Topics in Two-Dimensional Field Theory and Heterotic String Theory

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by

Joshua Michael Lapan

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Thesis advisor | Author

Andrew Strominger | **Joshua Michael Lapan**

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Abstract

We study a myriad of topics related to string theories in two dimensions and/or to heterotic string theories. In chapter 2, we use the duality of two-dimensional string theory with matrix models to study arbitrary time-dependent backgrounds. As an example, we study the case of a Fermi droplet cosmology and analyze properties of the coordinates in which the metric is trivial; we also comment on the form of the interaction terms in these coordinates.

Next, in chapter 3, we study dynamical D0-branes in $\mathcal{N} = 1$, two-dimensional string theory as boundary states in the closed string sector. In particular, we find that there are four stable "falling" D0-branes (two branes and two anti-branes) in the type 0A projection and two unstable ones in the type 0B projection.

In chapter 4, we switch gears to study the heterotic string. We begin studying the massless spectrum of the *non*-Kähler, supersymmetric Fu-Yau compactification by counting zero modes of the linearized equations of motion for the gaugino. This can be rephrased as a cohomology problem which for a trivial gauge bundle reduces to the Dolbeault cohomology of the manifold, which we then compute.

We continue the study of Fu-Yau compactifications (and generalizations) in chapter 5, where we implicitly construct a worldsheet CFT as the IR limit of an $\mathcal{N} = 2$ gauge theory. Spacetime torsion (non-Kählerity) is incorporated via a two-dimensional Green-Schwarz mechanism in which a doublet of axions cancels the gauge anomaly. We also argue that these models are smoothly extendable to solutions of the exact beta-function equations. By string dualities, these solutions provide a microscopic description of certain type IIB RR-flux vacua.

Finally, in chapter 6 we use recent developments to argue that there exists a holographic dual for the CFT living on a stack of N heterotic strings in $\mathbb{R}^{4,1} \times T^5$; this should also be describable by an exact worldsheet CFT. We use supergravity to show that the global supergroup of the background is $Osp(4^*|4)$, with an affine extension given, surprisingly, by a *non*linear, $\mathcal{N} = 8$ superconformal algebra. We also suggest a correspondence between supergroups with 16 supercharges and T^n compactification with $0 \leq n \leq 7$.

Contents

Citations to Previously Published Work

Most of the material in this thesis is derived entirely from articles published with various collaborators. Chapter 2 is from work with Morten Ernebjerg and Joanna Karczmarek which has appeared in

> M. Ernebjerg, J. L. Karczmarek and J. M. Lapan, "Collective Field Description of Matrix Cosmologies," JHEP **0409**, 065 (2004) [arXiv:hep-th/0405187].

Chapter 3 is from work with Wei Li which has appeared in

> J. M. Lapan and W. W. Li, "Falling D0-branes in 2D Superstring Theory," arXiv:hep-th/0501054.

Chapter 4 is from work done with Michelle Cyrier which has appeared in

> M. Cyrier and J. M. Lapan, "Towards the Massless Spectrum of Non-Kähler Heterotic Compactifications," Adv. Theor. Math. Phys. **10**, 853 (2007) [arXiv:hep-th/0605131].

Chapter 5 is from work done with Allan Adams and Morten Ernebjerg which has appeared in

> A. Adams, M. Ernebjerg and J. M. Lapan, "Linear Models for Flux Vacua," Adv. Theor. Math. Phys. **12**, 821 (2008) [arXiv:hep-th/0611084].

Finally, chapter 6 is from work done with Aaron Simons and Andy Strominger which has appeared in

> J. M. Lapan, A. Simons and A. Strominger, "Nearing the Horizon of a Heterotic String," arXiv:0708.0016 [hep-th].

Acknowledgments

My entrance into the graduate program at Harvard was not without initial challenges. In particular, being thrust into a cohort of such immense and impressive knowledge and intellect was quite intimidating. Fortunately, the group at Harvard has been full of friendly, unassuming personalities. First and foremost, I want to thank my advisor, Andy Strominger, who has undoubtedly played a huge role in setting the friendly atmosphere at Harvard. I would also like to thank him for sharing his penetrating insight into both physics and the art of developing coherent problems, and for his tutelage and friendship.

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Dedicated to my Parents
For teaching their Children
That our two most precious Gifts
Are Love and Curiosity.

