured by the combinatoric counting of three dimensional partitions subject to a certain truncation.

From the perspective of the remaining four dimensional spacetime, these bound states appear as BPS black holes, preserving half of the $\mathcal{N}=2$ supersymmetry, possessing a classical horizon with nonzero area in the regime of large charges. The Kahler moduli live in vector multiplets in IIA theory, hence they are driven to their attractor values at the horizon, which are determined entirely by the brane charges, via the quantum corrected prepotential. The Legendre transformation leading to a mixed ensemble, with chemical potentials ϕ^0 and ϕ^a for the D0 and D2 charges, results in attractor values that are independent of the prepotential,

$$g_{top} = \frac{4\pi^2}{\phi^0 - i\pi p^0}, \ t_a = g_{top}(p^a + \frac{i}{\pi}\phi^a), \tag{4.1}$$

as shown in [62]. The BPS partition function in this ensemble factorizes in the large charge limit according to the OSV relation,

$$Z_{BPS}(\phi^0, \phi^a; p^0, p^a) = |\psi_{top}(g_{top}, t_a)|^2, \tag{4.2}$$

to be interpreted as a perturbative expansion in the topological string coupling, $g_{top} \sim 1/\phi^0$. We will be concerned with the case of vanishing D6 charge, $p^0 = 0$.

The torus symmetry of a toric Calabi Yau induces an action on the moduli space of branes, enabling us to apply the equivariant localization theorem that the Euler character of the full moduli space is the same as that of the space of invariant configurations. Therefore we can assume that the D0 branes wrap the fixed points, and the D2 branes wrap fixed curves, which are the $\mathbb{P}^1$ legs of the toric diagram. Imagine cutting the Calabi Yau along the toric legs, so that it looks like copies of the local $\mathbb{C}^3$ vertex glued along $\mathbb{P}^1$'s over which the manifold is fibered as $\mathcal{O}(-p) \oplus \mathcal{O}(p-2)$ for an integer p sometimes called the framing. Then the bound states of D2 branes can be recovered by matching equivalent configurations at the cut, and D0 branes are localized to a single vertex. This is very reminiscent of the toric localization [4] of the topological string amplitudes, particularly as computed by the quantum foam D6 brane theory [37].

The generic points in the moduli space of very ample D4 branes can be reached by turning on VEVs for the $\mathcal{N} = 4$ adjoint field describing deformation in the normal direction inside the Calabi Yau, leading to a configuration of a single D4 brane wrapping a complicated smooth surface, $\mathcal{C} \subset X$, whose homology is fixed by the brane charges. In this phase, the bound states that were described in terms of D2 and D0 branes dissolved into flux of the $\prod U(N_i)$ quiver theory of intersecting D4 branes, become $U(1)$ flux configurations associated to the large number of new divisors in the deformed surface. The "D2" bound states wrapping curves in $H_2(\mathcal{C}) > H_2(X)$ which are homologically trivial in the ambient 6 dimensional geometry carry no D2 charge, and replace the D0 flux bound states present in the $\prod U(N_i)$ phase.

It is easy to see that such smooth D4 surfaces beak the $(\mathbb{C}^*)^3$ symmetry of the toric Calabi Yau. The crystal construction arises when we consider nilpotent VEVs for the adjoint field, which allows the use of the underlying toric symmetries to solve the theory. This corresponds to the limit in which the new divisors in $\mathcal{C}$ shrink to zero size, giving rise to the nontrivial D0 bound states at the singular points that will be counted by the crystal.

The bound states of D2 and D0 branes to a single stack of N D4 branes wrapping a noncompact cycle in a toric Calabi Yau was studied in [7], where the theory was reduced to q-deformed Yang Mills in two dimensions. The D2 bound states wrapping a $\mathbb{P}^1$ are parameterized by irreducible representations of $U(N)$, which are associated to the holonomy basis of the Hilbert space of the two dimensional topological gauge theory. These configurations can be readily understood in the Higgs phase of the D4 theory, describing the generic point in the moduli space of normal deformations where there is a single D4 brane wrapping a complicated surface. In that case, the D4 worldvolume is a N-sheeted fibration over the $\mathbb{P}^1$, and hence, locally, the transverse motion of k bound D2 branes is described by k points in the fiber, with simply looks like N copies of $\mathbb{C}$. Note that for the purpose of this discussion we can ignore the global obstruction to deforming a curve inside the D4 worldvolume. This is because the partition function separates into propagator and cap contributions, and the D2 bound states can be equivalently analyzed in the propagator geometry, over which the bundles are trivial. Counting the number of ways of distributing k objects among N choices gives the relevant $U(N)$ representations, associated to the moduli space

$$\frac{Sym^k(N\mathbb{C})}{S_N}. \tag{4.3}$$

We will see this feature more clearly by choosing a toric invariant D4 worldvolume in which to perform the analysis, where these bound states will be described by certain torus equivariant ideals.

Furthermore, there is an interesting and suggestive structure to the generating function of bound states in the vertex geometry of a stack of N D4 branes with only D0 branes. In the q-deformed Yang Mills language, this is the vacuum cap amplitude, that is, the amplitude with trivial holonomy, which has been calculated in [7] to be

$$\eta(q)^{-N\chi}S_{\cdot\cdot}^{(N)} = \frac{1}{\eta^N}\sum_{\sigma\in S_N}(-)^\sigma q^{-\rho_N\cdot\sigma(\rho_N)} = q^{-N/24}\prod_{n>0}\frac{1}{(1-q^n)^N}\prod_{1\leq i<j\leq N}[j-i]_q. \quad (4.4)$$

The BPS states in which the D0 charge is dissolved into $U(N)$ flux are captured by the q-deformed Yang Mills amplitude, while the η function counts one state for each k-tuple of point-like D0 branes bound to one of N identical D4 branes, noting that the effective Euler characteristic of the cap is $\chi = 1$. This amplitude can be expressed as

$$\prod_{n>0}\frac{1}{(1-q^n)^N}\prod_{j=1}^{N-1}(q^{j/2}-q^{-j/2})^{N-j} = \prod_{j=1}^{N}\frac{1}{(1-q^j)^j}\prod_{j>N}\frac{1}{(1-q^j)^N}, \quad (4.5)$$

associated to bound states of k-tuples of D0 branes with k ground states, truncated at N. This function nicely interpolates between the $1/\eta$ for a single D4 brane, and asymptotes to the McMahon function, which counts bound states to a D6 brane. The truncation has a transparent interpretation in terms of the crystal we will now proceed to describe.

Consider the generic situation of N, M, and K D4 branes wrapping the three intersecting 4-cycles in the local $\mathbb{C}^3$ vertex, with bound D2 branes in the legs, and n D0 branes at the vertex. We will first find that the equivariant BPS bound states of p D2 branes wrapping the intersection curve of N and M D4 branes are partitions of p that do not contain the point $(M+1, N+1)$. In the special case $M = 0$, this reduces to Young tableaux with N rows, which are related to the $U(N)$ representations that span the q-deformed Yang Mills Hilbert space. Such bound states have D2 charge

p, and induce some D0 charge which depends on the local bundle over the wrapped curve via the intersection form. Fixing such Young diagrams, R, Q, and P, as the asymptotic D2 brane configuration in the vertex geometry, we will show that each contribution to the index of BPS states carrying an additional n units of D0 brane charge can be associated to a three dimensional partition of n in the truncated region of the octant obtained by deleting all the points beyond $(N+1, M+1, K+1)$, with asymptotic behavior R, Q, P, as shown in figure 4.

The organization of this paper is as follows: In section 2, we review the theory of BPS D4 brane bound states, particularly in the local toric geometries studied in [69] [7] and [2], and the OSV relation to topological strings [62]. In section 3, we will examine the worldvolume theory on these branes in the Higgs branch. Giving a nilpotent VEV to the adjoint field, which controls normal deformations inside the Calabi Yau, reduces the partition function to a sum over toric invariant ideal sheaves of certain singular surfaces. In section 4, we study the bound states with D2 branes in the toric "leg". In section 5, we compute the index of D0 bound states in the vertex geometry. In section 6, we see that the OSV factorization is manifestly verified in all cases where these methods apply. Finally, in section 7, we make some concluding remarks.

4.2 BPS bound states of the D4/D2/D0 system

The remarkable conjecture of OSV [62] relates the large charge asymptotics of the indexed degeneracy of BPS bound states making up supersymmetric black holes in four dimensions with the Wigner function associated to the topological string wavefunction. Consider compactifying IIA theory on a Calabi Yau manifold, X, which gives rise to an effective $\mathcal{N}=2$ supergravity in the remaining four dimensions, with vector multiplets associated to the Kahler moduli. The microstates of $\frac{1}{2}$ BPS black holes in this theory are bound states of D6/D4/D2/D0 branes wrapping holomorphic cycles in X.

The effective theory in four dimensions has $U(1)$ gauge fields obtained by integrating the RR 3-form, C_3, over 2-cycles, $[D^a] \in H_2(X)$ for $a = 1, \ldots h^{1,1}(X)$, in the Calabi Yau. The D2 branes wrapping these 2-cycles and D4 branes on the dual 4-cycles, $[\check{D}_a]$, carry electric and magnetic charges respectively under these $U(1)$'s. There is one further $U(1)$ vector field, arising from the RR 1-form in ten dimensions, under which the D0/D6 branes are electrically (magnetically) charged.

We will investigate this partition function at 0 D6 charge in the local context of toric Calabi Yau, where the theory on the branes is given by twisted $\mathcal{N}=4$ Yang Mills on each stack of D4 branes, coupled along their intersections. The action of the $U(N)$ worldvolume theory of N D4 branes is BRST exact, and turning on the chemical potential for bound D2 and D0 branes inserts the closed (but not exact) operators,

$$\frac{\phi_0}{8\pi^2}\int F\wedge F + \frac{\phi_a}{2\pi}\int F\wedge k_a, \tag{4.6}$$

into the action. These topological terms count the induced D2 and D0 charge of a given gauge connection.

This Vafa-Witten theory [70] computes the indexed degeneracy of BPS bound states with partition function equal to

$$Z_{YM}({Q_4}^a, \varphi^a, \varphi^0) = \sum_{Q_{2\,a}, Q_0} \Omega(Q_4^a, Q_{2\,a}, Q_0) \quad \exp\left(-Q_0\varphi^0 - Q_{2\,a}\varphi^a\right), \tag{4.7}$$

in the mixed ensemble of fixed D4 (and vanishing D6) charges.

A collection of D4 branes in the OSV mixed ensemble will be well described by semiclassical ten dimensional supergravity as a black hole with large horizon area when the branes live in a homology class deep inside the Kahler cone [49]. Given D4 branes wrapping such a very ample divisor, $[D]$, one can show that the attractor values of the moduli are large and positive,

$$\mathfrak{Re}(t^a) >> 0, \tag{4.8}$$

as well as the leading classical contribution to the black hole entropy,

$$C_{abc} t^a t^b t^c > 0, \tag{4.9}$$

since the intersections numbers of $[D]$ with 2-cycles are positive. It is in this regime where we expect the brane partition function to factorize into the square of the topological string wavefunction. Moreover, these are precisely the conditions on the 4-cycle, with homology $[D]$ that is wrapped by the D4 branes, to have holomorphic deformations. The existence of such global sections of the $(2,0)$ bundle over the 4-cycle in the compact case would allow us to turn on a topologically exact mass deformation and solve the theory following [70]. For branes wrapping noncompact 4-cycles, the mass deformation results in the reduction to a theory on the two dimensional compact divisors.

This two dimensional theory was shown to be the q-deformed $U(N)$ Yang Mills on the divisor of genus g in [69], [7], who then solved the theory using techniques of topological gauge theory in two dimensions, obtaining the result

$$Z_{YM} = \sum_{\mathcal{R}} (\dim_q \mathcal{R})^{2-2g} q^{\frac{p}{2} C_2(\mathcal{R})} e^{i\theta|\mathcal{R}|}, \tag{4.10}$$

where $\mathcal{R}$ is an irreducible $U(N)$ representation. The more general situation of D4 branes intersecting in the fiber over the intersection point of two curves was solved in the toric context by [2], where the q-deformed Yang Mills theories found on each curve were coupled by operators inserted at the intersection point, obtained from integrating out bifundamentals. In this work, we will check the results of the crystal partition function against these examples, as well as generalizing to configurations of D4 branes intersecting along the base.

The BPS partition function in the large charge limit was shown to factorize, confirming the generalization of the structure $Z \sim |\Psi_{top}|^2$ to the noncompact case with an extra sum over chiral blocks parameterized by $SU(\infty)$ Young tableaux, P, P' controlling the noncompact moduli. In particular, for the case of N D4 branes in the Calabi Yau $\mathcal{O}(-p) \oplus \mathcal{O}(p-2) \to \mathbb{P}^1$ studied in [7], the large N 't Hooft limit gives

$$Z = \sum_{\ell \in \mathbb{Z}} \sum_{P,P'} Z^{qYM,+}_{P,P'}(t + pg_s\ell) Z^{qYM,-}_{P,P'}(\bar{t} - pg_s\ell), \tag{4.11}$$

where the topological string amplitudes are

$$\begin{aligned} Z^{qYM,+}_{P,P'}(t) &= q^{(\kappa_P + \kappa_{P'})/2} e^{-\frac{t(|P|+|P'|)}{p-2}} \\ &\qquad \times \sum_R q^{\frac{p-2}{2}\kappa_R} e^{-t|R|} W_{PR}(q) W_{P'^t R}(q) \\ Z^{qYM,-}_{P,P'}(\bar{t}) &= (-1)^{|P|+|P'|} Z^{qYM,+}_{P^t,P'^t}(\bar{t}), \end{aligned} \tag{4.12}$$

and

$$g_s = \frac{4\pi^2}{\phi_0} \qquad t = \frac{p-2}{2}\frac{4\pi^2}{\phi_0}N - 2\pi i \frac{\phi_2}{\phi_0}, \tag{4.13}$$

are the attractor values of the moduli.

Recall that in toric Calabi Yau all compact 4-cycles have positive curvature, and thus negative self-intersection number. Branes wrapping these rigid cycles will contribute negative real part to the attractor moduli. It is possible, however, that even in this rigid case a modified local analysis is still possible, in which an additional constraint for each wrapped compact 4-cycle is introduced on the sum of local bundles over its toric divisors. For example the twisted $\mathcal{N} = 4$ Yang Mills on the rigid divisor $\mathbb{P}^2$ in the Calabi Yau $\mathcal{O}(-3) \to \mathbb{P}^2$ is solved in [70], [43], and [42].[1] These partition functions do not factorize, and correspond to the regime in which nonperturbative baby universe effects are not suppressed [22] [2].

More applicably in the context of OSV, some of the branes might wrap rigid surfaces, as long as the total charge is the in Kahler cone. In this case one can still expect a factorization into the topological and anti-topological string, and a simplification of the Yang Mills theory on the compact rigid surfaces may occur due to the intersection with sufficiently large numbers of branes wrapping very ample divisors.

Another interesting feature of the D4 brane partition functions is their behavior under s-duality of the twisted theory in four dimensions. This is not manifest in the crystal formalism, since in our situation of noncompact D4 branes, the boundary conditions at infinity in the noncompact directions transform in a complicated way. It would be interesting to understand more completely how the duality is realized in our picture, although one aspect is already clear. Namely, we are really counting the vector bundles associated to the *dual* flux of the topological Yang-Mills theory. This explains why our expansions appear most naturally in terms of $q = e^{-g_s} = e^{-\frac{4\pi^2}{\phi^0}}$ rather then $\tilde{q} = e^{-\phi^0}$. We will also use the variable $\theta_a = \frac{g_s}{2\pi}\phi^a$.

The boundary conditions that are naturally associated to the crystal description

[1] We thank C. Vafa and M. Aganagic for discussions that led to the statements here.

are that no bound D2 branes should reach ∞ in the noncompact D4 worldvolume. This turns out to be the same condition used in the derivation of q-deformed Yang Mills [69] [7], but is the opposite of the standard "electric" convention from the four dimensional worldvolume point of view, which would require the holonomies of the gauge field to vanish asymptotically. These are exchanged by the action of s-duality. For example, the instantons contributing to the $U(1)$ Yang Mills amplitudes on $\mathcal{O}(-p) \to \mathbb{P}^1$, expressed in terms of the gauge fields, are given by the lift of line bundles on the $\mathbb{P}^1$, with noncompact toric divisors lying in the fiber over the fixed points. The dual configuration has the $\mathbb{P}^1$ itself as toric divisor, and it is this that we shall count.

4.2.1 Turning on a mass deformation for the adjoints

The deformations of the wrapped surface are encoding in the adjoint fields of the twisted $\mathcal{N} = 4 \;\; U(1)$ gauge theory living on the D4 brane. The torus symmetry of the underlying geometry can be used to localize the D4 brane to the equivariant configuration, by the usual method of turning on a Q-trivial mass deformation,

$$W = mUV + \omega T^2, \tag{4.14}$$

parameterized by the choice of ω, a global section of the $(2,0)$ bundle over $\mathcal{C}$, where U and T are scalar adjoint fields associated to motion in spacetime, while V governs deformation normal to $\mathcal{C}$ inside the Calabi-Yau. The superpotential, W, must live in a $(2,0)$ bundle over $\mathcal{C}$.

In our case, there is an important new feature to this story, due to the compactification of the normal direction in the Calabi-Yau. The adjoint field, V, will take values in some compact space, which, in general, would greatly complicate the analysis. As explained in [2] this issue can be avoided when the D4 branes wrap noncompact cycles, since one is free to choose boundary conditions at infinity corresponding to a particular surface in the given homology class. Changes of these boundary conditions are determined by non-normalizable modes in the four dimensional theory on the branes, and we should not sum over them, at least in the local case. It is sensible to choose this divisor to be torus invariant, for consistency with the toric symmetry of the local geometry (in fact, it is only in the equivariant sense that we can even appropriately distinguish various 4-cycles which would otherwise appear homologous in the noncompact Calabi-Yau).

It is gratifying to understand this point in greater detail, for concreteness in the example of local $\mathbb{P}^1 \times \mathbb{P}^1$. Let us consider a single D4 brane wrapped on the 4-cycle, $\mathcal{C}$, given by $\mathcal{O}(-2) \to \mathbb{P}^1$, whose normal direction is a compact $\mathbb{P}^1$. This seems to imply that the field, V, should be $\mathbb{P}^1$ valued, however this is modified by the fact that our 4-cycle, $\mathcal{C}$, is nontrivially fibred over this $\mathbb{P}^1$. The local coordinate normal to the brane is thus a section of the $\mathcal{O}(-2)$ bundle, rather then the trivial bundle. Globally, this implies that equation (4.14) will have two saddle points where this section vanishes. In order to preserve the torus symmetry, we must choose an equivariant section of the $\mathcal{O}(-2)$ bundle, which implies that the mass deformation localizes the D4 brane to exactly the two invariant surfaces with the homology class of $[\mathcal{C}]$, namely those sitting above the north and south pole of the other $\mathbb{P}^1$, exactly as expected.

4.3 Localization in the Higgs branch of the theory of intersecting branes

The worldvolume theory of D4 branes can be localized, due to the topological nature of the partition function, to a computation of the Euler characteristic of the moduli space of anti-self dual instanton solutions. This is because the action of the twisted theory is exact in the cohomology of the twisted supercharge, Q, and the chemical potentials for D0 and D2 branes couple to the instanton charge, $\int F \wedge F$, and $\int_{[D]_a} F$ respectively, which are topological invariants. It can be shown that these anti-self dual connections exist uniquely for each stable $U(N_i)$ vector bundle (or certain sheaves in a more general, singular context).

It is also necessary to take into account the contribution of additional bound D0 branes, which are given by sheaves with singularities at points where the D0 branes are located. This gives rise to a universal contribution of $\eta^{-\chi}$ of the eta function to the power of minus the Euler character [57]. The eta function appears as the generating function of Young tableaux associated to the Hilbert scheme of points in two dimensions. Such effects will be automatically included in our case, where all of the bound states will have to be described exclusively in the sheaf language, since the D4 branes will wrap singular surfaces.

Any collection of intersecting D4 branes whose total homology class is very ample can be deformed in the normal directions, generically producing a single D4 brane wrapping a complicated surface. Thus the quiver of $U(N_i)$ twisted $\mathcal{N} = 4$ Yang Mills theories with bifundamental interactions can be replaced by the $U(1)$ gauge theory on this high degree 2-fold, $\mathcal{C}$, which computes the same partition function. In both cases the adjoint field associated to motion of the brane in the Calabi-Yau directions will generally be compactified in a highly nontrivial manner.