Chapter 1

Introduction and Summary

The first question that nonexperts are entitled to ask of a string theorist—and believe me, they do—is, why should we be interested in string theory? A slightly more refined version of the question would be, why should we be interested in string theory when general relativity and the standard model of particle physics have explained experiment after experiment to an unprecedented degree of accuracy?

The first point is that they haven't *quite* explained every experiment; we already know from observational data that the standard model is incomplete because it does not explain dark matter or dark energy. For example, galactic rotation curves, which plot the angular velocity of matter in a galaxy as a function of distance from the center, do not exhibit the behavior predicted from general relativity when we assume that the only matter in the galaxies is that which we observe.

This first point, however, is not enough to warrant the invention of a whole new theoretical framework of physics; one can simply posit the existence of new matter that has small interactions with existing matter and add this to the standard model using the same language (quantum field theory). The second point is much deeper: we need a quantum theory of gravity or else quantum mechanics would be violated.

To see why general relativity and quantum mechanics are incompatible, we need only consider an application of the Heisenberg uncertainty principle. This principle tells us that particles do not *simultaneously* have an exact position and momentum. Note that while this *implies* that we cannot measure the position and momentum to arbitrary accuracy, the statement is that there does not *exist* a simultaneous position and momentum.

If classical general relativity were correct, an arbitrarily advanced civilization could conceivably invent devices that could "shoot" gravitational waves, the way a laser "shoots" light. We know that light is quantized, which means that for a given wavelength λ there is a minimal quanta that we can shoot, a photon, which carries energy $\frac{hc}{\lambda}$ and momentum $\frac{h}{\lambda}$ (h is Planck's constant and c is the speed of light in vacuum). If we want to shine light on a particle to discover its location accurately, we must use short wavelength light; but, if we do that then the minimal amount of momentum incident upon the particle will be very large, even for a single photon. Thus, with light we cannot measure the momentum and position of a particle to arbitrary accuracy.

The same would not be true for a classical gravitational wave, which is not quantized, because there would be no lower bound to the energy the wave could carry, regardless of wavelength. Thus, we would be able to measure the momentum and position of a particle to arbitrary accuracy, violating the underpinning of quantum mechanics. It is for this reason that we *know* our beloved theories to be mutually incompatible, and so we *know* that there must be a new theory that at least combines the two, which we lovingly (or "hatingly") call *quantum gravity.*

Great, so we need a quantum theory of gravity. What, you ask, is so hard about this? Well, classical gravity is described by an action principle, meaning that by extremizing the action S, we obtain differential equations (the "equations of motion") that determine the trajectories of objects in a curved spacetime. Quantum field theory involves a more glorified use of the action S where, instead of demanding that particles follow the trajectories determined by equations of motion, we imagine that particles can follow *any* path; but, to each path we assign the phase $e^{iS/\hbar}$ and we sum over all paths. In other words, the action determines a sort of "complex" *probability distribution* over the space of possible paths for a particle (actually, because of the i in the exponent, the sum over paths is more akin to calculating interference patterns for light). When we add up all the paths, we see that the exponent $iS/\hbar$ varies most slowly around paths where the action is extremized so that, if we let $\hbar \to 0$, the paths that dominate the sum are those that extremize the action, namely those that satisfy the equations of motion. Thus, in the limit that $\hbar \to 0$ quantum field theory returns us to classical field theory, which is the language of general relativity.

Why not add the action of general relativity to the action used in the standard model and treat the combination using quantum field theory? In fact, this can be done. The problem arises when we try to ask whether our theory could be "fundamental", which roughly means that the functional description is accurate at all length scales. We could ask a similar sort of question about the use of fluid dynamics to describe the flow of water down a river; the differential equations of fluid dynamics work well to describe this flow, but if we look closely at the river we see a furor of water molecules whose inner workings are not described by the equations of fluid mechanics. This attempt to quantize gravity is not so different; what we find out is that the "effective" description of quantum gravity breaks down at short distance scales because the coupling constant becomes infinite.[1] This leaves us with a vacuum for what the fundamental theory of nature is at all length scales.

String theory is an ambitious attempt to unify known physics into a more fundamental framework. It has met with multiple successes in resolving or shedding light on difficult questions from both general relativity and the standard model, such as giving a microscopic description of black hole entropy or a weakly coupled description of strongly coupled QCD. There are, however, many challenges to overcome before the contribution of string theory to science will peak. Chief among them is uncovering a set of underlying principles.