For example, consider wrapping N D4 branes on the cycle $\mathcal{O}(-3) \to \mathbb{P}^1$ in the Calabi Yau $\mathcal{O}(-3) \oplus \mathcal{O}(1) \to \mathbb{P}^1$, and M on the rigid $\mathcal{O}(1) \to \mathbb{P}^1$ cycle. Then the attractor value of the Kahler modulus, $\mathfrak{Re}\ t_{\mathbb{P}^1} = \frac{g_s}{2}(N-3M)$, will be positive for $N > 3M$, hence we should be able to write down a smooth surface with these charges. The projective coordinates are given by $(z, v; x, y) \sim (\lambda z, \lambda v; \lambda x, \lambda^{-3} y)$, where $z = v = 0$ is deleted. The deformed 4-cycle is described an equation of the form

$$x^N y^M + a x^{N-3} y^{M-1} + \cdots + b z^{N-3M} + \cdots + c v^{N-3M} = 0, \tag{4.15}$$

where each term has degree $N - 3M > 0$.

Therefore we really want to calculate the generating function for the Euler characters of the moduli space of ideal sheaves (ie. rank 1 stable sheaves) on $\mathcal{C}$. To do this, we will extensively use the toric geometry of the underlying Calabi Yau manifold. For such techniques to be applicable, the surface, $\mathcal{C}$, must be chosen carefully to be invariant itself. For a high degree surface with positive self-intersection of this type, the only possibility is a highly singular realization as a "thick" subscheme living on the original intersecting toric divisors. This is achieved in the worldvolume gauge theory by giving a nilpotent VEV to the adjoint field V that controls the normal deformations inside the Calabi Yau.

The ideal sheaves on such an affine space are equivalent to ideals of the algebra of functions, and can be counted in the equivariant setting as described below. By cutting the subscheme $\mathcal{C}$ into pieces, we can obtain the full answer by gluing vertices via propagators, analogously to [37]. Note that toric localization is crucial for this procedure to work, with D0 branes localizing to the vertices, and D2 branes to the legs. The vertex amplitudes, associated to bound states of D0 branes with the D4 and D2 branes, in the chemical potential, will turn out to be nicely encoded as the statistical partition function of a cubic crystal in a restricted domain.

4.3.1 Solving the twisted $\mathcal{N} = 4$ Yang Mills theory

The key point is that we have replaced the complicated system of intersecting stacks of N_i D4 branes described by bifundamental-coupled $U(N_i)$ Yang Mills theories by a single D4 brane wrapping a nonreduced subscheme of X. The $U(1)$ theory on that brane can be solved using toric localization, and the well known fact that the Vafa-Witten theory computes the generating function of stable bundles. The partition function is given by

$$Z_{YM} = \sum_{n \in H^0, [A] \in H^2} q^n e^{i \sum_a \theta_a A_a} \chi\left(\mathcal{M}_{n,[A]}\right), \tag{4.16}$$

where $\mathcal{M}_{n,[A]}$ is the moduli space of coherent sheaves, n is the instanton number (equivalently, the D0 charge), $[A]$ is the D2 charge, and a indexes H_2. The effective number of bound states with given charges is given by the Euler characteristic of the instanton moduli space. The brane charges are given exactly by the coefficients of the exponential Chern character, which is, in our situation,

$$\begin{aligned} q_{D0} &= 2ch_3 = c_3 - c_1 c_2 + \frac{1}{3} c_1^3 \\ q_{D2} &= ch_2 = -c_2 + \frac{1}{2} c_1^2 \\ q_{D4} &= ch_1 = [\mathcal{C}] \\ q_{D6} &= 0, \end{aligned} \tag{4.17}$$

where we are ignoring the gravitation contribution from the first Pontryagin class of the Calabi Yau, which gives an overall contribution independent of the D2/D0 bound state, and is in fact somewhat ambiguous in the noncompact setting.

We want to determine the contribution of D0 branes bound to the vertex in the toric geometry the looks locally like $\mathbb{C}^3$. In order to be able to apply the machinery of equivariant localization, we must choose a torus invariant representative of the homology class $[\mathcal{C}]$. This is impossible for smooth surfaces, thus we are forced to consider the singular, nonreduced subscheme described by the polynomial equation $x^N y^M z^K = 0$, where N, M, K are the number of D4 branes wrapping the equivariant homology classes associated to the three planes in $\mathbb{C}^3$. Because the worldvolume is so singular, we need to be careful when determining the moduli space of line bundles. Looking at the ring of functions on the surface, it is clear that it will approach the algebra

$$\mathcal{A} = \frac{\mathbb{C}[x, y, z]}{(x^N y^M z^K)} \tag{4.18}$$

as we deform a smooth surface to the singular limit. This corresponds to a non-reduced subscheme of $\mathbb{C}^3$, since the polynomial $x^N y^M z^K$ does not have separated roots, which translates the fact that the adjoint fields of the gauge theory have nilpotent, non-diagonalizable VEVs.

Given any very ample line bundle on the affine scheme $\mathcal{C}$, we can associate to it in the usual way an ideal of $\mathcal{A}$ by considering its sections. For a general line bundle, $\mathcal{E}$, corresponding to asymptotic representations with negative row lengths along a given curve, we first take the tensor product with a line bundle, $\mathcal{L}_n$, over the D4 worldvolume of sufficiently positive first Chern class, n, to obtain a very ample bundle. The bundle $\mathcal{L}_n$ must descend from a bundle, $\mathcal{O}(n)$, on the physical 4-cycle, namely, its divisor is simply the multiply wrapped curve in question. The instanton number can easily be calculated to be

$$ch_2(\mathcal{E} \otimes \mathcal{L}_n) = ch_2(\mathcal{E}) + c_1(\mathcal{L}_n)c_1(\mathcal{E}) + \frac{1}{2}c_1(\mathcal{L}_n)^2, \tag{4.19}$$

since $ch(\mathcal{E}) \otimes ch(\mathcal{L}) = ch(\mathcal{E})ch(\mathcal{L})$. Hence we can effectively concentrate on the $SU(N)$ component of the partition function, from which the full answer can be trivially reconstructed by summing over the choice of such line bundles, $\mathcal{L}_n$.

Now we can use the action of the toric symmetry to considerably simplify the problem. There is a equivariant localization theorem which relates the cohomologies of the moduli space $\mathcal{M}_{n,[A]}$ of bundles (or rather sheaves) on $\mathcal{C}$ with Chern classes $n \in H_0 = \mathbb{Z}$ and $[A] \in H_2$ to those of invariant sheaves,

$$\chi\left(\mathcal{M}_{n,[A]}\right) = \chi\left(\mathcal{M}^T_{n,[A]}\right), \tag{4.20}$$

that is, the Euler character of the moduli space is captured by the considering only the equivariant bundles. In more physical terms, the partition function involves summing over BPS states of D2 branes wrapping the legs of the toric diagram, that is the torus invariant $\mathbb{P}^1$'s, and D0 branes at the equivariant points, which are the vertices, bound to the D4 brane wrapping the nonreduced cycle $\mathcal{C}$.

In the next section, we will determine the number of equivariant bound states with wrapped D2 branes, and compute the induced D0 charge. They will correspond to a choice of representation along each leg, with rank determined by the D4 charges (for example, we will have $U(N)$ representations for D2 branes bound to a stack of N D4 branes). By cutting the Calabi Yau along the legs of the toric diagram, to effectively obtain glued $\mathbb{C}^3$ geometries. Clearly this cutting of the Calabi Yau is only valid in the toric context, since we have seen that the sum over D2 branes localizes to those wrapping only the legs. The situation here is analogous to that studied in [4] where the topological string amplitudes on toric Calabi Yau were shown to reduce to gluing copies of the $\mathbb{C}^3$ vertex, because the worldsheet instantons localized to equivariant curves. Similar localization arguments were used by [37] in the more precisely related context of counting instantons of the $U(1)$ gauge theory living on a D6 brane, and we will not repeat them here. This defines a vertex, $\mathcal{V}_{RQP}$, depending on up to three intersecting stacks of D4 branes intersecting in $\mathbb{C}^3$, and the asymptotic representations, R, Q, P, associated to the D2 configuration. Gluing these vertices by the propagators along the legs, we obtain the full partition function.

In the vertex geometry, the torus invariant bound states of D0 branes to the D4 worldvolume in the nilpotent Higgs phase are given by the equivariant ideals of the algebra (4.18). The toric symmetry acts by $(\mathbb{C}^*)^3$ multiplication on the coordinates, $(x, y, z) \to (\lambda_1 x, \lambda_2 y, \lambda_3 z)$, hence the only invariant ideals are those generated by monomials. Depicting the monomials, $x^n y^m z^k$, of the polynomial algebra, $\mathbb{C}[x, y, z]$, as the lattice of points, (n, m, k), in the positive octant, the algebra relation $x^N y^M z^K = 0$ is represented by deleting all points (monomials) set to zero. The remaining points give a three dimensional partition, which has been "melted" from the corner of the cubic lattice.

Consider a D0 bound state described by a toric ideal, $\mathcal{I} \lhd \mathcal{A}$, spanned by monomials, $x^n y^m z^k$, for $(n, m, k) \in S$. Clearly if $x^n y^m z^k \in \mathcal{I}$, then so is $x^{n+1} y^m z^k$ and so on, hence the quotient algebra $\frac{\mathcal{A}}{\mathcal{I}}$ will be depicted as a three dimensional partition, in the truncated region described by the D4 worldvolume algebra, $\mathcal{A}$. This quotient is the natural algebra associated to the D0 worldvolume, the codimension 2 analog of the divisor of the sheaf on $\mathcal{C}$. Hence the D0 charge of this bound state is exactly the dimension of the quotient algebra, which is the number of points in the partition. The region in which the configurations are restricted to live is defined by the polynomial relations of the algebra given by (4.18), shown graphically in figure 4.

4.4 Propagators and gluing rules

Typically, our D4 branes will wrap 4-cycles containing compact toric divisors joining two fixed point vertices. These geometries will always look locally like $\mathcal{O}(-p) \to \mathbb{P}^1$, for some p, possibly intersecting other D4 branes along the toric divisors in the Calabi Yau, X. For simplicity, we will first analyze this geometry in isolation, with a single stack of N wrapped D4 branes. They can be described in the above language as the subscheme, $\mathcal{C} \subset X$, associated to the graded algebra

$$\mathcal{A} = \frac{\mathbb{C}[x : t; y, z]}{z^N}, \tag{4.21}$$

with weights defined by

$$(x : t; y, z) \sim (\lambda x : \lambda t; \lambda^{-p} y, \lambda^{p-2} z), \tag{4.22}$$

for $\lambda \in \mathbb{C}^\times$.

Rank N vector bundles on the 4-cycle can now be described as rank 1 sheaves over $\mathcal{C}$, and thus by ideals of $\mathcal{A}$. Moreover, invariant bundles whose corresponding equivariant divisors are (possibly non-reduced) multi-wrappings of the $\mathbb{P}^1$ are equivalent to homogeneous ideals of the form

$$\mathcal{I} = (\{y^{R_i} z^{i-1}\}), \tag{4.23}$$

for $i = 1, \ldots N$. These ideals are in one to one correspondence with irreducible representations of $U(N)$ with positive row lengths. More general representations can be obtained as explained above in equation (4.19). We proceed to calculate the Chern characters of such bundles, which will serve as the background configuration of our crystals. This might seem difficult, since the base manifold is now non-reduced, so we will need to relate these sheaves to ones living over the full Calabi-Yau geometry.

In a similar manner, the propagator along legs where stacks of D4 branes intersect can be determined by counting equivariant ideals of the algebra

$$\mathcal{A} = \frac{\mathbb{C}[x:t; y, z]}{(y^M z^N)}. \tag{4.24}$$

They will be of the form

$$\mathcal{I} = (\{y^{R_i} z^{i-1}\}), \tag{4.25}$$

where the Young tableau R lies in the region restricted by N and M, that is, $0 \leq R_i < M$ for $i > N$. We will refer to such Young diagrams as being of type (N, M). The following analysis of the induced D0 charge for rank N bundles is easily extended to include this case as well.

The subscheme, $\mathcal{C}$, is embedded via a map

$$i : \mathcal{C} \hookrightarrow X, \tag{4.26}$$

allowing us to push forward a sheaf, $\mathcal{E}$, to $i_*\mathcal{E}$, a sheaf on X with support only along the 4-cycle. This corresponds to the fact that the ideals of $\mathcal{A}$ defined above are clearly also ideals of the structure algebra of X itself. The Chern characters of $i_*\mathcal{E}$ can be determined using equivariant techniques to be [37]

$$\begin{aligned} 2ch_3 &= ||R^t||^2 + \frac{p}{2}\kappa_R \\ ch_2 &= |R|[\mathbb{P}^1]. \end{aligned} \tag{4.27}$$

This is not quite the whole answer, since those Chern characters have contributions not only from the bundle, $\mathcal{E}$, but from the embedding of the subscheme $\mathcal{C}$ itself. Therefore we must essentially subtract the corrections arising from the normal bundle to $\mathcal{C}$. More precisely, the Chern character and the push forward do not commute, in a way measured by the Grothendieck-Riemann-Roch formula [34],

$$ch\left(f_*(\mathcal{E})\right) = f_*\left(ch(\mathcal{E}) \cdot Td(\mathcal{T}_f)\right), \tag{4.28}$$

where $f : Y \to X$, ch is the Chern character, Td is the Todd class, and $\mathcal{T}_f$ is the relative tangent bundle. Our situation of an immersion into a Calabi-Yau manifold is a significant simplification of the special case of the GRR formula considered in [39].

We find that the Chern characters are related by

$$\begin{aligned} ch_3(i_*\mathcal{E}) &= i_*\left(ch_2(\mathcal{E})\right) - \frac{1}{2}[\mathcal{C}] \cdot i_*\left(ch_1(\mathcal{E})\right) = ch_2(\mathcal{E}) - N\frac{p-2}{2}|R| \\ ch_2(i_*\mathcal{E}) &= i_* ch_1(\mathcal{E}) = |R|\ [\mathbb{P}^1], \end{aligned} \tag{4.29}$$

up to trivial terms that are independent of R. Applying this in reverse to the Chern characters from the three dimensional point of view written in (4.27), we conclude that the brane charges are

$$q_{D0} = \frac{p}{2}\kappa_R + ||R^t||^2 + \frac{p-2}{2}N|R| = \frac{p}{2}C_2(R) + ||R^t||^2 - N|R| \\ q_{D2} = |R|. \tag{4.30}$$

It will be useful later to separate these terms into a propagator depending on the bundle, $\mathcal{O}(-p)$, and a piece that will be included in the vertex amplitude calculated below. Therefore we find the toric propagator to be

$$q^{\frac{p}{2}C_2(R)}e^{i\theta|R|}, \tag{4.31}$$

as expected from q-deformed Yang Mills, and the remaining pieces of the second Chern character give the corresponding factors,

$$\exp\left\{-g_s\left(\frac{1}{2}||R^t||^2 - \frac{N}{2}|R|\right)\right\}, \tag{4.32}$$

in the two vertices being glued.

These bound D2 branes also contribute to the induced D0 charge when they intersect at the vertices. In particular, the intersection numbers give terms of the form

$$\sum_{i=1}^{N} R_i Q_i, \tag{4.33}$$

for D2 branes bound to the same stack of N D4 branes along two intersecting legs of the toric diagram. We will see in the next section that these terms exactly cancel a correction to the vertex associated to the number of boxes deleted by the background D2 branes, as expected.

If there are two intersecting stacks of D4 branes, the induced D0 charge can be similarly computed as

$$q_{D0} = \frac{p}{2}||R||^2 + \frac{2-p}{2}||R^t||^2 + \frac{p-2}{2}N|R| - \frac{p}{2}M|R|, \tag{4.34}$$

where the D2 bound state is of the type found in (4.25).

4.5 The crystal partition function

We will use the transfer matrix method, described, for example, in [59], to evaluate the partition functions of these crystal ensembles. Recall that cutting a three dimensional partition along diagonal planes, $y = x + t$, gives a sequence of interlacing Young diagrams, $\{\nu(t)\}$, indexed by t. In particular,

$$\nu(t+1) \succ \nu(t), t < 0, \tag{4.35}$$

and

$$\nu(t+1) \prec \nu(t), t \geq 0, \tag{4.36}$$

where two Young diagrams, μ and ν, are said to interlace, denoted by $\mu \prec \nu$, if

$$\nu_1 \geq \mu_1 \geq \nu_2 \geq \mu_2 \geq ... \tag{4.37}$$

It is now convenient to use the correspondence between Young diagrams and states of the NS sector a free complex fermion (or boson, via bosonization) in two dimensions, to express the crystal partition function as a sequence of operators acting between the states associated to the asymptotic Young diagrams.

Given a Young diagram, ν, define its Frobenius coordinates as

$$a_n = \nu_n - n + \frac{1}{2}, \quad b_n = \nu_n^t - n + \frac{1}{2}, \tag{4.38}$$

where the index runs from 1 to d, the length of the diagonal of ν. Then the fermionic state,

$$|\nu >= \prod_{n=1}^{d} \psi_{a_n}^* \psi_{b_n} |0 >, \tag{4.39}$$

is nontrivial and uniquely associated to ν, since the a_n and b_n are distinct and uniquely determine ν.

The key point in this transfer matrix approach is that the operators

$$\Gamma_{\pm}(z) = \exp\left(\sum_{n>0} \frac{z^{\pm n} J_{\pm n}}{n}\right), \tag{4.40}$$

constructed out of the modes, J_n, of the fermion current, $\psi^* \psi$, generate the interlacing condition. That is,

$$\Gamma_+(1)|\nu >= \sum_{\mu \prec \nu} |\mu >, \text{ and} \tag{4.41}$$

$$\Gamma_-(1)|\nu >= \sum_{\mu \succ \nu} |\mu > . \tag{4.42}$$

The number of boxes in a single slicing is easily computed using the Virasoro zero mode, L_0, to be

$$q^{L_0}|\nu >= q^{|\nu|}|\nu >, \tag{4.43}$$

which can be applied sequentially to evaluate the crystal action of a given partition.

It is important to take into account a finite boundary effect arising from the difference between the number of boxes falling inside a region bounded by diagonal slicings as opposed to a rectangular box. If we regulate the crystal partition function by placing it in a rectangular box of length L along some axis, then for L sufficiently large, only the background asymptotic representation, R, will contribute near the boundary. We will count the total number of boxes in the three dimensional partition, and hence must subtract off the background contribution, $q^{L|R|}$. The diagonal slicing includes an additional number of boxes, which are easily counted to be

$$\frac{1}{2}||R||^2 - \frac{1}{2}|R| = \frac{1}{2}\sum_i R_i(R_i - 1), \tag{4.44}$$

or its transpose, depending on the orientation of R as in [60].

This is still not quite right when there are asymptotic representations along more then one leg, since the subtraction of $L_1|R| + L_2|Q| + L_3|P|$ over counts the number of boxes in the background by the intersection at the origin.[2] For example, the intersection of two partitions with the parallel orientation has

$$\sum_i R_i Q_i \tag{4.45}$$

points. It will turn out that this effect is cancelled by the contribution of the Chern classes of the background divisors themselves.

Define the normalized vertex in terms of the crystal partition function, $P(R^a, N_a)$, to be, schematically,

$$\mathcal{V}_{R^a}^{(N_a)} = \exp\left\{\frac{g_s}{2}\left(\sum_a N_a|R^a| - \sum_a ||R^a||^2 + 2\sum_{a<b} R^a \cdot R^b - 4R^1 \cdot R^2 \cdot R^2\right)\right\} P(R^a, N_a), \tag{4.46}$$

where a ranges over the nontrivial asymptotics, $R^a \cdot R^b$ is given by (4.45), the final term is the triple intersection, and transposes must be used appropriately depending of the orientation of the Young tableaux. These terms are perfectly cancelled by the Chern classes of the background bundle, which we determined in (4.32).

We will make extensive use of various identities involving the operators, Γ, and skew Schur functions, which are collected in Appendix A.

4.5.1 One stack of N D4 branes

Let us apply these methods first to the simplest case of a single stack of D4 branes. Our analysis in this subsection will parallel that of Okuda [61], where the

[2]This mistake has actually appeared frequently in the literature.