String theory is in a state similar to that of electromagnetism in the middle of the nineteenth century; we have amassed a great many facts about the theory, but we have not yet been able to fit them into a fundamental set of principles, such as those encapsulated by Maxwell's equations (which themselves follow from an action principle). There are many tools that have been developed to answer questions in various regimes of the, as yet "unknown", theory. For now, we must content ourselves with studying questions in or near these regimes until we amass enough puzzle pieces to see what the picture on the puzzle is![2]

This thesis includes work that touches on some of the methods used to study string

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[1]In fact, this is worse than fluid dynamics: while *physically* wrong at small length scales, the theoretical framework of fluid dynamics does not break down and so it *could* have been correct.

[2]It's really unfortunate that someone lost the puzzle box.

theory in different regimes. As such, we feel it best to leave detailed introductions to the individual chapters. However, chapter 2 utilizes a relationship between two-dimensional string theory and fermi fluids, chapters 2 and 3 are based on Liouville field theory, and chapter 5 invokes gauged linear sigma models, so we have included some background here. The other chapters are more self-explanatory.

1.1 The Worldsheet Action

One of the definitions of string theory involves 1+1-dimensional, modular invariant conformal field theories (CFTs)—these CFTs describe the dynamics of a string in a first-quantized formalism. Their connection with reality is most evident when there is a direct geometric interpretation of the CFT. CFT methods have a somewhat limited range of usefulness in type II theories, where Ramond-Ramond (RR) fluxes are difficult to incorporate; however, in heterotic theories there are no RR fluxes, so it is possible to apply CFT techniques to a wide variety of circumstances. In a background involving nonzero vacuum-expectation-values (VEVs) of the lowest string excitations, namely the metric $G_{MN}(X)$, the B-field $B_{MN}(X)$, the vector potential $A_M(X)$, and the dilaton $\Phi(X)$, we can write the worldsheet action as

$$\begin{aligned} S_\Sigma \;=\; & \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{-g(\sigma)}\Big\{\frac{1}{\alpha'}\Big(G_{MN}(X)g^{\alpha\beta}(\sigma)+B_{MN}(X)\epsilon^{\alpha\beta}(\sigma)\Big)\partial_\alpha X^M\partial_\beta X^N \\ & +R(g)\Phi(X)+\eta_{\hat{M}\hat{N}}\bar{\psi}_+^{\hat{M}}\not{\mathscr{D}}\psi_+^{\hat{N}}+\delta_{IJ}\bar{\gamma}_-^I\not{\mathscr{D}}\gamma_-^J+\tfrac{1}{2}F_{\hat{M}\hat{N}}^{IJ}(X)\bar{\psi}_+^{\hat{M}}\psi_+^{\hat{N}}\bar{\gamma}_{-I}\gamma_{-J}\Big\} \end{aligned} \tag{1.1}$$

where the covariant derivatives $\mathscr{D}$ are defined by

$$\mathscr{D}_\alpha\psi_+^{\hat{M}} \;=\; \Big(\delta^{\hat{M}}{}_{\hat{N}}\nabla_\alpha+\partial_\alpha X^M\big(\omega_M{}^{\hat{M}}{}_{\hat{N}}(X)-\tfrac{1}{2}H_M{}^{\hat{M}}{}_{\hat{N}}(X)\big)\Big)\psi_+^{\hat{N}} \tag{1.2}$$

$$\mathscr{D}_\alpha\gamma_-^I \;=\; \Big(\delta^I{}_{\hat{J}}\nabla_\alpha-i\partial_\alpha X^M A_M{}^I{}_J(X)\Big)\gamma_-^J \tag{1.3}$$