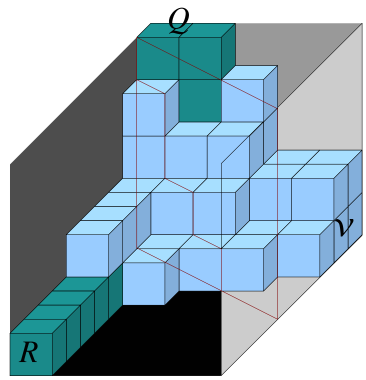

Figure 4.1: Shown in red is a diagonal slice of the crystal describing 5 coincident D4 branes in the vertex. The meaning of the representation ν will be discussed in section 6.

special case of a single nontrivial asymptotic representation was discovered in the context of Chern-Simons theory.[3]

Regulating the crystal by placing it in a box of size $L \times N \times \infty$, we apply the transfer matrix method as shown in figure 1, in a background determined by the representation Q along the vertical leg, with asymptotic R to the left and $\cdot$ required by the truncation on the right. The pattern of interlacing operators is altered by the presence of Q, as explained in [60], thus we see that the normalized partition function defined by (4.46) is

$$\mathcal{S}_{RQ} = q^{\frac{N}{2}(|R|+|Q|)} q^{\frac{1}{2}||Q^t||^2 + \frac{1}{2}|R|} q^{-L|R|} < R| \prod_{n>0}^{L} \left(q^{L_0} \Gamma_{\pm}(1) \right) q^{L_0} \prod_{m=1}^{N} \left(\Gamma_{\mp}(1) q^{L_0} \right) |0 >, \tag{4.47}$$

where the pattern of pluses are minuses is determined by the shape of Q. Commuting

[3] In fact, the original motivation of this work was to understand the underlying structure of that relation!

the q^{L_0}'s to the outside, and splitting the middle one in half, we obtain

$$q^{\frac{1}{2}||Q^t||^2} < R| \prod_{n>0} \Gamma_\pm(q^{-n+\frac{1}{2}}) \prod_{m=1}^{N} \Gamma_\mp(q^{m-\frac{1}{2}})|0>, \tag{4.48}$$

in the $L \to \infty$ limit. Next, we commute all of the Γ_+ to the right, where they act trivially on $|0>$, noting that the commutator, Z, depends only on the representation Q. This gives

$$q^{\frac{1}{2}||Q^t||^2} < R| \prod_{m=1}^{N} \Gamma_-(q^{-Q_m+m-\frac{1}{2}})|0> Z(Q) = q^{\frac{1}{2}||Q^t||^2} s_R(q^{-Q_m+m-\frac{1}{2}})Z(Q). \tag{4.49}$$

To determine $Z(Q)$, note that

$$\mathcal{S}_{\cdot Q} = q^{\frac{1}{2}||Q^t||^2 - \frac{N}{2}|Q|} Z(Q) = \mathcal{S}_{Q\cdot} = Z(\cdot) s_Q(q^{-\rho_N}). \tag{4.50}$$

Therefore, pulling everything together, we find that

$$\mathcal{S}_{RQ} = \mathcal{S}_{\cdot\cdot}\; s_Q(q^{-\rho_N}) s_R(q^{-\rho_N - Q}), \tag{4.51}$$

which is exactly the Chern Simons link invariant obtained from q-deformed Yang Mills in [7] and [2]!

For future reference, we will derive another useful formula for $\mathcal{S}_{RQ}$ by slicing the crystal with a different orientation as depicted in figure 2. Applying manipulations similar to the above, we find that

$$\mathcal{S}_{RQ} = q^{\frac{1}{2}(-N|R| - N|Q| + ||R||^2 + ||Q||^2)} q^{-\frac{1}{2}(||R^t||^2 + ||Q^t||^2)} \times < R^t| \prod_{n>0} \Gamma_+(q^{-n+\frac{1}{2}}) \mathcal{P}_N^t \prod_{m>0} \Gamma_-(q^{m-\frac{1}{2}})|Q^t>, \tag{4.52}$$

where $\mathcal{P}_N$ ($\mathcal{P}_N^t$) is the projection operator onto the space of representations with at most N rows (columns). This implements in the transfer matrix language the fact

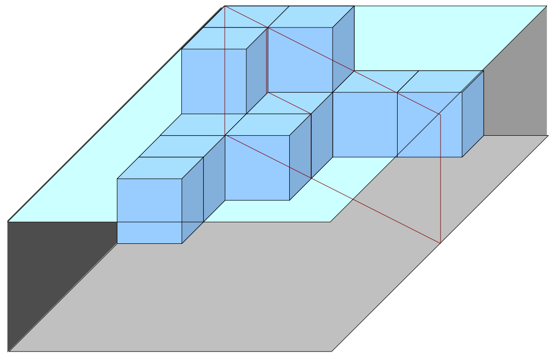

Figure 4.2: Here we rotate the crystal with trivial asymptotics associated to the $U(2)$ theory with no bound D2 branes.

that the three dimensional partition has height truncated by N, by enforcing the constraint on the largest diagonal tableaux. Writing this in terms of Schur functions, we find that

$$\mathcal{S}_{RQ} = \mathcal{S}_{\cdot\cdot}\,(-)^{|R|+|Q|}q^{-\frac{1}{2}(C_2(R)+C_2(Q))}\sum_{P\in U(N)} s_{P/R}(q^\rho)s_{P/Q}(q^\rho), \tag{4.53}$$

where we have used (4.79) to remove the transposes.

4.5.2 N D4 branes intersecting M D4 branes

Suppose we have N D4 branes wrapping the $x-y$ plane, M in the $y-z$ plane, with asymptotic conditions defined by a $U(N)$ representation, R, along the x-axis, and a $U(M)$ representation, Q, in the z direction. Then we will proceed to check that the D4 brane partition function is given by the vertex $\mathcal{V}_{RQ}$ of the statical ensemble of three dimensional partitions restricted to lie within the volume of the D4 branes with asymptotics R and Q, normalized according to (4.46) as before. Using the transfer matrix method, with diagonal slicings oriented in the planes $y = x + n$, indexed by n

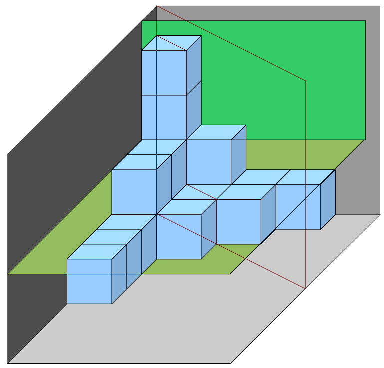

Figure 4.3: The crystal for $N = 2$ intersecting $M = 1$ D4 branes is oriented as shown, where the green planes are the boundaries of the allowed region.

(see figure 3), we obtain

$$\mathcal{V}_{R,Q}^{(N,M)} = q^{\frac{1}{2}||R||^2+\frac{1}{2}||Q^t||^2-\frac{N}{2}|R|-\frac{M}{2}|Q|} q^{-L|R|} q^{-\frac{1}{2}||R||^2+\frac{1}{2}|R|} \times \\ < R^t| \prod_{n=1}^{L-M} \left(q^{L_0}\Gamma_+(1)\right) q^{L_0}\mathcal{P}_N^t \prod_{m=1}^{M} \left(\Gamma_\pm(1)q^{L_0}\right) \prod_{k>0} \left(\Gamma_\mp(1)q^{L_0}\right) |0>, \tag{4.54}$$

where Q determines the pattern of $\Gamma_\pm$, as before, and the position of the projection, $\mathcal{P}_N^t$, is fixed by the truncation constraint that the point $(N+1, M+1, 1)$ is not in the partition. We commute the L_0's to the outside, dividing them at the plane $y = M + z$ where we enforce the projection, to obtain in the infinite L limit,

$$q^{-M|R|-\frac{1}{2}C_2(R)+\frac{1}{2}||Q^t||^2-\frac{M}{2}|Q|} \sum_{A\in U(N)} < R^t| \prod_{n>0} \Gamma_+(q^{-n+\frac{1}{2}})|A^t > \times \\ < A^t| \prod_{m=1}^{M} \Gamma_\pm(q^{m-\frac{1}{2}}) \prod_{k>0} \Gamma_\mp(q^{M+k-\frac{1}{2}})|0> . \tag{4.55}$$

Now, commuting the remaining Γ_+ to act trivially on $|0>$, we pick up a factor, $Y(Q)$, independent of R. Writing the resulting expression is terms of Schur functions, we have shown that

$$\mathcal{V}_{RQ} = q^{-M|R|-\frac{1}{2}C_2(R)+\frac{1}{2}||Q^t||^2-\frac{M}{2}|Q|}Y(Q)\sum_{A\in U(N)} s_{A^t/R^t}(q^{-\rho})s_{A^t}(q^{-Q^t-\rho})q^{M|A|}. \quad (4.56)$$

Note that the commutator term, $Y(Q)$, can be expressed as

$$< 0|\prod_{m=1}^{M}\Gamma_\pm(q^{m-\frac{1}{2}})\prod_{k>0}\Gamma_\mp(q^{M+k-\frac{1}{2}})|0>, \quad (4.57)$$

which, up to a shift in the origin of slicings by M units, is exactly the M-truncated crystal with asymptotics Q and $\cdot$. Thus the results of the previous subsection imply that

$$Y(Q) = q^{\frac{M}{2}|Q|}q^{-\frac{1}{2}||Q^t||^2}s_Q(q^{\rho_M}). \quad (4.58)$$

Manipulating our expression with the Schur function identities (4.79) and (4.80), we find

$$q^{-M|R|}q^{-\frac{1}{2}C_2(R)}\sum_\mu\left((-)^{|R|+|\mu|}\sum_{A\in U(N)} s_{A/R}(q^\rho)s_{A/\mu}(q^\rho)\right)q^{\frac{M}{2}|\mu|} \times\left(s_Q(q^{\rho_M})s_\mu(q^{Q+\rho_M})\right). \quad (4.59)$$

We can now recognize the terms in parenthesis in (4.59) to be exactly the Chern-Simons link invariants, $q^{\frac{1}{2}C_2(R)+\frac{1}{2}C_2(\mu)}S^{(N)}_{R\mu}(q)$ and $S^{(M)}_{\mu Q}(q^{-1}) = S^{(M)}_{\bar{\mu}Q}$. Therefore, recalling that $C_2^{(N)}(P) = C_2^{(M)}(P) + (N-M)|P|$ for a $U(M)$ representation P, we find that

$$\mathcal{V}_{RQ} = q^{-M|R|}\sum_{P\in U(M)} S^{(N)}_{RP}q^{\frac{1}{2}C_2^{(M)}(P)}q^{\frac{N}{2}|P|}S^{(M)}_{\bar{P}Q}, \quad (4.60)$$

noting that $S^{(M)}_{PQ}$ vanishes unless P is a $U(M)$ representation. This is in complete agreement, up to overall $U(1)$ factors that were not carefully taken into account, with the result of [2], where this partition function was derived by reducing the D4 brane worldvolume theories to q-deformed Yang-Mills in two dimensions, coupled by bifundamental fields living at the intersection point.

Thus far we have only considered the case of trivial asymptotics along the intersection plane of the branes, which is natural assuming that direction is noncompact in the toric Calabi-Yau. Lifting that restriction, we proceed much as before after replacing $|0>$ by $|P^t>$ in (4.55), to obtain

$$q^{-M|R|-\frac{1}{2}||R||^2-\frac{1}{2}||P||^2+\frac{M}{2}|Q|-\frac{1}{2}||Q^t||^2}(-)^{|R|}\times$$
$$\sum_{\mu,\nu}\left(\sum_{A\in U(N)} s_{A/R}(q^\rho)s_{A/\nu}(q^\rho)\right)\; q^{\frac{M}{2}|\nu|}s_Q(q^{\rho_M})s_{\nu/\mu}(q^{Q+\rho_M})(-)^{|\mu|}s_{P^t/\mu^t}(q^{\bar{Q}+\rho_M}). \tag{4.61}$$

By definition of the skew Schur functions and the Chern Simons invariants, $s_Q(q^{\rho_M})s_{\nu/\mu}(q^{Q+\rho_M}) = N^\nu_{B\mu}S^{(M)}_{BQ}(q^{-1}) = N^\nu_{B\mu}S^{(M)}_{B\bar{Q}}$, giving the final result

$$\frac{q^{-M|R|}}{S_{\bar{Q}\cdot}}\sum_{D,B,C} S^{(N)}_{RD}q^{\frac{1}{2}C^{(N)}_2(D)}q^{\frac{M}{2}|D|}\hat{N}^{DP}_{BC^t}S^{(M)}_{B\bar{Q}}S^{(M)}_{CQ}, \tag{4.62}$$

where D is a representation of $U(N)$, B and C are representations of $U(M)$, and

$$\hat{N}^{DP}_{BC^t} = (-)^{|D|}\sum_\mu N^D_{B\mu}(-)^{|\mu|}N^P_{C^t\mu}. \tag{4.63}$$

4.5.3 The general vertex

The most general possibility for intersecting toric D4 branes is the triple intersection with representations running along all three legs of the vertex. The partition function is significantly more complicated, and, although it can still be computed schematically with the transfer matrix technology, we will not be able to express it in terms of the Chern-Simons link invariants. In the context of OSV, the existence

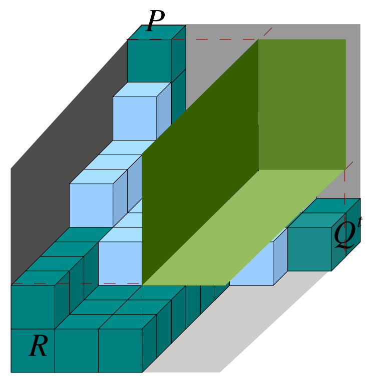

Figure 4.4: The restricted region of general vertex describing $U(N)\times U(M)\times U(K)$, for $N=3$, $M=1$, and $K=2$, in the local $\mathbb{C}^3$ geometry is demarcated by the green planes, and the asymptotes are labelled.

of a large N factorization remains clear, simply from the crystal picture itself, as explained in detail in the next section. We will also consider a special case that can be solved more explicitly.

The partition function for the crystal oriented as in figure 4, but with $M \geq N$, can be determined using the same methods to be given by

$$\begin{aligned} & q^{-\frac{1}{2}||R^t||^2-\frac{1}{2}||Q||^2} q^{(N-M)(|R|-|Q|)} \times \\ & \qquad < R|\prod_{n>0}\Gamma_{\pm}(q^{-n+\frac{1}{2}})\mathcal{P}\prod_{m=1}^{M-N}\Gamma_{\pm}(q^{m-\frac{1}{2}})\prod_{l>0}\Gamma_{\mp}(q^{M-N+l-\frac{1}{2}})|Q^t>, \end{aligned} \tag{4.64}$$

where we have chosen the framing with parallel rectangular walls for convenience. The pattern of signs of Γ is given by P, and the projection $\mathcal{P}$ forces the Young diagram to live in the restricted region determined by K in the vertical direction, and, in the horizontal direction, by the distance along the diagonal from the background representation, P, to the point (N, M). We orient the representations along the intersecting pairs of D4 branes such that R is of type (N, K), Q is (K, M), and P is (M, N) in the notation of (4.25).

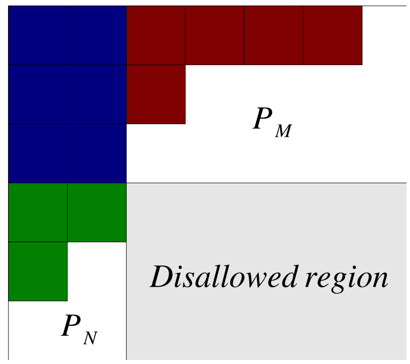

Figure 4.5: A special kind of (M,N) Young diagram is which the $U(N)$ and $U(M)$ degrees of freedom are decoupled is shown.

Let us consider the special case that all of the boxes in the bifundamental region of P are filled, and joined to a $U(N)$ representation, P_N, and a $U(M)$ representation, P_M, as depicted in figure 5, for which a more detailed analysis is possible. This form of P implies that $\mathcal{P} = \mathcal{P}_K^t$, and commuting the $\Gamma_\pm$'s appropriately, it is convenient to define

$$W = q^{-\frac{1}{2}||R^t||^2 - \frac{1}{2}||Q||^2} q^{(N-M)(|R|-|Q|)} X(P_N) X(P_M), \tag{4.65}$$

in terms of the commutator term $X(P_N) = q^{-\frac{1}{2}||P_N^t||^2 + \frac{N}{2}|P_N|} s_{P_N}(q^{-\rho_N})$, and the framing prefactor. Then we find that the statistical partition function is

$$W \sum_{L\in U(K)} \sum_{\mu,\nu} s_{R/\mu}(q^{-\frac{N}{2}+\rho_N+\bar{P}_N}) s_{L^t/\mu}(q^{-\rho+N-P_N^t}) s_{L^t/\nu}(q^{M-\rho-P_M^t}) s_{Q^t/\nu}(q^{-\frac{M}{2}+\rho_M+\bar{P}_M}). \tag{4.66}$$

Now apply various identities to expand and simplify Schur functions evaluated at expressions of the form $q^{\rho+P_N^t}$, to obtain,

$$\begin{aligned} W \sum_{L\in U(K)} \sum_{\mu,\nu,\alpha,\beta} q^{-\frac{N}{2}|R| - \frac{N}{2}|\mu| + (M+N)|L| - \frac{M}{2}|\nu| - \frac{M}{2}|Q|} (-)^{|\mu|+|\nu|} s_{R/\mu}(q^{\rho_N+\bar{P_N}}) \times \\ s_{\alpha/\mu^t}(q^{-\frac{N}{2}+\rho_N+P_N}) s_{L/\alpha}(q^{\rho-N}) s_{L/\beta}(q^{\rho-M}) s_{\beta/\nu^t}(q^{-\frac{M}{2}+\rho_M+P_M}) s_{Q^t/\nu}(q^{\rho_M+\bar{P_M}}). \end{aligned} \tag{4.67}$$

Recognizing the appearance of the Chern-Simons link invariant from (4.53), we can simplify the partition function to find

$$W \sum_{A,B\in U(K)} \sum_{\mu,\nu} q^{-\frac{N}{2}|R|+\frac{N}{2}|A|+\frac{M}{2}|B|-\frac{M}{2}|Q|}(-)^{|\mu|+|\nu|+|A|+|B|} s_{R/\mu}(q^{\rho_N+\bar{P_N}}) \times \\ s_{A/\mu^t}(q^{\rho_N+P_N}) S^{(K)}_{AB} q^{\frac{1}{2}C_2(A)+\frac{1}{2}C_2(B)} s_{B/\nu^t}(q^{\rho_M+P_M}) s_{Q^t/\nu}(q^{\rho_M+\bar{P_M}}) \tag{4.68}$$

Therefore putting everything together, and rewriting the result in terms of Chern-Simons invariants, we conclude that,

$$Z = q^{(\frac{N}{2}-M)|R|+(\frac{M}{2}-N)|Q|-\frac{1}{2}||R^t||^2-\frac{1}{2}||Q||^2} q^{-\frac{1}{2}||P_N^t||^2-\frac{1}{2}||P_M^t||^2+\frac{N}{2}|P_N|+\frac{M}{2}|P_M|} \sum_{A,B\in U(K)} \\ \sum_{C,D\in U(N)} \sum_{E,F\in U(M)} \hat{N}^{AR^t}_{CD^t} \hat{N}^{BQ}_{EF^t} q^{\frac{N}{2}|A|+\frac{M}{2}|B|+\frac{1}{2}C_2(A)+\frac{1}{2}C_2(B)} S^{(N)}_{C\bar{P_N}} S^{(N)}_{DP_N} S^{(M)}_{E\bar{P_M}} S^{(M)}_{FP_M}, \tag{4.69}$$

where the fusion coefficients are defined by equation (4.63).

4.6 Chiral factorization and conjugation involution

The chiral factorization of the indexed entropy at large D4 charge can be seen very elegantly in the melting crystals. The appearance of one factor of the topological string is immediate, since in the limit that N goes to infinity, the restrictions on the tableaux become vacuous, and we exactly reproduce the propagators and crystal vertices of the topological A-model in the same toric geometry. Referring to the derivation of the induced brane charges (4.29), we see that one obtains exactly the framing factors and vertex amplitudes of the topological A-model on the Calabi Yau [37]. Moreover, the extra contribution to the induced D0 charge is precisely

$$\frac{1}{2}[\mathcal{C}] \cdot [D], \tag{4.70}$$

the intersection number of the D4 branes on $[\mathcal{C}]$ with the homology class, $[D]$, wrapped by the D2 branes. This combines with the chemical potential for the D2 charge to give the attractor value of the Kahler moduli,

$$t_a = \frac{1}{2} g_{top}[\mathcal{C}] \cdot [D_a] - i\theta_a, \tag{4.71}$$

for the curve $[D_a] \in H_2(X)$.

To see what the anti-chiral block arises from, recall that the propagator depends on the quadratic Casimir, so we expect both small representations and their conjugates to contribute, following

$$C_2(\mathcal{R}) = \kappa_{R+} + \kappa_{R-} + N(|R_+| + |R_-|) + N{\ell_R}^2 + 2\ell_R(|R_+| - |R_-|), \tag{4.72}$$

where $\mathcal{R} = (R\bar{Q})[\ell_R]$ is the $U(N)$ representation with chiral part R_+, anti-chiral component R_-, and overall $U(1)$ factor ℓ_R.

There is an involution on restricted three dimensional partitions of the type we have been discussing which is inherited from the conjugation of $U(N)$ representations. Consider two successive Young tableaux in the transfer matrix approach, $R \succ Q$, that is $R_1 \geq Q_1 \geq R_2 \geq Q_2 \geq \ldots$. Then, when both are $U(N)$ representations, under conjugation, one can see that $\bar{Q}_N \geq \bar{R}_N \geq \ldots$, and thus $\bar{Q} \succ \bar{R}$, up to a shift by a sufficiently large and positive $U(1)$ factor. Therefore this defines an involution on the chiral fermion Hilbert space, sending

$$|R> \longrightarrow |\bar{R}>, \; q^{L_0} \to q^{-L_0} \text{ and } \Gamma_- \to \Gamma_+, \tag{4.73}$$

since Γ generates the interlacing relation, and $|\bar{R}| = -|R|$. Naturally, these relations make sense only when we are restricted to $U(N)$ representations.

The involution also acts naturally when the slicing is oriented differently, for example, suppose R and Q are $U(N)$ representations such that $R^t \succ Q^t$. It is easy to see that $(\bar{R})^t_n + (R)^t_{A-n} = N$, for conjugation with shifting by some sufficiently large $U(1)$ factor A. Thus $(\bar{R})^t \prec (\bar{Q})^t$, again up to an appropriate $U(1)$ shift, chosen so that all column lengths are positive.

Applying this transformation to the interlacing representations of the crystals, one can show that they are invariant. For a single stack of D4 branes, for example, we have

$$\begin{aligned} S^{(N)}_{\bar{R}\bar{Q}} &= q^{-\frac{1}{2}C_2(\bar{R})-\frac{1}{2}C_2(\bar{Q})} < (\bar{R})^t| \prod_{n>0} \Gamma_+(q^{-n+\frac{1}{2}}) \mathcal{P}^t_N \prod_{m>0} \Gamma_-(q^{m-\frac{1}{2}}) |(\bar{Q})^t > \\ &= q^{-\frac{1}{2}C_2(R)-\frac{1}{2}C_2(Q)} < R^t| \prod_{n>0} \Gamma_-(q^{n-\frac{1}{2}}) \mathcal{P}^t_N \prod_{m>0} \Gamma_+(q^{-m+\frac{1}{2}}) |Q^t >= S^{(N)}_{RQ}, \end{aligned} \tag{4.74}$$

by commuting the Γ's in the obvious way, using crucially the fact that we are free to shift by $U(1)$ factors to avoid having rows of negative length.

Therefore the factorized form $Z_{YM} \sim |\psi_{top}|^2$, follows automatically, by breaking all of the representations associated to D2 branes wrapping the legs into their chiral and anti-chiral pieces. The sum over chiral blocks (ie. ghost branes) first found in [7] arises because crystals with order $N|A|$ boxes can connect the chiral and anti-chiral regions with the ghost Young tableau A via a $N \times A$ transversal configuration, contributing $e^{-g_s N|A|}$ to the partition function, which remains finite even at large N. The simplest example of this is the vertex partition function of N coincident D4 branes, where in the large N limit diagrams, ν, along the truncated direction, as shown in figure 1, are exactly the ghost tableaux, which clearly have no restrictions on their rank. The additional D0 branes is the corner contributing to the chiral block will experience an effective background described by the topological vertex $C_{Q^t_+ R_+ \nu}$, as expected from the factorization formulae of [7], [6], and [2].

4.7 Concluding remarks and further directions

We have solved the theory of BPS bound states of D2 and D0 branes to D4 branes wrapping deformable cycles in toric Calabi-Yau by localization, obtaining a truncated crystal partition. This calculation is exact, even for small charges, and smoothly asymptotes, in the OSV large charge limit, to the square of the topological string amplitude in the crystal vertex expansion. The crystal configurations we have found count the extra D0 brane bound states at the singular points of the non-reduced surface obtained by giving nilpotent VEVs to the worldvolume adjoint fields controlling the normal deformations of the D4 branes in the Calabi Yau. These are the toric invariant, zero size limit of the extra 2-cycles in $H_2(\mathcal{C}) > H_2(X)$, in the phase where the D4 charge results from a single brane wrapping the high degree, smooth surface $\mathcal{C}$.

The index of BPS bound states should be the same in all three cases, since it is protected by supersymmetry. We check that the results found here agree with the topological gauge theory calculations in known examples. This gives a nice realization of the underlying geometric nature of the interacting topological Yang Mills theories, since the crystal picture is not at all manifest in the quiver of four dimensional $U(N_i)$ gauge theories with bifundamental couplings along the intersections.

We view this work as a step in the development of a nonperturbative topological vertex, that is a general method to compute the partition function of D4 branes on toric Calabi Yau manifolds in the OSV ensemble, which defines the nonperturbative completion of the topological A-model amplitudes. We have computed the relevant vertex and gluing rules, whose analogs in the topological string were sufficient to solve the theory, since the A-model only involves holomorphic maps from curves to the Calabi Yau.

The main problem in the exact OSV calculation on toric Calabi Yau which remains to be solved is the effect of branes wrapping compact 4-cycles. In toric Calabi Yau, compact 4-cycles always have positive curvature, and therefore negative self-intersection. As emphasized in [2], we must wrap D4 branes on very ample divisors to create black holes with large positive values for the geometrical moduli at the attractor point. This is the regime of validity of the OSV conjecture. Moreover, the theory living on D4 branes instead wrapping rigid divisors with negative self-intersection cannot be solved by the $\mathcal{N} = 1$ mass deformation of [70], and it exhibits background moduli dependance and lines of marginal stability where the number of BPS states jump.

This might seem to preclude the analysis of any more complicated geometries with multiple compact 4-cycles in the toric context, such as the resolution of A_3 ALE fibred over $\mathbb{P}^1$. However, if the D4 branes wrap toric 4-cycles, including some rigid cycles, which still sum to a very ample divisor, or equivalently, give rise to positive Kahler moduli at the attractor point, then there is no reason to expect a breakdown of the OSV conjecture. Therefore it seems likely that the effect of intersection with sufficiently many noncompact branes wrapping "good" 4-cycles will eliminate the problems of the twisted $\mathcal{N} = 4$ theory on compact rigid surfaces. It would be very interesting to see how this works in detail, and to understand the new contributions arising from such branes. This could lead to a more complicated set of gluing rules, involving additional contributions from "loops" in the toric diagram as well as vertex and propagator terms that we have found here.

It would also be interesting to investigate the relation between our crystals and those studied in [36], which have a different kind of truncation, arising purely in the context of the q expansion of (perturbative) topological string theory. This might further elucidate why the Chern Simons amplitudes $S^{(N)}_{RQ}$ appear, even at finite N, in the solution of D4 brane $U(N)$ gauge theories.

Appendix A

All Γ_+ commute, as do Γ_-, however one can show that

$$\Gamma_-(x)\Gamma_+(y) = \Gamma_+(y)\Gamma_-(x)\left(1 - \frac{y}{x}\right). \tag{4.75}$$

Also,

$$\Gamma_\pm(x)q^{nL_0} = q^{nL_0}\Gamma_\pm(xq^{-n}). \tag{4.76}$$

These operators are conjugate; following the definition (4.40) it is easy to see that

$$(\Gamma_+(x))^\dagger = \Gamma_-(x^{-1}). \tag{4.77}$$

The repeated application of the $\Gamma_\pm$ operators results in the appearance of skew Schur functions, since

$$\prod_n \Gamma_-(x_n)|\nu >= \sum_{\mu\supset\nu} s_{\mu/\nu}(x_n)|\mu > . \tag{4.78}$$

We will often come across skew Schur functions evaluated at special values of the form $x_n = \rho_n + \nu_n$, where $\rho = \{-\frac{1}{2}, -\frac{3}{2}, \dots\}$, and ν is a Young diagram, as well as the finite N version, $x_i = \rho_i^N + R_i,\ i = 1, \ldots N$, where R is a $U(N)$ representation, and $\rho_i^N = \frac{N+1}{2} - i$ is the Weyl vector. These functions obey many beautiful relations, including

$$\begin{aligned} s_{\mu/\nu}(q^{\rho+\eta}) &= (-)^{|\mu|+|\nu|} s_{\mu^t/\nu^t}(q^{-\rho-\eta^t}) \\ s_{\mu/\nu}(c\{x_n\}) &= c^{|\mu|-|\nu|} s_{\mu/\nu}(x) \\ s_{\mu/\nu}(x) &= \sum_\eta N^\mu_{\nu\eta} s_\eta(x), \end{aligned} \tag{4.79}$$

where $N^\mu_{\nu\eta}$ are the tensor product coefficients. There are various useful summation formulae as well,

$$\begin{aligned} \sum_\eta s_{\mu/\eta}(x) s_{\nu/\eta}(y) &= \prod_{n,m} (1 - x_n y_m) \sum_\lambda s_{\lambda/\mu}(y) s_{\lambda/\nu}(x) \\ \sum_\nu s_{\mu/\nu}(x) s_\nu(y) &= s_\nu(x, y), \end{aligned} \tag{4.80}$$

where (x, y) denotes the conjunction of the strings of variables x and y.

We also make use of the quadratic Casimirs,

$$\kappa_\mu = ||\mu||^2 - ||\mu^t||^2, \quad \text{and} \quad C_2^{(N)}(R) = N|R| + \kappa_R, \tag{4.81}$$

for any Young diagram μ and $U(N)$ representation R.

Chapter 5

Quantum foam from topological quiver matrix models

5.1 Introduction

The idea that the topological A-model involves quantized fluctuations of the Kahler geometry of a Calabi-Yau was introduced in [37]. They argued that the topological string partition function could be reproduced by summing over non-Calabi-Yau blow ups along collections of curves and points. The equivalent Donaldson-Thomas theory, given by a topologically twisted $\mathcal{N} = 2\ \ U(1)$ gauge theory on the Calabi-Yau, involves a sum over singular instantons, which can be blown up to obtain the fluctuations of the geometry.

The connection between this theory of BPS bound states of D2 and D0 branes to a 6-brane was further explained in [24] by lifting to M-theory. The D6-brane lifts to a Taub-NUT geometry in 11 dimensions, and the Donaldson-Thomas theory results in precisely that repackaging of the Gopakumar-Vafa invariants counting bound states of D2/D0 at the center of the Taub-NUT which is required to reproduce the A-model.

In this work, I will try to elucidate the role of Kahler quantum geometry when the topological A-model is embedded in the full IIA theory. We will show that the effective internal geometry experienced by BPS 0-brane probes of bound states of 2-branes and a D6 is exactly the blow up of the Calabi-Yau along the wrapped curves! This will be done in the context of the quiver matrix models that describe the low energy dynamics, so we will see the topological string theory emerge from a matrix model, with quite a different flavor than [23].

Therefore, if such a bound state in the mixed ensemble were constructing in $\mathcal{R}^{3,1}$, then in the probe approximation, the effective geometry of the Calabi-Yau would be literally the "foam" envisioning in [37]. Moreover, the work of [21] relating D4/D2/D0 bound states with classical black hole horizons to multi-centered D6/$\overline{D6}$ configurations means this might have implications for BPS black holes with non-vanishing classical entropy.

More subtle probes of the geometry using N 0-branes are able to discern the presence of a line bundle associated to the exceptional divisors of the blow up. Thus we see that the 2-branes are blown up to 4-branes, which can be dissolved into $U(1)$ flux on the D6. Moreover, only a small number of 0-branes can "fit" comfortably on the exceptional divisors, which might indicate that their Kahler size is of order g_s, as predicted in [37]. We will exactly reproduce the know topological vertex amplitudes from the quivers we discover, by counting fixed points of the equivariant T^3 action.

The topological quiver matrix models we shall discuss are quite interesting in their own right. Consider the BPS bound states of D6/D4/D2/D0 system in IIA theory compactified on a Calabi-Yau 3-fold. For large Kahler moduli, the D6 brane worldvolume theory is described by the 6+1 dimensional nonabelian Born-Infeld action. At low energies, we can integrate out the Kaluza-Klein modes in the internal manifold, presumably giving rise to a 0+1 dimensional superconformal quantum mechanics, which is the gauge theory dual to the AdS_2 near horizon geometry. The full superconformal theory remains mysterious, but there have been numerous attempts to understand it, see for instance [28].

Our interest is in the index of BPS ground states, which is captured by the topological twisted version of the physical theory. In the case of D6 branes, this is the twisted $\mathcal{N} = 2$ Yang Mills in six dimensions. The low energy description is now a 0-dimensional topological theory, that is a matrix model with a BRST trivial action. This topological version of the dual theory is far simpler, and can be written down exactly in non-trivial situations.

The physical interpretation of the quiver matrix model changes dramatically as one moves from the Gepner point to the large volume limit in the Kahler structure moduli space. It is very satisfying to see that the quiver description, which is most at home at the Gepner point, gives the correct result for the index of BPS states even in the opposite noncompact limit we will consider. At large volume, the quiver should be thought of as the collective coordinates of instantons in the worldvolume theory on the D6 brane.

Note that the moduli on which the theory depends are really the background moduli, defined at infinity in spacetime. In the case of 6-brane bound state we are examining, there *are* no BPS states at the naive attractor value of the moduli, so the attractor flow is always a split flow. Amazingly, this doesn't impair the ability of the quiver theory to capture the index of bound states, and furthermore we are able to read off certain aspects of the attractor flow tree [21] from the shape of the matrix model potential.

One of the most powerful worldsheet techniques to explore string theory compactified on a Calabi-Yau given by a complete intersection in projective space are gauged linear sigma models with superpotential. These two dimensional worldsheet theories can be constructed explicitly and exactly, and flow in the IR to the conformal non-linear sigma on the Calabi-Yau. For the purpose of understanding BPS saturated questions, the linear model itself already gives the exact answer, just as in the context of the topological string; finding the IR fixed point is unnecessary.

The holomorphically wrapped D-branes we are interested in are described by boundary linear sigma models, which have been studied extensively in the context of the B-model. The branes of the topological B-model can be encoded in the boundary conditions for the open B-model worldsheet, which are naively associated to vector bundles on the Calabi-Yau. In fact, a much richer structure of boundary data has been discovered, associated to D6 and anti-D6 branes with tachyons condensed. Mathematically speaking, the holomorphic branes are objects in the derived category of coherent sheaves [25], with the tachyons becoming maps between the sheaves associated to the 6-branes. The BPS branes of IIA theory require a stability condition (for generic Kahler moduli, π-stability of the triangulated category) not present for branes of the topological B-model.

At the Gepner point in the Calabi-Yau moduli space, this description simplifies, and one can generate all of the configurations of branes by condensing tachyons between a complete collection of fractional branes. The chain maps of the derived category are then encoded in the linear maps between a quiver of vector spaces, as shown in [26]. In general there will be a nontrivial superpotential, which must be minimized to obtain the supersymmetric ground states [10]. The effective topological theory of BPS branes is therefore the topological quiver matrix model, whose partition function localizes to the Euler characteristic of the anti-ghost bundle over the Kahler quotient of the quiver variety.

As we move in the Calabi-Yau moduli space out to the large volume, certain degrees of freedom will become frozen in the local limit. We find the quiver for the remaining dynamical degrees of freedom, and discover that there are a few extra terms in the effective action. In the local limit, branes with different dimensions, or those wrapping cycles with different asymptotics in the local Calabi-Yau, become sharply distinguished. This corresponds to the breaking of some gauge groups $U(N+M) \to U(N) \times U(M)$ of the Gepner point quiver. There are residual terms in the action from the off-diagonal components of the original D-term, which need to retained even in the local limit. Crucially, however, the local limit is universal, and the action we find is independent of the specific global geometry into which is was embedded, so our analysis is consistent.

The index of BPS states always looks, from the large volume perspective, like the Euler character of some bundle over the resolution of the instanton moduli space. The classical instanton moduli are described by the zero modes of the instanton solutions in the topologically twisted gauge theory living on the 6-branes. We will find this moduli space more directly, using the quiver description of the derived category of coherent sheaves, and taking the appropriate large volume limit.

The resulting holomorphic moduli space literally gives the moduli space of B-branes as the solution to some matrix equations (5.7), moded out by the gauge group. There are exactly the pieces of the open string worldsheet gauged linear sigma model which survive the topological string reduction to the finite dimensional Q-cohomology. This moduli space is explicitly independent of the Kahler moduli.

It is crucial that much of this moduli space does not correspond to BPS bound states of holomorphically wrapped branes in IIA theory. This is already clear because the stability of bound states depends on the background Kahler moduli. There is a clear realization of this fact in the gauge theory living on the branes, whose instantons satisfy the Hermitian Yang-Mills equations (5.1) (5.2). The holomorphic moduli space only imposes the F-term conditions, namely

$$F^{2,0} = 0, \tag{5.1}$$

and gauges the complexified gauge group, $GL(N, \mathbb{C})$, while the physical instanton moduli space is the Kahler quotient, obtained from the D-term condition,

$$F^{1,1} \wedge \omega^{d-1} = r\omega^d, \tag{5.2}$$

which plays the role of the moment map, together with the compact gauge group $U(N)$.

In the case of D4/D2/D0 bound states on local Calabi-Yau, whose BPS sector is described by the Vafa-Witten twist of $\mathcal{N} = 4$ Yang Mills theory on a 4-manifold, the partition function reduces to the Euler character of the instanton moduli space. There the quiver realization is exactly the ADHM construction of this finite dimensional space. This can be promoted to a matrix model with the same partition function, moreover the structure of the fermionic terms will be associated to the tangent bundle over the classical moduli space. Satisfyingly, this theory has exactly the 10 expected scalars and 16 fermions of the dimensional reduction of the maximally supersymmetric gauge theories in higher dimensions. Moreover, the action looks like a topologically twisted version of the usual D0 worldvolume theory. In flat space, there are no dynamical degrees of freedom associated to the motion of the noncompact D2 and D4 branes, but there are near massless bifundamentals associated to the 4-0 and 2-0 strings.

Our matrix models are the analogous construction for 6-branes. There is no reason for the antighost bundle of the matrix model to be the tangent bundle, although their dimensions are equal when the virtual dimension of the moduli space is zero. In fact, we will find that this obstruction bundle is indeed different in the case of D6/D0 bound states.

The quivers we will find provide are sensitive probes of the effective geometry, enabled one to go beyond calculations of the Euler characteristic. An interesting structure emerges, in which different effective quivers for the dynamical 0-brane fields are related by flops in the blow up geometry of the Calabi-Yau they probe. This leads to a more detailed understanding of the quantum foam picture of the topological A-model.

5.2 Review of quivers and topological matrix models

5.2.1 A quiver description of the classical D6-D0 moduli space

We will begin by constructing the classical moduli space of N D0 branes in the vertex geometry. The gauged linear model can be obtained by dimensional reduction of the instanton equations in higher dimensions. Consider bound states of holomorphically wrapped branes which can be expressed as flux dissolved into the worldvolume of the top dimensional brane. Then the classical moduli space of these instantons is determined by the solutions of the Hermitian Yang-Mills equations,

$$\begin{aligned} F^{(2,0)} &= 0 \\ F^{(1,1)} \wedge \omega^{d-1} &= r\omega^d. \end{aligned} \tag{5.3}$$

The reduction of the 6d gauge field to zero dimensions results in 6 Hermitian scalar fields, which can be conveniently combined into complex Z_i, for $i = 1, 2, 3$. In terms of these matrices, the F-term condition is the zero dimensional reduction of $F^{(2,0)} = 0$,

$$[Z_i, Z_j] = 0. \tag{5.4}$$

We find the D-term constraint as the analog of $F^{(1,1)} = r\omega$, where ω is the Kahler form,

$$\sum_{i=1}^{3} [Z_i, Z_i^\dagger] = 0, \tag{5.5}$$

which one can think of as the moment map associated to the $U(N)$ action on adjoint fields. Here it is impossible to find solutions with nonzero r, since the left hand side is traceless. This is equivalent to the fact that one cannot add a Fayet-Iliopoulos term when there are only adjoint fields. The F-term condition (5.4) comes from the superpotential

$$W = \mathrm{Tr}\, Z_1[Z_2, Z_3], \tag{5.6}$$

by the usual BPS condition that $\partial W = 0$.

The gauged linear sigma model description of the topologically twisted version of the worldvolume theory of N D0 branes in $\mathbb{C}^3$ can be obtained by computing the appropriate *Ext* groups in the derived category description. The fields in the quiver theory are exactly the 0-0 strings that in this case give three complex adjoint fields, Z_i, of $U(N)$, which can be naturally interpreted as the collective coordinates of N points in $\mathbb{C}^3$.