$\alpha, \beta, \ldots = 0, 1$	Worldsheet coordinate indices
$M, N, \ldots = 0, 1, \ldots, 9$	Spacetime coordinate indices
$\hat{M}, \hat{N}, \ldots = 0, 1, \ldots, 9$	Tangent space coordinate indices
$I, J, \ldots = 1, 2, \ldots, 32$	Spacetime vector bundle indices
σ^α, $g_{\alpha\beta}$, ∇_α, ...	Worldsheet coordinates, metric, covariant derivative, *etc.*
$X^M(\sigma)$	Worldsheet scalar fields describing spacetime embedding (*i.e.* they are spacetime coordinates): $\vec{X}$: $\Sigma \to M_{10}$
G_{MN}, $\omega_M{}^{\hat{M}}{}_{\hat{N}}$, ...	Spacetime metric, spin connection, *etc.*
$A_M{}^I{}_J$	Spacetime gauge field with field-strength $F = dA$ corresponding to an $SO(32)$ or $E_8 \times E_8$ gauge group
B_{MN}	"B-field": 2-form field with field-strength $H = dB$
Φ	"Dilaton": spacetime scalar field
$\psi_+^{\hat{M}}(\sigma)$	Worldsheet +-chirality Majorana-Weyl fermion transforming as sections of the pullback to the worldsheet of the spacetime tangent bundle[3]—superpartner of X^M
$\gamma_-^I(\sigma)$	Worldsheet −-chirality Majorana-Weyl fermion transforming as sections of the pullback of a vector bundle associated with the spacetime gauge symmetry

Table 1.1: Notation and definitions relevant to the action S_Σ in (1.1).

and table 1.1 explains the rest.

The action S_Σ is invariant under worldsheet conformal transformations, which are composed of worldsheet diffeomorphisms as well as Weyl transformations

$$g_{\alpha\beta} \to e^{2\omega(\sigma)} g_{\alpha\beta}\,, \tag{1.4}$$

but the quantum theory, where we integrate over worldsheet fields, need not be conformally invariant. The functions of the X^M, namely G_{MN}, B_{MN}, A_M, and Φ, can be expanded around some constant X^M and can thus be thought of as an infinite set of worldsheet coupling constants. As such, these constants may run under renormalization group (RG) flow, in which case the quantum theory would not be conformal. The

[3]This is correct when $H = 0$, but when $H \neq 0$ the connection is actually the pullback of the spacetime spin connection minus the H-field.

condition that the theory be conformally invariant is that the beta functions vanish (see *e.g.* [23]) and these lead to constraints on G, B, A, and Φ, that can be recast as equations of motion that derive from a low-energy, effective spacetime action for the fields.

The action S_Σ is also invariant under a *spacetime* gauge transformation $\Omega(X)$

$$\gamma_-^I \to (1+i\Omega)^I{}_J\gamma_-^J\,, \qquad A_M{}^I{}_J \to \left(A_M - \partial_M\Omega + i[\Omega, A_M]\right)^I{}_J \tag{1.5}$$

as well as a *spacetime* Lorentz transformation $\Lambda(X)$

$$\psi_+^{\hat{M}} \to (1+i\Lambda)^{\hat{M}}{}_{\hat{N}}\psi_+^{\hat{N}}\,, \qquad \omega_M{}^{\hat{M}}{}_{\hat{N}} \to \left(\omega_M - i\partial_M\Lambda + i[\Lambda, \omega_M]\right)^{\hat{M}}{}_{\hat{N}}\,, \qquad etc. \tag{1.6}$$

However, since we have chiral fermions, the quantum action need not be invariant under these classical symmetries. As a result, we have a potential anomaly (see *e.g.* [84]) proportional to

$$\propto \int_\Sigma X^*\Big[\mathrm{Tr}\big(\Omega\, dA\big) - \mathrm{Tr}\big(\Lambda\, d(\omega - H)\big)\Big] \tag{1.7}$$

where $R(\omega - H)$ is the Ricci two-form derived from the "minus" connection $\omega - H$ and X^* refers to the pullback to the worldsheet of the spacetime quantities. In a seminal paper by Green and Schwarz [72], it was realized that this anomaly can be consistently canceled by allowing B to shift

$$B \to B + \frac{\alpha'}{4}\mathrm{Tr}\big(\Lambda\, d(\omega - H)\big) - \frac{\alpha'}{4}\mathrm{Tr}\big(\Omega\, dA\big)\,, \tag{1.8}$$

which means that we have a globally defined, invariant 3-form

$$H' \equiv dB + \frac{\alpha'}{4}\mathrm{Tr}\Big((\omega - H)\wedge R(\omega - H) + \tfrac{1}{3}(\omega - H)^3\Big) - \frac{\alpha'}{4}\mathrm{Tr}\Big(A\wedge F + \tfrac{i}{3}A^3\Big)\,. \tag{1.9}$$

Consistency of the theory can thus be phrased in terms of the modified Bianchi identity

$$dH' = \frac{\alpha'}{4}\Big(\mathrm{Tr}\big(R(\omega - H)^2\big) - \mathrm{Tr}\big(F^2\big)\Big)\,. \tag{1.10}$$

This will be important in non-Kähler compactifications, which we study in chapters 4 and 5.