Adding the background D6 brane extends this quiver by a new $U(1)$ node, with a single field, q, coming from the 0-6 strings, transforming in the bifundamental, as shown in [72]. As described by Witten, the new quiver data which determines the zero dimensional analog of the instanton equations (5.3) is

$$\begin{aligned} [Z_i, Z_j] &= 0 \\ \sum_{i=1}^{3} [Z_i, Z_i^\dagger] + q^\dagger q &= r I_N \end{aligned} \tag{5.7}$$

The Fayet-Iliopoulos parameter, r, will turn out to be the mass of the 0-6 fields, as we will see more clearly in the matrix model. This mass is determined by the asymptotic B-field needed to preserve supersymmetry in the D6/D0 bound states [72]. In the absence of a B-field, $r < 0$, and q is a massive field, hence there would be a stable nonsupersymmetric vacuum. For $r > 0$, which is the case of interest for us, the now tachyonic q condenses to give a supersymmetric bound state. Note that there are no new possible terms in the superpotential (5.6) because of the shape of the quiver, hence the bifundamental can only appear as we have written it in the D-term.

The holomorphic quotient space that is naturally determined by the F-term is $\mathcal{M}_{hol} = X/GL(N, \mathbb{C})$, where X is the space of commuting complex matrices. This is exactly the moduli space of branes in the B-model, which depends purely on holomorphic data. It turns out to be a larger space, with less structure, than the Kahler quotient $\mathcal{M}_{Kahler} = X//U(N)$ that describes the physical BPS bound states in II theory. If an additional stability condition, which in this case is the existence of a cyclic vector for the representation of the quiver, is imposed on $\mathcal{M}_{hol}$ then it can be shown to be identical to the Kahler quotient, following the reasoning of [52].

5.2.2 Quiver matrix model of D0 branes

The index of BPS states of 0-branes can be found using the topologically twisted reduction of the D0 matrix model of [12], where the time directions drops out for the obvious reasons when considering the structure of ground states. This results in a zero dimensional theory, ie. a matrix model, with a BRST like supersymmetry.

Following [56] and [55], we will first consider the theory describing only D0 branes in $\mathbb{C}^3$, by utilizing the formalism developed by [70] to study Euler characteristics by means of path integrals. The action of the topological supersymmetry is given by

$$\begin{aligned} \delta Z_i = \psi_i,\ \delta\psi_i = [\phi, Z_i],\ \delta\phi = 0 \\ \delta\bar{\phi} = \eta,\ \delta\eta = [\phi, \bar{\phi}] \\ \delta\varphi = \zeta,\ \delta\zeta = [\phi, \varphi] \\ \delta\chi_i = H_i,\ \delta H_i = [\phi, \chi_i] \\ \delta\chi = H,\ \delta H = [\phi, \chi], \end{aligned}$$

where all fields are complex $U(N)$ adjoints, except for χ and H which are Hermitian. This corresponds to the $d = 10$ case of [55] in a slightly different notation that is more convenient in the context of compactification on a Calabi-Yau 3-fold. The Z_i are the D0 collective coordinates in the internal manifold, while φ morally lives in the bundle of $(3, 0) + (0, 3)$ forms over the Calabi-Yau, which practically means that it is a scalar.

This is the reduction to zero dimensions of the twisted $\mathcal{N} = 4$ Vafa-Witten theory, and equivalently, of the twisted $\mathcal{N} = 2$ gauge theory in six dimensions. We will write a Q-trivial action for this topological matrix model, which can be interpreted as the topological twisted version of the $0 + 1$ dimensional superconformal quiver quantum mechanics [28] describing these bound states. The topological property results from the fact we are only interested in computed the index of BPS states, which can be described in this zero dimensional theory.

Choose the sections, in the language of [70], to be

$$\begin{aligned} s_i &= \Omega_{ijk}\ Z_j Z_k + [\varphi, Z_k^\dagger] \\ s &= \sum_i [Z_i, Z_i^\dagger] + [\varphi^\dagger, \varphi], \end{aligned} \tag{5.8}$$

where Ω_{ijk} is the holomorphic 3-form, which we can assume to be given by the anti-symmetric ϵ_{ijk} for the $\mathbb{C}^3$ vertex geometry.

The action is given by $S = t\{Q, V\}$, where

$$V = \mathrm{Tr}\left(\chi_i^\dagger (H_i - s_i)\right) + \mathrm{Tr}\left(\chi(H - s)\right) + \mathrm{Tr}\left(\psi_i[\bar{\phi}, Z_i^\dagger]\right) + \mathrm{Tr}\left(\zeta^\dagger[\bar{\phi}, \varphi]\right) + \mathrm{Tr}\left(\eta[\phi, \bar{\phi}]\right).$$

Following [70], we write the bosonic terms after integrating out H_i and H, which appear quadratically in the action, to obtain

$$S_{bosonic} = \mathrm{Tr}\left(s_i^\dagger s_i + s^2 + [\phi, Z_i][\phi, Z_i]^\dagger + [\phi, \varphi^\dagger][\phi, \varphi^\dagger]^\dagger + [\phi, \bar{\phi}]^2\right), \tag{5.9}$$

where the $\mathrm{Tr}[\phi, Z_i][\phi, Z_i]^\dagger$ terms arise from the twisted superpotential, thus coupling the vector multiplet, ϕ, with the chiral matter fields charged under it.

This theory has a total of 9 real scalars, coming from the Z_i, φ, and ϕ, transforming the adjoint of $U(N)$, corresponding to the 9 collective coordinates of a point-like brane. It is clear, following the reasoning of [70], that the partition function will count, with signs, the Euler character of the moduli space of gauge equivalence classes of solutions to the equations, $s_i = 0$, $s = 0$. Moreover, the field ϕ can be easily eliminated from the theory, since it simple acts to enforce the $U(N)$ gauge symmetry on the moduli space. To relate this to the ADHM type data we discussed in the previous section, we need to understand what happens to the extra collective coordinate, φ. In fact, we will prove a vanishing theorem, and find that it does not contribute to the instanton moduli space. Its main function in the matrix model is to ensure that all of the solutions are counted with the same sign, so that the Euler character is correctly obtained, in the same spirit as [70].

Note that there are 4 dynamical complex adjoint fields in the matrix model, the Z_i and φ, which satisfy 3 holomorphic equations (5.8). The Kahler quotient by $U(N)$

gives 1 real D-term equation from the moment map, and quotients by $U(N)$. Hence the expected dimension of the moduli space is $8-6-1-1=0$, however it may not consist of isolated points in practice unless the sections are deformed in a sufficiently generic manner.

In order to fully make sense of this theory, we need to regulate the noncompactness of $\mathbb{C}^3$ in some way. For local analysis on toric Calabi-Yau, the natural choice is to add torus-twisted mass terms, which will further enable one to localize the path integral to the fixed points of the induced action of $U(1)^3$. It is possible to understand the toric localization in the matrix model language. We want to turn on the zero dimensional analog of the Ω - background explained in [37]. Exactly as in that case, we twist the $U(N)$ gauge group by the torus $U(1)^3$ symmetry. That is, we gauge a nontrivial $U(N) = SU(N) \times U(1)$ subgroup of $U(N) \times U(1)^3$.

The effect of this on the topological matrix model is to change the BRST operator so that Q^2 generates one of the new gauge transformations. That is, the fields transform as

$$\begin{aligned} \delta Z_i &= \psi_i \\ \delta \psi_i &= [\phi, Z_i] - \epsilon_i Z_i, \end{aligned} \tag{5.10}$$

where the ϵ_i determine the embedding of the gauged $U(1)$ inside $U(1) \times U(1)^3$.

This changes the terms involving ϕ in the bosonic action to become $\sum \mathrm{Tr}([\phi, Z_i] - \epsilon_i Z_i)([Z_i^\dagger, \bar{\phi}] - \epsilon Z_i^\dagger)$, which forces the adjoint fields Z_i to be localized with respect to the torus action. We will soon see that the partition function is independent of the choice of weights ϵ_i as long as the superpotential itself is invariant under the chosen group action, so that

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = 0.$$

This is natural, since only these torus actions are subgroups of the $SU(3)$ holonomy, and thus the equivariant twisted theory continues to preserve supersymmetry.

There is an induced action of $U(1)^3$ on the other fields in the theory, $\chi_i \to e^{i\alpha - i\epsilon_i}\chi_i$ and $\varphi \to e^{\alpha}\varphi$, where $\alpha = \sum \epsilon_i$, which changes the action of Q in the obvious way.

To understand the measure factor at the solutions that contribute to the partition function we need to examine the fermionic piece of the action,

$$S_f = \mathrm{Tr}\left(\phi[\chi_i, \chi_i^\dagger] + \bar{\phi}([\psi_i, \psi_i^\dagger] - [\zeta, \zeta^\dagger]) + \Omega_{ijk}\psi_i[Z_j, \chi_k^\dagger] + \zeta[Z_i^\dagger, \chi_i^\dagger] - \psi_i^\dagger[\chi_i^\dagger, \varphi]\right) \\ + \mathrm{Tr}\left((\chi + i\eta)([\psi_i, Z_i^\dagger] - [\zeta, \varphi^\dagger])\right). \tag{5.11}$$

The anti-ghost, χ, which enforces the moment map condition, naturally pairs with η, the fermionic partner of the $U(N)$ multiplet, to produce the Kahler quotient of the moduli space. We see that η appears linearly in the action, thus when it is integrated out, the delta function constraint $[\psi_i, Z_i^\dagger] - [\zeta, \varphi^\dagger] = 0$ is enforced. This forces the fermions ψ_i and ζ to lie in the sub-bundle of the tangent bundle normal to the D-term condition, $s = 0$. Therefore the Euler character of the Kahler quotient will be obtained, as in [70].

The topological nature of the partition function means that we are free to change the coupling, t, which results in a BRST trivial change of the theory, leaving the partition function invariant. Taking the limit $t \to \infty$, the matrix model localizes to the classical moduli space defined by

$$\begin{aligned} \Omega_{ijk} Z_j Z_k + [\varphi, Z_i^\dagger] = 0 \\ \sum_i [Z_i, Z_i^\dagger] + [\varphi^\dagger, \varphi] = 0, \end{aligned} \tag{5.12}$$

where we must mod out by the gauge group, $U(N)$.

Near the solutions to the BPS equations the potential can be approximated as a Gaussian, and the partition function will pick up a determinant factor. From the bosonic part of the action (5.9) expanded near a solution, there are quadratic terms of the form $\mathrm{Tr}(\hat{\delta} s_i^\dagger \hat{\delta} s_i)$, where here $\hat{\delta}$ is the variation. Terms such as $\mathrm{Tr}(s_i^\dagger \hat{\delta}(\hat{\delta} s_i))$ cannot arise since $s_i = 0$ for the BPS solutions. For each such contribution, there is an analogous fermionic term of the form $\mathrm{Tr}(\chi_i^\dagger \delta s_i)$. After integrating out the anti-ghosts, the quadratic terms involving the ghosts ψ_i will exactly match those of the fields Z_i. Call the resulting quadratic form $\mathcal{A}$.

In addition there are the twisted superpotential terms of the form $|[\phi, Z_i] + \epsilon_i Z_i|^2$, which give a total contribution to the one loop determinant

$$\frac{1}{\det(\mathrm{ad}\,\phi + \alpha)} \frac{\det(\mathrm{ad}\,\phi + \alpha - \epsilon_i)}{\det(\mathrm{ad}\,\phi + \epsilon_i)}, \tag{5.13}$$

which exactly cancels, up to sign when we include the measure factor for diagonalizing ϕ, if $\alpha = \sum \epsilon_i = 0$, as in [55]. This is all there is for the bound state of N 0-branes, since there are no nontrivial fixed points, hence $\mathcal{A} = 0$. For future reference, the answer in general would be

$$\frac{\det(A + T)}{\det(A + T')}, \tag{5.14}$$

where T and T' are the ϵ_i dependent pieces in the quadratic approximation. Thus for $\sum \epsilon_i = 0$, we still have exact cancellation, even at the nontrivial fixed points. This is no surprise, since we designed the theory for precisely this effect.

Naively one would expect that the anti-ghost bundle whose Euler character the matrix model will compute is simply spanned by χ_i and χ fibered trivially over the moduli space. However, by looking at the fermion kinetic terms involving ψ_i and ζ, one finds that the bundle is obstructed when restricted to the manifold obtained after imposing the F-flatness condition. In particular, there are terms in the action,

$$\mathrm{Tr}\left(\psi_i\left(\Omega_{ijk}[\chi_j^\dagger, Z_k] - [\chi, Z_i^\dagger]\right) + \zeta\left([\chi_i^\dagger, Z_i^\dagger] + [\chi, \varphi^\dagger]\right)\right), \tag{5.15}$$

which require that

$$\begin{aligned}\Omega_{ijk}[Z_i^\dagger, \chi_j] &= [Z_k, \chi] \\ [Z_i, \chi_i] &= [\chi, \varphi].\end{aligned} \tag{5.16}$$

This defines the anti-ghost bundle, or rather complex of bundles, whose Euler character is the index of BPS states computed by the matrix integral. It is obviously distinct from the tangent bundle, although it has the same rank, and their Euler characters may thus differ.

Now we proceed to find the vanishing theorem that shows the field φ does not contribute to the instanton equations. First, notice that using the F-term condition,

$$\sum_k |[\varphi^\dagger, Z_k]|^2 = \mathrm{Tr}[\varphi^\dagger, Z_k]\Omega_{ijk}Z_jZ_k = \frac{1}{2}\,\mathrm{Tr}\,\varphi^\dagger\Omega_{ijk}[Z_k,[Z_j,Z_k]] = 0, \tag{5.17}$$

by the Jacobi identity, and the antisymmetry of Ω_{ijk}. This means that

$$[\varphi^\dagger, Z_k] = 0, \tag{5.18}$$

and moreover $[Z_i, Z_j] = 0$, as we had hoped.

What can we learn from the D-term? Rewriting it as

$$0 = \left|\sum[Z_i, Z_i^\dagger]\right|^2 + \left|[\varphi^\dagger, \varphi]\right|^2 + 2\sum\mathrm{Tr}[\varphi^\dagger, \varphi][Z_i, Z_i^\dagger], \tag{5.19}$$

we see that all the terms are positive, since

$$\mathrm{Tr}\,\varphi^\dagger\left[\varphi,[Z_i,Z_i^\dagger]\right] = -\,\mathrm{Tr}[\varphi^\dagger,Z_i][Z_i^\dagger,\varphi] - \mathrm{Tr}[\varphi^\dagger,Z_i^\dagger][\varphi,Z_i] = |[Z_i,\varphi]|^2\,, \tag{5.20}$$

by the Jacobi identity and equation (5.18). Therefore we see that φ commutes with both the Z_i and their conjugates, and can be trivially factored out of the theory.

At this point, all of the fields are on the same footing, since we can use the same reasoning to show that $[Z_i, Z_j^\dagger] = 0$, and the theory localizes to the trivial branch of moduli space. This is exactly the marginally supersymmetric state of N independent D0 branes. Note that if we rewrite the adjoint fields in terms of 9 Hermitian collective coordinates, and expand all of the terms appearing in the action, we will see the full $SO(9)$ rotational symmetry of non-relativistic D0 branes in flat space. This will soon be broken by the addition of a D6 brane, and the presence of a B-field.

5.3 Solving the matrix model for D6/D0 in the vertex

Now we will proceed to find the matrix model description of the quantum foam theory of bound states of 1 D6 brane and N D0 branes in the vertex. The only new field comes from the 0-6 strings, which results in a chiral scalar in the low energy description. This appears in the matrix model as a new topological multiplet,

$$\begin{aligned} \delta q &= \rho \\ \delta\rho &= \phi q, \end{aligned} \tag{5.21}$$

living in the fundamental of $U(N)$. See the left side of Figure 5.1 for the quiver diagram of the internal Calabi-Yau degrees of freedom.

It is clear that moving the D0 branes off the D6 brane in the normal directions will give a mass to the 0-6 strings. Thus there should be a term in the quiver action of the form $q^\dagger \varphi \varphi^\dagger q$. The similar mass term, $q^\dagger \bar{\phi} \phi q$, is automatically included in the twisted superpotential, as we shall see below. Moreover, the BPS bound states must therefore have $\varphi^\dagger q = 0$, which, however cannot obviously be implemented by any superpotential. This requires the addition of new anti-ghosts,

$$\begin{aligned} \delta\xi &= h \\ \delta h &= \phi\xi, \end{aligned} \tag{5.22}$$

living in the fundamental of $U(N)$. The number of new fields is thus cancelled by the new equations, and we are still left with a moduli space of expected dimension 0.

As we will see later in similar examples, this extra equation should be understood as a symptom of working in a noncompact geometry. Quiver theories normally arise by considering the theory of fractional branes at a Gepner point where the central charges are all aligned. The exact nature of that quiver depends upon the compactification, but in the local limit, will be reduced to those found here. The general pattern is that a larger gauge group gets broken to $U(N) \times U(M)$ where the N and M are the numbers of branes that are distinguished (for example as D0 and D6) in the large volume limit we are interested in, far from the Gepner point. There will typically be some off-diagonal D-term which persists even in the large volume limit, and $\varphi^\dagger q = 0$ is just such an example.

The $U(N)$ D-term now includes the contribution of the bifundamental, replacing the section s by

$$s = \sum_i [Z_i, Z_i^\dagger] + [\varphi^\dagger, \varphi] + qq^\dagger - rI, \tag{5.23}$$

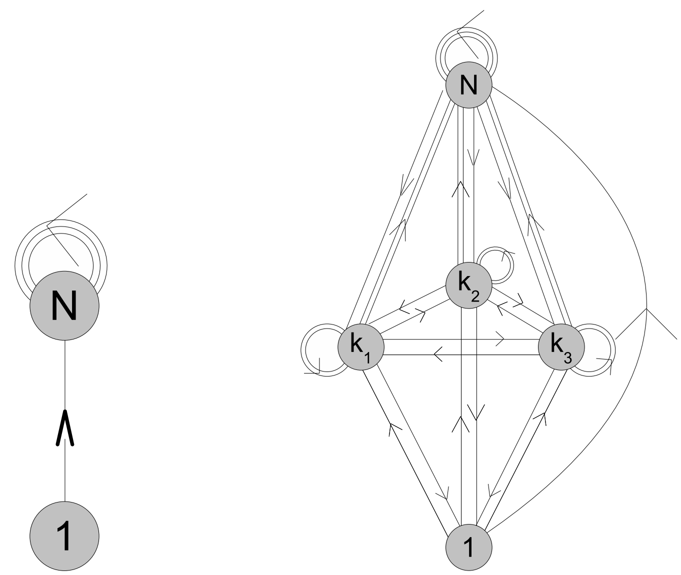

Figure 5.1: The quiver diagrams for the topological vertex with trivial and generic asymptotics.

where r is the mass of the field q. The anti-ghost ξ appears in the action in the form

$$\{Q, \xi^\dagger(h - \varphi q)\}. \tag{5.24}$$

Putting everything together, we find that the bosonic part of the action is

$$\begin{aligned} S_b = &\left|\Omega_{ijk} Z_i Z_j + [\varphi, Z_k^\dagger]\right|^2 + \left|\sum_i [Z_i, Z_i^\dagger] + [\varphi^\dagger, \varphi] + q q^\dagger - rI\right|^2 \\ &+ q^\dagger \varphi \varphi^\dagger q + |[\phi, Z_i]|^2 + \left|[\phi, \varphi^\dagger]\right|^2 + \mathrm{Tr}[\phi, \bar{\phi}]^2. \end{aligned} \tag{5.25}$$

It is clear that the only effect of the Fayet-Iliopoulos parameter, r, is to make the 0-6 strings tachyonic by adding the term $-r\ qq^\dagger$ to the action. As explained in [72] the mass of these bifundamentals is determined by the asymptotic B-field. In particular, they are massive, $r < 0$, in the absence of a magnetic field, so that the minimum of the action has nonzero energy and there are no BPS states. For a sufficiently strong B-field, q will become tachyonic and can condense into a zero energy BPS configuration.

The argument we used before implies that the theory localizes on solutions to the equations,

$$\begin{aligned}[Z_i, Z_j] &= 0 \\ [Z_i, \varphi^\dagger] &= 0 \\ \sum_i [Z_i, Z_i^\dagger] + [\varphi^\dagger, \varphi] + q^\dagger q &= rI \\ q^\dagger \varphi &= 0.\end{aligned} \tag{5.26}$$

Note that equation (5.18) still holds, since the superpotential it was derived from is unchanged.

We find a vanishing theorem from the D-term as before, computing

$$\begin{aligned}0 = \left| \sum_i [Z_i, Z_i^\dagger] + [\varphi^\dagger, \varphi] + q^\dagger q - rI \right|^2 = \left| [Z_i, Z_i^\dagger] + q^\dagger q - rI \right|^2 + \mathrm{Tr}[\varphi^\dagger, \varphi]^2 + \\ 2 \sum \mathrm{Tr}[\varphi^\dagger, \varphi][Z_i, Z_i^\dagger] + 2q^\dagger[\varphi^\dagger, \varphi] q.\end{aligned} \tag{5.27}$$

Only the last term is different, and using the fact that $q^\dagger \varphi = 0$, as required by the potential for moving the D0 branes off the D6 in the spacetime direction, we see that all terms on the right hand side of (5.27) are positive squares. Hence the moduli space consists of solutions to the quiver equations (5.7), with the field φ totally decoupling from the classical moduli space. It will, however, give a crucial contribution to the 1-loop determinant at the fixed points.

5.3.1 Computing the Euler character of the classical moduli space

The moduli space of BPS states is given by the solutions to (5.7) up to $U(N)$ gauge equivalence. We want to determine the Euler characteristic of this quiver moduli space using equivariant techniques. There is a natural action of the $U(1)^3$ symmetry group of the vertex geometry on this classical instanton moduli space, given by $Z_i \to \lambda_i Z_i,\ q \to \lambda q$. The Euler character is localized to the fix points, which are characterized by the ability to undo the toric rotation by a gauge transformation, that is,

$$\begin{aligned} [\phi, Z_i] &= \epsilon_i Z_i \\ q\phi - \alpha q &= \epsilon q, \end{aligned} \tag{5.28}$$

for some $\phi \in su(N)$, and where $\alpha \in u(1)$ implements the $U(1)$ gauge transformation, which can obviously be re-absorbed into the $U(N)$ transformation parameterized by ϕ.

This is implemented in the matrix model as before, by changing the $\mathcal{Q}$ action so that it squares to the new $U(N)$ subgroup of $U(N) \times U(1)^3$ that is being gauged. Consistency requires that φ and ξ have weights $\epsilon_1+\epsilon_2+\epsilon_3$ and $\epsilon-\epsilon_1+\epsilon_2+\epsilon_3$, respectively, under the torus action. The twisted superpotential terms in the path integral are modified in the obvious way, schematically, from $\mathrm{Tr}[\phi,\cdot][\phi,\cdot]^\dagger$ to $|[\phi,\cdot]+\epsilon\ \cdot|^2$.

The weight at the fixed point can be determined as follows, assuming that $\epsilon_1+\epsilon_2+\epsilon_3=0$ to preserve the $SU(3)$ holonomy. It is easy to check as before that contributions to the quadratic terms in the variation of Z_i away from the fixed point exactly match those of ψ_i. The twisted superpotential gives rise to additional contributions

$$\mathrm{Tr}\left([\delta Z_i, \phi] + \epsilon_i \delta Z_i\right)\left([\bar{\phi}, \delta Z_i^\dagger] + \epsilon_i \delta Z_i^\dagger\right) + |[\varphi, \phi]|^2 + q^\dagger(\bar{\phi} + \epsilon)(\phi + \epsilon)q, \tag{5.29}$$

to the bosonic fields, and for the anti-ghosts,

$$|[\chi_i, \phi] - \epsilon_i \chi_i|^2 + \xi^\dagger(\bar{\phi} + \epsilon)(\phi + \epsilon)\xi, \tag{5.30}$$

where we used the fact that our torus action lives in $SU(3)$. Therefore, everything cancels exactly after including the Vandermonde determinant from diagonalizing ϕ.

To begin finding the fixed points, chose the gauge by requiring that ϕ of (5.28) is diagonal, denoting the eigenvalues by ϕ_a. This is a far more judicious choice then trying to diagonalize the Z_i, since they are not Hermitian, and all of the interesting bound states in fact require them to be non-diagonalizable. The noncommutivity of Z_i and $Z_i^\dagger$ thus implied can be understood physically as resulting from the background B-field we have turned on to produce supersymmetric bound states in the D6/D0 system. This field is indeed proportional to the Kahler form, $\omega = \sum_i dz_i^* \wedge dz_i$.

In this gauge, the equivariant condition gives the strong constraint that

$$(Z_i)_{ab}(\phi_a - \phi_b - \epsilon_i) = 0,\ \ q_a(\phi_a - \alpha - \epsilon) = 0. \tag{5.31}$$

For generic ϵ_i, this forces most of the components of Z_i to vanish, and it is useful to think of the nonzero elements as directed lines connecting a pair of the N points indexed by ϕ_a.

It is convenient to represent the action of the Z_i as translation operators on a finite collection, η, of N points in $\mathbb{Z}^3$ associated to the eigenvectors of ϕ, with adjacency determined by the condition

$$\phi_a - \phi_b = \epsilon_i, \tag{5.32}$$

which must hold whenever $(Z_i)_{ab} \neq 0$. More precisely, associating each eigenvector of ϕ to a point x^a in $\mathbb{Z}^3$, whenever (5.32) is satisfied, the points are related spatially by

$$x_i^a = x_i^b + 1.$$

In relating the quiver description to the six dimensional Donaldson-Thomas theory, recall that the fixed points of the induced $(\mathbb{C}^\times)^3$ action on equivariant ideals of the algebra of polynomials on $\mathbb{C}^3$ can be encoded as three dimensional partitions in the positive octant in the lattice $\mathbb{Z}^3$ [37]. These are equivalent in the standard way to coherent, torus invariant sheaves with point-like support at the origin in $\mathbb{C}^3$.

What does the F-flatness condition, $[Z_i, Z_j] = 0$, mean for the torus localized configurations? Clearly this implies the fact that if $p, q \in \eta$ can be connected by a particular sequence of positive translations, Z_i, remaining within η at each stage, then all such paths generated by other orders of the same Z_i must also lie in η. It is easy to see that this means that $\eta = \pi - \pi'$ is the difference between two three dimensional partitions, $\pi' \subseteq \pi$. Moreover, the values of the nonzero $(Z_i)_{ab}$ can be chosen such that the matrices indeed commute.

This has a nice interpretation in the language of the derived category of coherent sheaves used to describe all equivariant B-model boundary states in the vertex geometry. Let $\mathcal{E}_\pi$ be the equivariant sheaf associated to the partition π following [37]. Then the configuration for general η is associated to the complex

$$\mathcal{E}_\pi \to \mathcal{E}_{\pi'},$$

which, although perfectly fine as a B-model brane, is always an unstable object in the full theory, for any value of the B-field. It has D0 charge of N, and vanishing

D6 charge, since we have condensed the $D6/\overline{D6}$ tachyon. At the level of modules, the worldvolume of this B-brane is described by $M = \mathcal{I}_{\pi'}/\mathcal{I}_{\pi}$. This is exactly what we should have expected, since we haven't yet imposed the D-term constraint, which gives the stability condition in the quiver language.

The dimensional reduction of the $(1,1)$ part of the field strength that appears in (5.7) is naturally diagonal in the basis we have chosen. For most points in η it is possible to find Z_i satisfying all the conditions, such that $[Z_i, Z_i^\dagger] = Z_i Z_i^\dagger - Z_i^\dagger Z_i$ has a positive eigenvalue at that point. The exception are the points $p \in \eta$ on the interior boundary, π', since they are killed by $Z_i^\dagger$ and have only the negative contribution $-Z_i^\dagger Z_i$.

Therefore to satisfy the D-term condition, these negative eigenvalues must be cancelled by the contribution of the D6/D0 bifundamental, $q^\dagger q$. The single D6 brane only gives us one vector $q^\dagger \in \mathbb{C}^N$ to work with, hence only one negative eigenvalue of $[Z_i, Z_i^\dagger]$ can be cancelled. Hence there are only solutions when the interior has a single corner, namely when π' is trivial, and η is a three dimensional partition. Therefore we have reproduced the crystals first related to the A-model in [60]

In this case, the Z_i act as multiplication operators in the algebra $\mathcal{A} = \mathbb{C}[x, y, z]/\mathcal{I}_\pi$, which is the algebra of polynomials on the nonreduced subscheme found in the non-abelian branch of N points in three dimensions. The holomorphic version of the stability condition is the existence of a cyclic vector, $q \in \mathbb{C}^N$, such that polynomials in the Z_i, acting on q, generate the entire vector space. This is obvious for Z_i given by translation operators in the dimension N unital algebra, $\mathcal{A}$, with $q = 1 \in \mathbb{C}[x, y, z]$. Moreover, it can be shown that the D-term constraint is equivalent to this algebraic stability requirement.

5.3.2 Generalization to the $U(M)$ Donaldson-Thomas theory

The $U(M)$ Donaldson-Thomas theory describes the bound states of M D6 branes with D2 and D0. The instanton moduli space is described by the same quiver equations (5.7) as before, where q is now a $(N, \bar{M})$ bifundamental.

Applying the analysis of section 2, we find that the Z_i act as translations on a nested partition, $\eta = \pi - \pi'$, with N boxes. In the same way as before the moment map constraint can be saturated only when $q^\dagger q$ can cancel the negative contributions to $\sum [Z_i, Z_i^\dagger]$ on the interior corners. Therefore η can have at most M interior corners. Equivalently, choose M points in $\mathbb{Z}^3$, and construct overlapping three dimensional partitions based on each point. This will define a permissible η.

Consider the example of $U(2)$ Donaldson-Thomas invariants. Then it is clear from the quiver description of the moduli space that the equivariant bound states are associated to partitions in an "L-shaped" background, as shown. This background is the most general with two corners, moreover configurations consisting of decoupled partitions resting independently on the corners are obviously not included. Therefore we can explicitly determine the partition function using the results of [60] and [38] about the statistical mechanics of melting crystals, obtaining

$$Z_{U(2)} = \sum_{n,m,k>0} \left(M(q)\; C_{[nm][kn]\cdot}\; q^{(nm^2+n^2k)/2} - S^{(n)} S^{(m,\;k)} \right) + \sum_{n,m>0} \left(M(q)\; C_{[nm]\cdot\cdot}\; q^{nm^2/2} - S^{(n)} S^{(m)} \right), \tag{5.33}$$

where $[nm]$ is the rectangular $n \times m$ Young diagram.

The first term is the generating function of crystals on the L-shape given by the asymptotic rectangular Young diagrams, where the power of q is present because of the framing factor in the topological vertex relative to the crystal partition function. This term leads to an over-counting of contributions to the $U(2)$ theory because of the inclusion of decoupled partitions supported independently in the two corners, hence these are subtracted off by the second term. It is also possible for the two interior corners to lay in the same plane, which is captured by the final two terms in (5.33). The partition functions $S^{(n,\ m)}$ are defined in [38].

Although the result (5.33) is not very transparent, it is notable as the first calculation of the $U(2)$ Donaldson-Thomas theory, even in the vertex. It would be very interesting to try to confirm this formula mathematically, as well as the more implicit (although still fully calculable order by order) answer for $U(M)$.

5.4 The full vertex $C_{\mu\nu\eta}$

The topological A-model partition function has been shown to be given by a dual description in terms of the bound states of a single 6-brane with chemical potentials turned on for D2 and D0 branes at large values of the background B-field [37]. In toric Calabi-Yau this has been checked explicitly, since both the A-model and the Donaldson-Thomas theory localize onto equivariant contributions in the vertex glued along the legs of the toric diagram. The general vertex, which thus determines the entire partition function of the A-model on any toric Calabi-Yau, can also be described by a quiver matrix model.

We will construct this quiver, and see that it reproduces the known answer for the topological partition function. In addition, this model gives much additional interesting information about the moduli space of these bound states then merely the Euler character. We will find that in general, the effective geometry seen by the dynamical D0 branes depends in an intriguing way on the background Kahler moduli, and undergoes flop transitions as one crosses walls where the attractor trees change shape, as in [21]. In the next section, we will see how the quantum foam picture of fluctuating Kahler geometry arises naturally in the quiver description, and further surprises of the effective geometry will become apparent.

The effective geometry we explore with 0-branes is, of course, the geometry of the moduli space of BPS states. This can receive various corrections both in α' and g_s. In the local case, however, the situation is more under control, since the Calabi-Yau moduli are always at large volume. Moreover, the torus action extends to the effective geometry, and we see no reason that this T^3 symmetry should be violated by corrections, at least in the local limit. Hence the corrections to the Kahler structure on the effective moduli appear likely to be simple renormalizations of the already present Kahler parameters, ie. the dependence of the FI terms on the background moduli receives α' corrections, which is no surprise.

5.4.1 One nontrivial asymptotic of bound D2 branes

We want to determine the low energy effective quantum mechanics of BPS D6/D2/D0 bound states. Note that the D6/D2 system is T-dual to a 4-brane with bound 0-branes, and is described by the same ADHM quiver [11]. As one dials the background values of the Kahler moduli, the D6/D2/D0 configurations are more naturally regarded as either point-like instantons in a D6 brane with D2 (singularly supported) "flux" turned on, a collection of 2-branes bound to a D6 with D0 charge, or a D2/D0 bound state attached to the D6. Only the first interpretation will be relevant for us, as we will soon see.

The low energy spectrum of 2-0 strings in flat space consists of a single tachyonic bifundamental multiplet, denoted here by B, which remains tachyonic for all values of the background B-field. The BPS states are obtained by condensing this field to cause the D0 branes to dissolve into flux on the D2. In our situation, however, if we turn on a B-field, the 6-2 strings can become far more tachyonic. As we will see quantitatively below, this field condenses first, melting the D2 branes into $U(1)$ flux on the D6, and giving a large positive contribution to the mass of the usual D2/D0 and D6/D0 bifundamentals. They become irrelevant to the moduli space of BPS states, but, surprisingly, an initially massive multiplet in the 2-0 spectrum receives an opposite contribution and becomes tachyonic.

A particularly simple way of determining the relevant spectrum of low energy 2-0 fields motivated by string theory is to look at the large volume limit of a D2-D0 bound state wrapping a compact 2-cycle. Consider k D2 branes on the S^2 of the resolved conifold, which are described by the quiver [40] shown in Figure 5.2, with total D0 charge $N + k/2$ (including induced charge). The superpotential implies that

$$C_1 D_a C_2 = C_2 D_a C_1, \qquad D_1 C_a D_2 = D_2 C_a D_1, \tag{5.34}$$

for the supersymmetric vacua.

The motion of the 0-branes is governed by the collective coordinates on the conifold, $c_1 d_2 = c_2 d_1$, where $(d_1 : d_2)$ are projective coordinates on $\mathbb{P}^1$. At first, we want

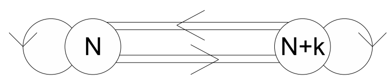

Figure 5.2: The quiver diagram for the resolved conifold

to focus on the case where there are N point-like bound D0 branes near one vertex. I will work holomorphically to avoid considerations of stability, which will be strongly affected by the D6 brane we will introduce shortly . This can be done by fixing the gauge,

$$D_2 = \begin{pmatrix} I_N & 0_{N\times k} \end{pmatrix},$$

which implicity assumes that all of the D0 branes are in the patch $d_2 \neq 0$, consistent with the desired noncompact limit we will take to obtain the vertex geometry.

For this gauge choice, we have broken the $U(N) \times U(N+k)$ symmetry down to $U(N) \times U(k)$, and obtained the "localized" quiver shown. This gives the quiver for D2-D0 branes in flat space, including the A_a fields which are massive without the presence of the D6 and background B-field. They will play a crucial role for us, as we will soon see.

The remaining quiver fields for the conifold decompose under the breaking of the gauge group as

$$C_a = \begin{pmatrix} Z_a \\ A_a \end{pmatrix}, \qquad D_1 = \begin{pmatrix} Z_3 & B \end{pmatrix}. \tag{5.35}$$

The F-flatness equation (5.34) implies that

$$[Z_1, Z_2] = 0,\ [Z_2, Z_3] = BA_1,\ [Z_3, Z_1] = BA_2,\ A_1 Z_2 = A_2 Z_1. \tag{5.36}$$

To obtain the full quiver for 2-0 in $\mathbb{C}^3$, one needs to modify these equations in the natural way to include the motion of the D2 branes, which are rigid in the conifold example. These equations naturally result from extremizing the superpotential (5.40) below.

In order to obtain the correct moduli space for $N > 1$ and $k > 1$ these is one extra ingredient we have so far neglected. This can be motivated in two different ways. First, in the above derivation of the quiver by taking a limit of the resolved conifold geometry, the D-term of the original $U(N+k)$ symmetry has an off-diagonal component in the decomposition to $U(N) \times U(k)$. This would imply the existence of an additional constraint,

$$A_a Y_a^\dagger + Z_a^\dagger A_a = 0, \tag{5.37}$$

where we have included the generalization to a D2 brane bound state in a nontrivial configuration encoded in the Y_a adjoints of $U(k)$.

Moreover, this term is needed to describe the ordinary contribution to the mass of the 2-0 strings when the D2 and D0 branes are separated in transverse directions. That is, there must be a term in the low energy effective theory of the form

$$\mathrm{Tr}\left(A_a Y_a^\dagger Y_a A_a^\dagger + Z_a^\dagger A_a A_a^\dagger Z_a \right). \tag{5.38}$$

We will soon see that the F-term, $|\partial W|^2$, only gives an appropriate mass to the bifundamentals A_1 when there is a distance $(Y_2)_i - (Z_2)_j \neq 0$ between a pair of eigenvalues, however the A_2 mode must also receive a mass. Later, it will become clear the condition $Z_a^\dagger A_a = 0$ is redundant, and hence the associated terms in the action can be absorbed into the already existing ones, but the constraint

$$A_a Y_a^\dagger = 0 \tag{5.39}$$

is new, and must be imposed in addition to the ordinary quiver conditions. This is similar to the mass term needed for the 6-0 strings generated by separation of the branes in the direction transverse to the Calabi-Yau. Note that there is no off-diagonal residual gauge symmetry, since the form of B_2 breaks it to exactly $U(N) \times U(k)$; this would be an issue for $GL(N+k, \mathbb{C})$, however.

The correct bundle over the classical moduli space can be found by including the fields that are relevant for motion normal to the Calabi-Yau, playing an analogous role to φ and ϕ in the D0 theory. By the transversal $SO(7)$ rotational symmetry broken from $SO(9,1)$ by the presence of the D2 brane, there must exist 3 additional low energy 0-2 modes, a complex chiral field, $\widetilde{A}$, and a real vector, analogous to ϕ. The vector multiplet is associated to the unbroken off-diagonal component of $GL(N+k, \mathbb{C})$, and only its associated D-term survives in the Kahler description, as we have seen. The presence of $\widetilde{A}$ is important for finding the correct 1-loop determinant at the fixed, although it is vanishing for all BPS solutions.

The dynamical topological multiplets, describing the motion (adjoint fields) and tachyons (bifundamentals) of the D0 branes are

$$\delta Z_i = \psi_i,\ \delta\psi_i = [\phi, Z_i]$$

$$\delta\varphi = \zeta,\ \delta\zeta = [\phi, \varphi]$$

$$\delta q = \rho,\ \delta\rho = \phi q$$

$$\delta B = \beta,\ \delta\beta = B\phi - \phi' B$$

$$\delta A_a = \alpha_a,\ \delta\alpha_a = A_a\phi' - \phi A_a$$

$$\delta\widetilde{A} = \widetilde{\alpha}$$

$$\delta\widetilde{\alpha} = \widetilde{A}\phi' - \phi\widetilde{A}$$

where $i = 1, 2, 3$ and $a = 1, 2$, and B, A_a are the lowest lying 2-0 string modes, with masses $m_B = -m_A < 0$ at all values of the background Kahler moduli. There are auxiliary multiplets

$$\delta\phi = 0$$

$$\delta\overline{\phi} = \eta$$

$$\delta\eta = [\phi, \overline{\phi}],$$

as before. We regard the D2 brane moduli as frozen due to the noncompactness of their worldvolume, but they can still be derived from the T-dual D4/D0 system as:

$$\delta Y_a = \xi,\ \delta\xi = [\phi', Y_a]$$

$$\delta J = \upsilon,\ \delta\upsilon = \phi' J$$

$$\delta K = \kappa,\ \delta\kappa = K\phi'$$

$$\delta\phi' = 0,$$

where the completion to the full ten dimensional theory is ignored, as we are not concerned with with anti-ghost bundle over this moduli space, having already chosen a particular point due to the noncompactness of the wrapped 2-cycles.