The difference for type II theories is that the $-$-chirality fermions γ_-^I, $I = 1, \ldots, 32$, are replaced by $\psi_-^{\hat{M}}$, $\hat{M} = 0, \ldots, 9$, which are superpartners of the X^M under a left-moving worldsheet supersymmetry. Additionally, there is no gauge field A_M; instead, the connection to which the $\psi_-^{\hat{M}}$ couple is almost the same as for the $\psi_+^{\hat{M}}$ (1.2) except that the opposite sign appears in front of H. This means that in the four fermion term in (1.1), F is replaced by the Riemann tensor. This seemingly "simple" change leads to a world of difference in the spacetime theory; in particular, in flat space the massless spectrum of the worldsheet theory includes various p-form fields (RR-fields) instead of the gauge field. In the heterotic theory, it is explicit in (1.1) how to include VEVs for the spacetime fields in the worldsheet CFT; unfortunately, in type II theories it is incredibly difficult to incorporate VEVs for the RR-fields. It is for this reason that CFT methods are more useful in heterotic theories than in type II theories.

1.2 String Theory in Two Dimensions

In chapters 2 and 3, we concern ourselves with problems in $1 + 1$ spacetime dimensions. The reason we are interested in $1 + 1$-dimensional physics is because there are powerful tools that can be brought to bear there, as we will see. The hope is that problems with solutions in two dimensions may provide analogies useful to solving problems in higher dimensions.

In chapter 2, we will be interested in the bosonic string and can forget the fermionic terms in (1.1) (in chapter 3, we will describe the appropriate modifications to the case of the superstring, but the ideas will be the same). Consider a background

$$G_{MN}(X) = \eta_{MN}\,, \qquad B_{MN}(X) = 0\,, \qquad \Phi(X) = \frac{2}{\sqrt{\alpha'}} X^1\,, \tag{1.11}$$

where now $M, N, \ldots = 0, 1$. Under a Weyl transformation (1.4), the action S_Σ is variant as is the path-integral measure; however, both can be canceled by letting

$$X^1 \to X^1 - \sqrt{\alpha'}\omega\,. \tag{1.12}$$

(Said another way, if we started with just the free X^0 action, the Weyl transformation would fail to be a quantum symmetry of the theory, so the conformal mode of the metric would remain a dynamical variable and become X^1 with the linear dilaton background—see [65], for example.)

The string coupling is $g_s = e^\Phi$, so we see here that the string coupling diverges as X^1 becomes large, but this means that the path-integral appears to be badly divergent. There is a resolution, though, which is to note that the transformation of X^1 under a Weyl transformation allows us to include a "tachyon"[4] vertex operator in the action

$$\Delta S_\Sigma \propto \mu \int d^2\sigma \sqrt{-g}\; e^{2X^1/\sqrt{\alpha'}} \tag{1.13}$$

so that we have the *Liouville* action

$$S_\Sigma = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{-g}\left\{\frac{1}{\alpha'}\eta_{MN} g^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N + \frac{2}{\sqrt{\alpha'}} R(g) X^1 + 4\mu\, e^{2X^1/\sqrt{\alpha'}}\right\}. \tag{1.14}$$

[4]In two dimensions, the lowest excitation of the string—the would-be tachyon—is massless.

For the appropriate sign of μ, this potential term ensures that the action goes to $+\infty$ when $X^1 \to \pm\infty$, so the Euclidean path integral (with integrand e^{-S_Σ}) can be convergent. Physically, this term provides a potential that shields strings from the strongly coupled region $X^1 \to +\infty$, so a string coming from $X^1 = -\infty$ will be reflected and never encounter infinite string coupling.