The topological nature of the partition function means that the Euler character of the obstruction bundle is a deformation invariant when it satisfies the proper convergence properties. In particular, the cubic terms in the superpotential are sufficient to determine the exact result, as higher order corrections, even if they exist, will not affect the answer, although they may correct the geometry of moduli space. The superpotential can be read off from the quiver diagram, including the usual D0 Chern-Simons like term as well as the natural superpotential of the D2/D0 system, giving

$$W = \mathrm{Tr}\left(\Omega_{ijk} Z_i Z_j Z_k + \epsilon_{ab} Z_a A_b B + \epsilon_{ab} A_a Y_b B + qKB\right), \tag{5.40}$$

which implies the F-flatness conditions,

$$\begin{aligned} [Z_1, Z_2] = 0,\ [Z_3, Z_a] = A_a B,\ BZ_a = Y_a B, \\ Z_1 A_2 + A_1 Y_2 = Z_2 A_1 + A_2 Y_1 + qK,\ KB = 0,\ Bq = 0. \end{aligned} \tag{5.41}$$

The moment maps are

$$\begin{aligned} \sum [Z_i, Z_i^\dagger] + \sum A_a A_a^\dagger - B^\dagger B + qq^\dagger = rI_N, \\ \sum [Y_a, Y_a^\dagger] - \sum A_a^\dagger A_a + BB^\dagger + K^\dagger K - JJ^\dagger = r' I_M, \end{aligned} \tag{5.42}$$

which square to give the D-term contribution to the topological matrix model. Note in particular that $r' > r$ so that the usual 2-0 string is classically tachyonic, as it must be.

The collective coordinates, Y_a, of the D2 brane will encode their configuration as bound flux in the D6, which we will regard as a fixed asymptotic condition, due to the noncompactness of the D2 worldvolume. There is also the F-term constraint for the D6/D2 system,

$$[Y_1, Y_2] = JK. \tag{5.43}$$

First let us understand why these are physically the correct conditions. The FI parameters serve to give masses to the bifundamentals via terms of the form $-2r \operatorname{Tr} A_a A_a^\dagger$, and we find that the 0-2 tachyon has bare mass $-2(r' - r) < 0$ for all values of the background B-field. There is a pair of massive fields, A_a, in the 0-2 spectrum with mass $2(r' - r)$ that allow us to write the usual superpotential. The 6-2 string K, which is also tachyonic with mass $-2r' < -2(r' - r)$, will be the first to condense, dissolving the D2 branes into flux on the D6.

Recall that the D6/D2 system is T-dual to D4/D0, and the bound D0 flux is described by the ADHM construction [11]. Therefore the combination $\sum[Y_a, Y_a^\dagger] + K^\dagger K - JJ^\dagger$ is an $M \times M$ matrix with positive eigenvalues, which results in a large positive contribution, from the term $\operatorname{Tr} B^\dagger(\sum[Y_a, Y_a^\dagger] + K^\dagger K - JJ^\dagger)B$ in the action, to the effective mass of B in the vacuum with nonzero K. Therefore we find that $B = 0$ in the supersymmetric vacuum, which also agrees with the F-term condition. This can be thought of as a quantum corrected mass for B after integrating out the heavy field, K.

As can easily be seen from the structure of the D-terms, the effective masses are $m_A = -m_B$, hence these bifundamentals now condense, binding the D0 branes to the D2. Putting everything together, we see that the background D2 configuration is described by a Young diagram, λ, with M boxes, encoded in the matrices Y_a, as expected. The D0 collective coordinates obey $[Z_i, Z_j] = 0$, giving a difference, η, of three dimensional partitions, and the D-term can only be satisfied when the interior corners are exactly the image of the map A_a from $\mathbb{C}^M \to \mathbb{C}^N$.

Moreover, there is a holomorphic equation which implies that

$$Z_1 A_2 + A_1 Y_2 = Z_2 A_1 + A_2 Y_1. \tag{5.44}$$

It is also clear from the discussion of the effective mass of the bifundamentals that $Y_a A_a^\dagger = 0$ for $a = 1,\ 2.$ This means that only one of the terms on each side of (5.44) can be nonvanishing. The geometric interpretation in terms of crystal configurations is that the $z_3 = 0$ plane of η together with λ forms a Young diagram, and by the commutativity of the Z_i, this implies that η is exactly a three dimensional partition in the background of $\lambda \times z_3$ axis, as expected from [60]!

To further check that (5.44) is the correct F-flatness condition even for non-torus invariant points in the BPS moduli space, note that shifting the positions of the D0 and D2 brane together by sending $Z_i \mapsto Z_i + x_i I_N$ and $Y_a \mapsto Y_a + x_a I_k$ doesn't change the solutions for the A_a, as expected. It is easy to confirm that in the generic branch of moduli space, specifying $3N$ independent eigenvalues of mutually diagonalizable Z_i and likewise $2k$ distinct eigenvalues of Y_a totally determines the A_a up to gauge equivalence. This is because (5.44) essentially allows one to solve for A_2 in terms of A_1, and the condition (5.39) together with the D-terms fix A_1 up to the $U(1)^{N+k-1}$ symmetry left unbroken by the choice of Z_i and Y_a.

The path integral can be calculated as before by introducing a toric regulator. The fields transform as before under the $U(1)^3$ action on $\mathbb{C}^3$, with

$$A_a \to e^{i\epsilon_a} A_a, \qquad B \to e^{i\epsilon_3} B, \qquad \widetilde{A} \to e^{i\sum \epsilon_i} \widetilde{A}, \tag{5.45}$$

and consistently for the anti-ghosts. Therefore the full result for the determinant is given by

$$\frac{(\mathrm{ad}\,\phi+\epsilon_1+\epsilon_2)(\mathrm{ad}\,\phi+\epsilon_1+\epsilon_3)(\mathrm{ad}\,\phi+\epsilon_2+\epsilon_3)(\phi-\phi'+\epsilon_1+\epsilon_2)(\phi'-\phi+\epsilon_3+\epsilon_1)}{(\mathrm{ad}\,\phi+\epsilon_3)(\mathrm{ad}\,\phi+\epsilon_2)(\mathrm{ad}\,\phi+\epsilon_1)(\phi-\phi'+\epsilon_3)(\phi'-\phi+\epsilon_2)}$$
$$\times\frac{(\phi'-\phi+\epsilon_3+\epsilon_2)}{(\phi'-\phi+\epsilon_1)}\frac{(\phi-\epsilon_1-\epsilon_2-\epsilon_3)(\phi-\phi')}{(\phi)(\phi-\phi'+\epsilon_1+\epsilon_2+\epsilon_3)(ad\phi+\epsilon_1+\epsilon_2+\epsilon_3)},$$

which precisely cancel when the toric weights sum to zero, and we include the Vandermonde from the ϕ integral. The measures from the fields governing motion of noncompact objects have not been included since they are frozen, not integrated.

5.4.2 Multiple asymptotics and a puzzle

Now we would like to understand the quiver matrix model that describes the bound states of D0 branes to our D6 when D2 charge is turned on in more than one of the three toric 2-cycles. The index of BPS states on this theory, with a fixed configuration of the nondynamical fields associated to motion of noncompact objects, should give exactly the topological vertex of [4], [37]. Generalizing the quiver, it is again clear that the frozen D2 adjoint fields should give a stable representation of the constraint $[Y_1, Y_2] = 0$; these are the asymptotic Young diagrams appearing in the topological vertex, $C_{\mu\nu\eta}$. See Figure 5.1 for the complete quiver diagram including all fields.

The quiver is given as before, with a node, $U(k_i)$, for each of the three stacks of frozen D2 branes, and bifundamentals 2-2' strings, which have the same low energy field content as a T-dual pair of 4-0 bifundamentals. Only the cubic terms in the superpotential are relevant, and the possible terms allowed by the quiver (which also respect the obvious rotational symmetries in $\mathbb{C}^3$) are

$$W_0 = \epsilon_{ijk}\,\mathrm{Tr}\,(Z_iZ_jZ_k)+\epsilon_{ijk}\,\mathrm{Tr}\left(B^i\left(Z_jA^i_k+A^i_jY_k\right)\right)+\epsilon_{ijk}\,\mathrm{Tr}\left(J^i_jB^iA^j_k\right)+K_iB_iq, \quad (5.46)$$

for the D0 degrees of freedom and

$$W_2 = \mathrm{Tr}\left(J_2^1 J_3^2 J_1^3 - J_3^1 J_2^3 J_1^2\right) + \epsilon_{ijk}\,\mathrm{Tr}\left(J_j^i Y_k^i J_i^j\right) + K_i J_i^j \widetilde{K}_j, \tag{5.47}$$

where we have only included couplings of the dynamical fields. The frozen 6-2 modes must satisfy the equations of motion obtained from $\partial W_{frozen} = 0$, where

$$W_{frozen} = K_i \widetilde{Y}^i \widetilde{K}_i + \epsilon_{ijk}\,\mathrm{Tr}\left(\widetilde{Y}^i Y_j^i Y_k^i\right), \tag{5.48}$$

where the $\widetilde{Y}^i$ are exactly 0 in the vacuum, being associated to motion transverse to the D6 brane, and have been introduced simply to be able to write the superpotential. Note that it is possible to rescale the fields, while preserving rotational invariance in $\mathbb{C}^3$, in such a way to set all of the relative coefficients in the superpotential to unity.

The physical moduli space is the Kahler quotient of the resulting algebraic variety by $U(N) \times U(k_1) \times U(k_2) \times U(k_3) \times U(1)$, where the overall $U(1)$ acts trivially on all of the fields. This means that there is a D-term in the action given by the square of the equations

$$\begin{aligned} \sum_{i=1}^{3}[Z_i, Z_i^\dagger] + \sum_{i\neq a=1}^{3}\left(A_a^i\right)\left(A_a^i\right)^\dagger + qq^\dagger &= r_N I_N \\ \sum_{a\neq i}\left([Y_a^i, Y_a^{i\dagger}] - A_a^{i\,\dagger} A_a^i + J_i^a J_i^{a\dagger} - J_a^{i\,\dagger} J_a^i\right) + I^i I^{i\dagger} - \widetilde{I}^{i\dagger}\widetilde{I}^i &= r_i I_{k_i} \\ q^\dagger q + \sum_{i=1}^{3} I^{i\dagger} I^i - \widetilde{I}^i \widetilde{I}^{i\dagger} &= r_6, \end{aligned} \tag{5.49}$$

where the FI parameters, r, are determined by the background values of the Kahler moduli in a complicated way. We will later graph the loci where various combinations of these Kahler parameters of the moduli space vanish, which can be found by looking for walls of marginal stability where the central charges of some of the constituent branes align.

The fact that we are working in a local geometry obtained as a noncompact limit far from the Gepner point of the global Calabi-Yau again means that there will exist residual off-diagonal D-terms. Collecting all of the relevant equations, one has that

$$\begin{aligned} A^i_j Y^{i\dagger}_j + Z^\dagger_j A^i_j + A^j_i J^{j\dagger}_i = 0 \\ J^i_j Y^{i\dagger}_k + Y^{j\dagger}_i J^i_k = 0 \end{aligned} \tag{5.50}$$

The 2-2' bifundamental strings are localized in $\mathbb{C}^3$, stretched along the minimum distance between the orthogonal 2-branes (and thus living at any non-generic intersection), so they are also dynamical fields. Looking at the form of W_2, it is possible to see that the 2-6 system without D0 branes has no unlifted moduli after fixing the frozen degrees of freedom, that is, the semi-classical BPS moduli space is a point. This can be easily seen, since if the D2 branes are separated by a large distance in $\mathbb{C}^3$, all of the J^i_j become heavy. Thus the moduli space we are interested in is indeed the effective geometry as seen purely by the 0-brane probes.

There is a new feature in this quiver, since it is not immediately obvious which of the 6-2 and 2-2' strings should given background VEVs. We will find that depending on the values of the background Kahler moduli and B-field, the BPS ground states will live in different branches of the moduli space of these fields, which are frozen from the point of view of local D0 dynamics. The resulting effective quivers for the D0 degrees of freedom will be different, although the final computation of the Euler character turns out to be independent. In the next section, it will become clear what this means in terms of quantum foam. The alternative quiver realizations will exactly correspond to the resolutions, related by flops, of the blown up geometry experienced by the D0 branes, viewed as probes. It is thus very natural that the effective Kahler structure depends on the background Kahler moduli, via the Fayet-Iliopoulos parameters.

To get a feel for the way this works, consider the simplest example with two nontrivial asymptotics, $C_{\square\square\cdot}$. There are two equivariant configurations of the D6-D2 system that are consistent with the F-term condition, $\partial W_{frozen} = 0$, namely $Y^i_j = 0$ and either $J^1_2 = 0, J^2_1 = 1$ or $J^2_1 = 0, J^1_2 = 1$. Only one of these solutions can satisfy the D-term constraint for the D2 $U(1)$ gauge groups, for any given value of the FI parameters. For definiteness, let us focus on the latter solution. Then N D0 branes probing this background will be described by the quiver as before, with the new condition that

$$Z_3 A^1_2 + A^2_3 J^1_2 = Z_2 A^1_3, \tag{5.51}$$

whose equivariant fixed points can be interpreted as a skew three dimensional partition generated by the commuting Z_i in the usual way. The vectors A^1_3 and A^2_1 form the interior corners of the partition, saturating the moment map condition, while $A^1_2 = 0$ by (5.37), and A^2_3, interpreted as a vector in $\mathbb{C}^N$, is simply given be $Z_2 A^1_3$ by equation (5.51). Using the further holomorphic equation that

$$Z_1 Z_2 A^1_3 = Z_1 A^2_3 J^1_2 = Z_3 A^2_1 J^1_2, \tag{5.52}$$

we see that this means the skew partition exactly fits into the empty room crystal configuration associated to $C_{\square\square\cdot}$.

The essential idea extends to arbitrary background D2 brane configurations. In particular, there are usually many choices in determining the frozen fields, which depend of the FI parameters, and thus on the background Kahler moduli. One way to solve the constraints is to start with one asymptotic, and condense the tachyonic 2-6 field, being sure to tune the FI parameters so that it is along the direction of steepest descent in the potential. This procedure will give new contributions to the effective masses of the 2-2' strings, and there will again be some region in the background moduli space where they become unstable and condense. Finally, some of the 2-2" and 2'-2" fields condense. The D-term condition for each first $U(k_1)$ is supported by the 2-6 fundamental, while the D-terms of the two successive $U(k_i)$ are saturated by the condensed 2-2 bifundamental modes.

The above rather complicated sounding procedure, when unraveled, simply acts as a mechanism to obtain the correct equation relating the vectors of $U(N)$ corresponding to the edge of the background D2 branes in the crystal, starting with only a cubic superpotential (and thus quadratic equations $\partial W = 0$). If we regard the dynamical 2-0 bifundamentals only as vectors of $U(N)$, then relations are schematically of the form $Z_1^a Z_2^b Z_3^c A = Z_1^d Z_2^e Z_3^f A'$, where the powers of Z_i are determined by the shape of the intersecting asymptotic Young diagrams. The various quivers for a given vertex, $C_{\mu\nu\eta}$, are associated to different configurations of the frozen fields which lead to the same final conditions of this form. Physically speaking, the effective theory for the Z_i, obtained by integrating out all the bifundamental matter, is independent of the attractor tree, as long as no walls of marginal stability are crossed. However the full action with cubic superpotential that "completes" the higher order effective action can differ as we vary the background moduli.

A particularly simple example is $C_{\square\square\square}$, which turns out to have four quiver representations, one of which is $A^i_j \propto A^j_i$, as $\mathbb{C}^N$ vectors, which obey the equations $Z_k A^i_j = Z_j A^i_k$. We will soon see how all of these facts come together in a compelling physical picture, which gives a robust realization of the quantum foam geometries of

[37] as the physical geometry experienced by probe D0 branes.

5.5 Effective geometry, blowups, and marginal stability

Consider the quiver describing a single D0 brane in the vertex with one asymptotic 2-brane bound state encoded in the Young diagram, $|\mu| = k$. Then the relevant dynamical fields are the D0 coordinates and the 2-0 tachyons, which satisfy the F-flatness relation

$$A_1 \left(Y_2 - z_2\right) = A_2(Y_1 - z_1), \tag{5.53}$$

where $Y_a(\mu)$ are fixed matrices. It is easy to check that the projectivization of this holomorphic moduli space obtained by moding out by the residual nontrivial $\mathbb{C}^\times$ gauge symmetry left unbroken by the Y_a is exactly the blow up of $\mathbb{C}^3$ along the ideal defined by μ. Suppose that all of the parallel 2-branes are separated in the transverse z_1-z_2 plane. Then, working in the holomorphic language, we can use $GL(k, \mathbb{C})$ to simultaneously diagonalize the Y_a, and in that basis we therefore find

$$(A_1)_m \left((y_2)_m - z_2\right) = (A_2)_m \left((y_1)_m - z_1\right), \qquad m = 1, ..., k$$

These k equations define the blow up of $\mathbb{C}^2$ at k distinct points; the z_3 direction is a simple product with the resulting smooth, non Calabi-Yau 4-manifold. As we imagine bringing the D2 branes on top of each other, exploring other branch of the Hilbert scheme of k points in $\mathbb{C}^2$, the effective geometry experiencing by the 0-brane will be the blow up along that more complicated ideal sheaf. The smoothness of the

Hilbert scheme in two complex dimensions (which is the moduli space of the 2-branes bound to a D6) assures one that no subtleties can emerge in this procedure, as can be confirmed in specific examples.

More generally, using N 0-branes as a probe, we would see a geometry related to, but more complicated than, the Hilbert scheme of N points in this blow up variety. In particular, the 0-branes are affected by residual flux associated to the blown up 2-branes, so that globally they live in sections of the canonical line bundle associated to the exceptional divisors, rather than the trivial bundle. More abstractly, the associated sheaves fit into an exact sequence

$$0 \to \mathcal{I} \to \mathcal{L} \to i_* \mathcal{O}_Z \to 0,$$

where $\mathcal{L}$ is the nontrivial line bundle, and Z encodes the bound point-like subschemes. This doesn't change the nature of the Euler character of the moduli, which is determined by the local singularities. Intriguingly, there is a further refinement which naturally associates a size of order g_s to the exceptional divisors. It is impossible for more than a small number (typically two in our examples) of D0 branes to "fit" on the exceptional divisor. It would be interested to see if this feature also emerged in the study of the exact BPS moduli space of N 0-branes on a Calabi-Yau in the small volume limit.

In many examples with two or three nontrivial asymptotics, there exist multiple resolutions of the blown up geometry, as shown in the simple example of $C_{\square\square}$. (see Figure 5.3). We can see this as well in trying to write down the quiver description of these moduli spaces: there are different choices of background D2 moduli which, however, all have equal Euler characteristic. In fact, there is a clear physical reason

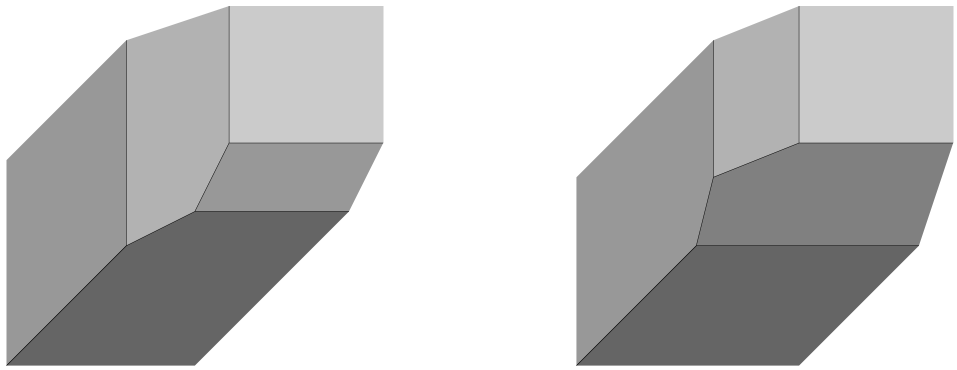

Figure 5.3: The two blowups associated to $C_{\square\square\cdot}$, related by a flop.

for this, to be found in a more detailed analysis of the lines of marginal stability. Depending on the values of the background Kahler moduli, different bifundamental tachyons will condense first. The index of BPS states is independent of which attractor tree [21] the system follows, as long as no lines of marginal stability are crossed.

The partition function for the general vertex reduces to the Euler characteristic of the anti-ghost bundle over the classical moduli, because of the topological nature of the matrix integral. As we saw in the previous section, this Euler character can be localized to the fixed points of a T^3 action induced on the moduli space, and the weights at the fixed points are all equal when the torus twist preserves the $SU(2)$ holonomy, as shown by computing the 1-loop determinant near the fixed point.

We now investigate the properties of the moduli space itself, which it the Kahler quotient of a holomorphic variety defined by quadratic equations, resulting from the cubic superpotential. The three F-term equations,

$$Z_i A_j^k + A_i^k Y_j^k + A_i^j J_j^k = Z_j A_i^k + A_j^k Y_i^k + A_j^i J_i^k, \tag{5.54}$$

where i, j, k are distinct and *not* summed over, together with $[Z_i, Z_j] = 0$ serve to define the holomorphic moduli space of the dynamical low energy degrees of freedom.

In our noncompact setup, we have not specified the boundary conditions for the 2-branes, which are frozen in the local analysis. If they are free to move in the transverse directions, generically they will not intersect, and the effective geometry probed by a BPS 0-brane will be the blowup along the wrapped 2-cycles, as can be seen by going to a complex basis in which Z_i and A^i_j are all diagonal. The 2-2' strings will become massive, and exit the low energy spectrum, when the branes are separated, and (5.54) becomes $k_1 + k_2 + k_3$ equations describing the local geometry $\mathbb{C} \times \mathcal{O}(-1) \to \mathbb{P}^1$ near each of the blown up 2-cycles.

It is also very interesting to consider compactifications in which the D2 branes wrap rigid or obstructed 2-cycles. In that case, they will have a nontrivial intersection, and a more sophisticated analysis is needed. As we will show in the following examples, the effective geometry is again the blow up along the wrapped subschemes. It is here that multiple smooth resolutions of the geometry may exist, corresponding to different attractor trees in which the order in which the tachyonic fields condense changes. No walls of marginal stability are crossed, hence the index of BPS remains constant.

The supersymmetric bound states we are discussing only exist when the asymptotic B-field is sufficiently large. From the perspective of Minkowski space, as the B-field flow down to the attractor value, various walls of marginal stability will be crossed. Thus the full geometry is multi-centered. Certain features of the attractor flow tree are visible in the quiver matrix model, so we first examine them from supergravity.

The central charge of a D6/D4/D2/D0 bound state described by a complex of sheaves $\mathcal{E}^*$ over the Calabi-Yau is, at large radius,

$$Z(\mathcal{E}^*) = \int e^{-\omega} ch(\mathcal{E}^*) \sqrt{Td(X)}, \tag{5.55}$$

where $\omega = B + iJ$ is the Kahler $(1,1)$-form on X [9]. At small volumes, this would receive worldsheet instanton corrections, hence the following analysis should only be expected to indicate the qualitative behavior of the attractor flow. This is sufficient for our purposes, since the local limit automatically assumes large volume.

In our case of k_i 2-branes and N 0-branes bound to a D6 brane in the equivariant vertex, the central charge (5.55) is computed as

$$\int e^{\sum t_i \omega_i} \left(1 + k_1 \omega_2 \wedge \omega_3 + k_2 \omega_1 \wedge \omega_3 + k_3 \omega_1 \wedge \omega_2 + N \omega_1 \wedge \omega_2 \wedge \omega_3\right) = t_1 t_2 t_3 + \sum t_i k_i + N, \tag{5.56}$$

where the t_i are Kahler parameters associated to toric legs. They may be independent or obey some relations, depending on the embedding of the vertex into a compact 3-fold.

From this one can check the well known fact that the 6-0 BPS bound state only exists when $\arg(t^3) > 0$, that is $|B_{z\bar{z}}| > \left(1/\sqrt{3}\right) |g_{z\bar{z}}|$. Similar calculations have been done in [21] when there are also D2 branes. We are most interested in the behavior of the 2-branes as the moduli approach the attractor values. Note that in a local analysis, the attractor flow itself cannot be determined without knowing the compactification; this is obvious because when the t_i become small, the vertex is no longer a good approximation.

All of these details of the global geometry are hidden in the FI parameters and the VEVs of the frozen fields appearing in the effective action for the 0-brane degrees of freedom. This is consistent with the expectation that the theory is defined at the asymptotic values of the moduli, where the 2-brane fields are heavy and their fluctuations about the frozen values can be integrated out. The topological nature of the matrix models guarantees that the only effect will be a ratio 1-loop determinants, which in fact cancel up to a sign.

As the moduli flow according to the attractor equations from the large B-field asymptotic limit needed to support the 6-brane BPS states, there are various possible decays. To find the walls along which they occur, one finds the condition for the central charges of the potential fragments to align. We find that in our examples, the 0-branes first split off.

The remaining D6/D2 bound state will itself decay at another wall of marginal stability. Depending on the asymptotic value of the moduli, the 2-branes wrapped one of the three toric legs will fragment off. Because the relevant Kahler moduli space is six real dimensional even in the local context, the full stability diagram is difficult to depict on paper, so we will content ourselves with the representative cross-section shown in Figure 5.4. The charges in this example are denoted by

$$\begin{aligned}\Gamma &= \omega_1 \wedge \omega_2 \wedge \omega_3 + \omega_1 + \omega_2 + 1\\ \Gamma_{6/2} &= \omega_1 \wedge \omega_2 \wedge \omega_3 + \omega_1 + \omega_2\\ \Gamma_1 &= \omega_1\\ \Gamma_2 &= \omega_2.\end{aligned}$$

We choose the background Kahler moduli to be

$$t_1 = 16i + x, \quad t_2 = 14i + y, \quad t_3 = 4i + 4,$$

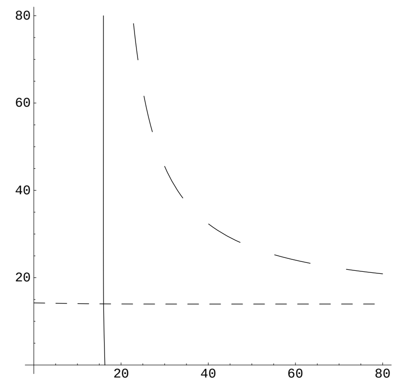

Figure 5.4: Walls of marginal stability for the decays $\Gamma \to \Gamma_0 + \Gamma_{6/2}$ (dashed line), $\Gamma_{6/2} \to \Gamma_1 + \Gamma'$ (dotted), and $\Gamma_{6/2} \to \Gamma_2 + \Gamma''$ (solid).

and plot the $x,\ y$ plane which parameterizes the asymptotic value of the B-field along the curves wrapped by the D2 branes.

It is immediately apparent that as the B-field in the z_1 and z_2 planes decrease along the attractor flow, the 0-branes separate along the first curve shown, and then the D6/D2 fragment intersects one of the two 2-brane curves. We shall see below how this phenomenon is imprinted on the matrix model. Therefore, in our local analysis, the 2-brane degrees of freedom are frozen in a pattern which depends on the attractor flow tree. It is natural that the indices calculated for the different effective 0-brane geometries are equal, since we see that they arise from the same theory in the global Calabi-Yau geometry.

5.5.1 Probing the effective geometry of a single 2-brane with N 0-branes

Let us first understand a very simple example from this point of view, namely the effective geometry induced by a single D2 brane, corresponding to the vertex $C_{\square\cdot\cdot}$

Consider probing the induced geometry wrapped by the D6 brane using N 0-branes. The F-term equation (5.44) tells us that $Z_1A_2 = Z_2A_1$, where we set the D2 position to 0 without loss of generality. Ignoring the z_3 direction for now, as it simply goes along for the ride, we see that this can be related to the quiver for a D4/D2/D0 bound state in $\mathcal{O}(-1) \to \mathbb{P}^1$, which lifts to D6/D4/D0 in the three dimensional blow up. The D4 brane is equivalent to the line bundle mentioned before, since it can be dissolved into smooth flux on the 6-brane worldvolume.

To see this, consider the two node quiver for $\mathcal{O}(-1) \to \mathbb{P}^1$ obtained by dropping one dimension from the quiver of resolved conifold. Setting the field $D = (I0)$ for quiver charges N and $N+1$ splits $U(N+1) \to U(N) \times U(1)$, and allows one to define

$$Z_a = C_a D, \tag{5.57}$$

and A_a to the be the $U(1)$ piece of C_a. Then the relation $C_1BC_2 = C_2BC_1$ implies that $Z_1A_2 = Z_2A_1$ and $[Z_1, Z_2] = 0$, as desired.

The two moduli spaces are not quite identical, however, because it is not always possible to choose a gauge such that $D = (I\ 0)$, and conversely, given such a form for D, it is impossible to satisfy the D-term condition when $Z_a = 0$. Therefore the structure of the moduli spaces near the zero section of $\mathcal{O}(-1)$ differ. In particular, it is easy to see that if $Z_1 = Z_2 = 0$ in some M dimensional subspace of $\mathbb{C}^N$ then there is a $\mathbb{P}^1$ of A_1, A_2 if $M = 1$, a single point if $M = 2$, and a violation of the D-term if $M > 2$. This is the sense in which the resolution is small - only a single 0-brane can fit conformably.

Lifting to the full $\mathbb{C}^3$ adds one complex dimension to the 2-brane, hence the even far from the exceptional divisor the 0-branes will see a $U(1)$ flux turned on in the D6 gauge theory. This has no effect on the moduli space of a single BPS D0, but will twist the global structure of the N D0 moduli space as follows.

Ordinarily, the Hilbert scheme of points is a kind of non-smooth "resolution" of the classical space of N points in a 3-fold, $\mathrm{Sym}^N(X)$. For simplicity, consider the case of $\mathrm{Sym}^2(X)$, which looks like the bundle $\mathcal{T}/\mathbb{Z}_2 \to \Delta$ near the diagonal embedding, $\Delta : X \hookrightarrow \mathrm{Sym}^2 X$, where $\mathcal{T}$ is the tangent bundle, with the natural $\mathbb{Z}_2$ action $z_i \to -z_i$ in local coordinates on the fiber. The z_i should be thought of as the small relative separation of the two points. In this special case, the Hilbert scheme $\mathrm{Hilb}^2(X)$, is locally the smooth resolution of the $\mathbb{Z}_2$ orbifold, which replaces the fiber by $\mathcal{O}(-2) \to \mathbb{P}^2$.

Therefore near the diagonal, the Hilbert scheme of two points on X is given by

$$\begin{array}{ccc} \mathcal{O}(-2) \otimes \mathcal{K} & \longrightarrow & \mathbb{P}\mathcal{T} \\ & & \downarrow \\ & & X, \end{array}$$

where the $\mathcal{O}(-2)$ bundle is fibered trivially over the base, and $\mathcal{K}$ is the canonical bundle over X. When a $U(1)$ flux on the D6 worldvolume is turned on, carrying D4 charge, the $\mathcal{O}(-2)$ of the Hilbert scheme becomes fibered over the diagonal in the sense of $\mathcal{L}$, the $U(1)$ bundle over X. That is, we should write $\mathcal{L}_X \otimes \mathcal{K}_X \otimes \mathcal{O}(-2)_{\mathbb{P}^2}$ in the above diagram of the geometry where the points are close together. This is compactified in a natural way to the full Hilbert scheme with a background line bundle, and a similar situation holds for $N > 2$ D0 branes.

Intriguingly, the effective canonical class of X, in the sense of the what line bundle appears in the above description of the Hilbert scheme, turns out to be the trivial bundle in our case, since $\mathcal{L}$ exactly cancels the fibration of $\mathcal{K}$ by the nature of the blow up construction. It would be interesting to understand if there is any more robust way in which the blow ups are effectively Calabi-Yau.

5.5.2 Example 2: the geometry of $C_{\square\square}$.

Let us work out the example of $C_{\square\square}$. in detail, as it already possesses most of the new features. The classical moduli of the D2 branes are the coordinates y_2^1, y_3^1 and y_1^2, y_3^2 respectively. Depending on the details of the compactification, which determine the superpotential for these fields, it may be possible to separate the branes in the z_3 direction. In that case, the 2-2' strings develop a mass and exit the low energy spectrum. Hence N probe D0 branes will have a BPS moduli space given by the $U(1)^2 \times SU(N)$ Kahler quotient of

$$\begin{aligned} \left(Z_2 - y_2^1\right) A_3^1 &= \left(Z_3 - y_3^1\right) A_2^1 \\ \left(Z_1 - y_1^2\right) A_3^2 &= \left(Z_3 - y_3^2\right) A_1^2 \\ [Z_i, Z_j] &= 0. \end{aligned} \tag{5.58}$$

For $y_3^1 \neq y_3^2$, this looks locally, near each of the D2 branes, like the geometry of N D0 branes in the "small" blowup discussed before. In particular, a single D0 probe sees exactly the blow up geometry along the two disjoint curves.

More interesting behavior occurs when the 2-branes cannot be separated in this way, for example in the closed vertex geometry in which the wrapped $\mathbb{P}^1$'s are rigid. This is more closely connected with the equivariant result, which requires the wrapped 2-cycles to have such a non-generic intersection by torus invariance. Here we must use the fact that

$$J^i_j \left(A^i_j\right)^\dagger = 0, \tag{5.59}$$

which results from the term (5.50) in the effective action.

The effective geometry of the $C_{\square\square\cdot}$ vertex, as probed by an individual BPS D0 brane is therefore determined by the Kahler quotient of the variety ,

$$\begin{aligned} z_2 a^1_3 &= a^2_3 j^1_2 \\ z_1 a^2_3 &= z_3 a^2_1, \end{aligned} \tag{5.60}$$

by $U(1) \times U(1)$ under which the fields have charges

z_i	a^1_3	j^1_2	a^2_1	a^2_3
0	1	1	0	0
0	1	-1	1	1

The first equation implies that (z_2, a^2_3) lives in the $\mathcal{O}(-1)$ bundle over the $\mathbb{P}^1$ with homogeneous coordinates $(a^1_3 : j^1_2)$. Moreover the second equation fixes (z_1, z_3) to be fibered in $\mathcal{O}(-1)$ on the compactification of the a^2_3 direction. This is shown in the toric diagram (see Figure 5.3), which is exactly the resolution of the blow up of $\mathbb{C}^3$ along the non-generically intersecting z_1 and z_2 lines!

Reversing the roles of the two D2 branes will result in a flopped geometry (which in this case is coincidentally geometrically equivalent to the first). It seems reasonable to conjecture that as one dials the background moduli, and thus implicitly the FI parameters, such that the roles of the two branes are exchanged, the attractor tree pattern changes without crossing a wall of marginal stability. This behavior was observed in a related system in [21]. We are using the intuitive identification between the attractor flow and the line of quickest descent in the potential of the matrix model.

5.5.3 An example with three 2-branes

The blowup of the vertex geometry along all three torus invariant legs has a total of four smooth resolutions, related by conifold flops. This is the right description when the legs are expected to become rigid spheres in the global geometry, so they cannot be holomorphically deformed away from the triple intersection. Let us see explicitly how one of these blowups arises from the quiver description of $C_{\square\square\square}$.

Suppose that the 2-brane wrapping the z_1 plane has melted first into the background D6, that is, $I^1 \neq 0$. Without any 0-brane probes, the dynamical fields are the J^i_j, which satisfy F-term conditions derived before requiring that $J^i_j J^j_i = 0$, $J^i_j J^k_i = 0$, and $J^i_j I^i = 0$ for all distinct i, j, k. It is easy to satisfy the D-term constrain, for appropriate values of the FI parameters, by taking $J^2_1, J^3_1 \neq 0$ and all others vanishing. These can be gauged away using the $U(1)$'s associated to the D2 worldvolumes, as expected, therefore the BPS configuration is completely determined by the frozen fields, and the moduli space is a point.

Probing this system with a D0 brane, we find that $\partial W = 0$ implies

$$z_1 a^2_3 = a^1_3 j^2_1$$

$$z_2 a^1_3 = z_3 a^1_2$$

$$z_1 a^3_2 = a^1_2 j^3_1.$$

The Kahler structure is determined by the D-terms,

$$|J^2_1|^2 + |J^3_1|^2 - |a^1_3|^2 - |a^1_2|^2 = r_1$$

$$|J^2_1|^2 + |a^2_3|^2 = r_2$$

$$|J^3_1|^2 + |a^3_2|^2 = r_3,$$

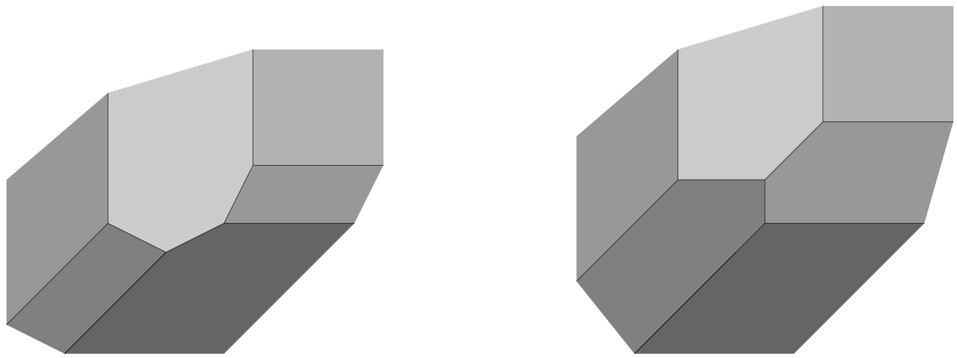

Figure 5.5: Toric diagrams of some blowups associated to $C_{\square\square\square}$, related by flops.

where we have absorbed the contributions of the frozen degrees of freedom associated to noncompactness into the FI parameters. Completing the Kahler quotient by $U(1)^3$, one obtains the toric web diagram shown in Figure 5.5.

As we have seen before, the fact that the Euler character is 4 instead of the expected 3 is not a contradiction, since if $r_1 > 0$, than one of the $J_1^a \neq 0$, and the central $\mathbb{P}^1$ gets shrunken to a point. It is reasonable to expect that this is not a wall of marginal stability, but rather a change in the attractor tree pattern, since the $J_1^a = 0$ branch of the moduli space for $r_1 < 0$ is connected to the configurations in which the 2-branes are separated. If that was the case, then a full analysis of the fermionic terms in the matrix model would show that the anti-ghost bundle is not the tangent bundle in this case, and only contributes 1 from the resolved $\mathbb{P}^1$. This seems more natural than a true jump in the index, because the $r_1 > 0$ classical moduli space is singular, and one would expect an Euler character of 2 from the conifold singularity if we were using the tangent bundle.

The other resolutions of the blow up are also related by changing the pattern of the attractor flow, condensing first one of the other 2-6 tachyons. These are related geometrically via flops through the symmetric resolution shown in Figure 5.5.

5.6 Conclusions and further directions

We constructed matrix models possessing a topological supersymmetry from the quiver descriptions of holomorphic branes in a Calabi-Yau by adding the associated fermions, introducing the multiplets resulting from motion in the Minkowski directions, and imposing the constraints using the anti-ghost field method of [70]. These matrix models are the topologically twisted version of the D-brane theory in the extreme IR limit; that is, they are theories of the BPS sector of the (unknown) dual superconformal quantum mechanics.

The partition function was shown to localize to the Euler character of an obstruction bundle over the classical moduli, by proving the appropriate vanishing theorems. We evaluated the partition function by regularizing the path integral by gauging part of the $U(1)^3$ torus symmetry, and using the localization of this equivariant version to the fixed points of the torus action, giving exact agreement with known results.

The toric vertex geometry we studied can be obtained as the local limit of various compact Calabi-Yau manifolds. The T^3 symmetry guarantees that the information of the global geometry only affects the local degrees of freedom that remain dynamical in the limit through the values of the FI parameters and the background VEVs of certain frozen fields. Quite generally, some gauge groups of global quiver will be broken by the frozen VEVs, and the associated residual off-diagonal components of the D-term can appear in the effective matrix model.

We saw that the geometry of the BPS moduli space of a single 0-brane in a D6/D2 background in the vertex is exactly the blowup along the curves wrapped by the 2-branes. The structure of the moduli space of N probe 0-branes was also investigated, revealing interesting behavior. This can be viewed as a first step in embedding the quantum foam picture of the A-model into the full IIA theory. In particular we found that effective internal geometry seen by 0-brane probes is indeed the blow ups of the Calabi-Yau predicted in [37].

It might be interesting to relate these ideas to the open/closed duality discussed in [29]. Although the discussion there is in the context of finding the closed string dual of Lagrangian branes in the A-model, they find a sum over Calabi-Yau geometries that depend on which term one one looks at in the Young diagram expansion of the open string holonomy. From the quantum foam point of view, these are related to the 2-brane bound states, so there could be some connection to the effective geometries we have investigated.

In this paper we have focused on the quiver for Donaldson-Thomas theory in the vertex geometry, but the extension of these ideas to many toric geometries should be straightforward. Likewise, even the bound states of branes on a compact Calabi-Yau such as the quintic, which have a quiver description, may be computed using similar topological matrix models. In those cases, the technical difficulty of finding all of the fixed points of a toric action on the classical BPS moduli space increases proportionally. Therefore the hope would be to find a way around performing the direct evaluation used here.

It would be very interesting to try to apply the techniques of matrix models at large N to the quiver theories constructed here. It is natural to expect that an intriguing structure should emerge in the expansion about that large charge limit. This would make sharp the idea that these matrix models give the "CFT" dual of the square of the topological string, at least in the OSV regime, since one would identify the perturbative topological string expansion with the $1/N$ corrections in the quiver matrix model.

Closer to the topic of the 6-brane theories studied in this paper, one would also hope to find a large charge expansion describing the analog of the limit shapes in [60]. The quiver theories would contain further information about the large charge moduli space away from the torus fixed points. Such a method of finding the asymptotic expansion of the partition function could have implications for the entropy enigma found in [21]. On the other hand, because our matrix models are topological, the naive perturbative expansion is trivial, cancelling between bosons and fermions, so novel techniques would be required to obtain the large N limit.

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