1.2.1 The Matrix Model

In chapter 2, we will use the "matrix model" to describe time-dependent backgrounds in $1+1$-dimensional, bosonic string theory. To see the connection between Liouville theory and matrix quantum mechanics, we first discretize the worldsheet by triangularizing it into equilateral triangles with side a, vertices labeled by $i, j, \ldots,$ and total number of vertices given by V (see [92, 65] for a more complete review). We work with the non-critical theory of a single free boson with cosmological constant, which we have seen is equivalent to the Liouville theory since the conformal mode of the metric is dynamical. Then, we make the substitutions

$$\begin{aligned}
\int \mathcal{D}g\big|_h &\longrightarrow \lim_{a\to 0} \sum_{\text{Triangularizations } \Lambda} \\
\int \mathcal{D}X &\longrightarrow \int_{\Lambda \text{ Embeddings in } \mathbb{R}^2} \\
\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{g} g^{\alpha\beta}\partial_\alpha X \partial_\beta X &\longrightarrow \frac{1}{2\alpha'}\sum_{\langle ij\rangle} \left(X_i - X_j\right)^2 \\
\int d^2\sigma\sqrt{g} &\longrightarrow \frac{\sqrt{3}}{4}a^2 V
\end{aligned} \tag{1.15}$$

where h refers to the number of handles of the closed string worldsheet and $\langle ij \rangle$ refers to nearest neighbors. Using the action (1.14), we can recast the partition function as

$$\ln Z = \ln \left\{ \sum_h \int \mathcal{D}[g, X] e^{-S_\Sigma} \right\} \longrightarrow \sum_{h,\Lambda} g_s^{2h-2} \kappa^V \prod_i \int dX_i \prod_{\langle ij \rangle} e^{-\frac{1}{2\alpha'}(X_i - X_j)^2} \quad (1.16)$$

where $\kappa = e^{-\frac{\sqrt{3}}{4\pi}\mu a^2}$.

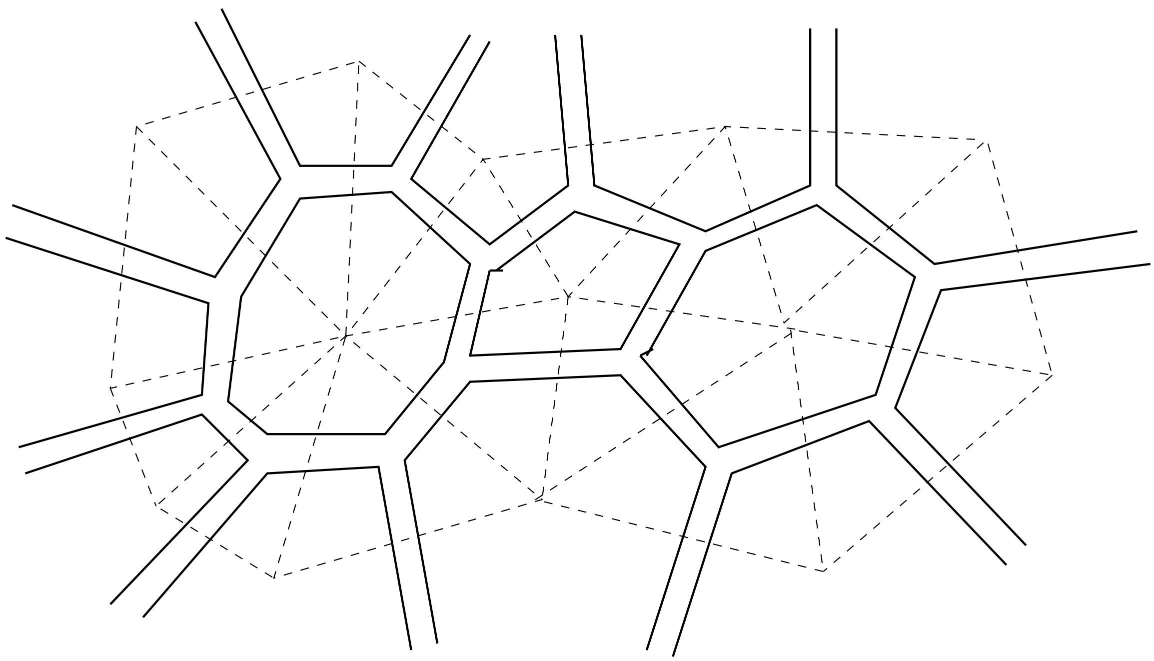

Figure 1.1: Triangularization of a surface (dotted lines) with dual lattice (solid double lines).

The dual lattice $\widetilde{\Lambda}$ contains only vertices that join three edges since the lattice Λ only contains triangular faces (see figure 1.1). As such, it is superficially similar to a Feynman diagram for a theory with only cubic interactions. In fact, we are about to see that the similarity is more than superficial.

Consider a matrix quantum mechanics of $N \times N$ Hermitian matrices $M(t)$ described by

$$Z_{MQM} = \int \mathcal{D}M e^{-N\int_{-T/2}^{T/2} dt \mathrm{Tr}\left[\frac{1}{2}\dot{M}^2 + \frac{1}{2\alpha'}M^2 - \frac{\kappa'}{6}M^3\right]} . \tag{1.17}$$

The matrix indices can be incorporated into Feynman diagrams by using 't Hooft's double-line propagators [121]. We see that propagators contribute a $\frac{1}{N}$ to a diagram, vertices contribute $N\kappa'$, and loops contribute an N from the trace over each internal propagator (see figure 1.1). Thus, a diagram has dependence

$$N^{\tilde{F}+\tilde{V}-\tilde{E}}\kappa'^{\tilde{V}} , \tag{1.18}$$

where $\tilde{F}$ is the number of internal momentum integrals (loops), $\tilde{V}$ is the number of vertices, and $\tilde{E}$ is the number of internal propagators. Diagrammatically, $\tilde{F}$ is the number of "faces", $\tilde{V}$ the number of "vertices", and $\tilde{E}$ the number of "edges", which exactly gives us the Euler characteristic $\chi(\widetilde{\Lambda})$. For an oriented Riemann surface, $\chi = 2 - 2h$, thus the generic behavior is

$$\left(\frac{1}{N}\right)^{2h-2} \kappa'^{\tilde{V}} . \tag{1.19}$$

In fact, we can say a bit more. Because there is a mass term for M, the two-point $\langle M(t)M(t')\rangle$ dies off exponentially as $e^{-|t-t'|/\sqrt{\alpha'}}$, giving us

$$\lim_{T\to\infty} \ln Z_{MQM} \propto \sum_{h,\widetilde{\Lambda}} \left(\frac{1}{N}\right)^{2h-2} \kappa'^{\tilde{V}} \prod_i \int dt_i \prod_{\langle ij\rangle} e^{-|t_i-t_j|/\sqrt{\alpha'}} . \tag{1.20}$$

This is clearly similar to (1.16), for example $g_s \sim 1/N$, but the integrands ($e^{-|t|}$ versus e^{-X^2}) are different. This need not be a problem because in the continuum limit we will wind up with a conformal theory (if necessary, after an RG flow); thus, as long as the two integrands are in the same "universality class" they will give us the same theory in the continuum limit. The evidence indicates that the two link factors are indeed in the same universality class [92, 65].

There are plenty of subtleties, but let us just expand on one, namely the relationship between κ and κ'. Large values of κ' mean that graphs with large numbers of vertices dominate the partition function, while small values of κ' mean that small numbers of vertices dominate; thus, there must be some critical value κ_c separating these phases. In order to obtain the continuum limit, we must take the *double-scaling* limit in which we simultaneously send $\kappa' \to \kappa_c$ and take $a \to 0$. In this limit, we find that

$$\mu \sim \frac{\kappa_c^2 - \kappa'^2}{a^2}\,, \tag{1.21}$$

where the limits are taken in such a way as to leave μ finite.

1.2.2 Free Fermions

Now that we have seen a relationship between Liouville theory and matrix quantum mechanics, we next describe a relationship with free fermions (see the reviews [92, 65] for more details). After a slight rescaling of the matrices $M \to \sqrt{\frac{\beta}{N}}M$, the action becomes

$$S_{MQM} = \beta \int_{-T/2}^{T/2} dt \mathrm{Tr}\left[\frac{1}{2}\dot{M}^2 + \frac{1}{2\alpha'}M^2 - \frac{1}{6}M^3\right] \tag{1.22}$$

where we see that $\kappa' = \sqrt{N/\beta}$.

We will be interested in the large T limit, which essentially means that we will be interested in properties of the ground state of the Hamiltonian. Using a unitary transformation U to diagonalize M, two things will happen: first, the action will split into terms independent of U and terms dependent on U; second, the change of variables will generate a Jacobian in the path-integral measure. That Jacobian is called the Vandermonde determinant $\Delta(\lambda)^2 = \prod_{i,j,i<j}(\lambda_i - \lambda_j)^2$—note that it only depends on the eigenvalues of M. The U-dependent terms make a positive definite contribution to the Hamiltonian, and so the ground state is invariant under unitary transformations.

The Hamiltonian acting on the ground state is, then,

$$H = \frac{1}{\Delta(\lambda)}\left[\beta N \sum_k \left(-\frac{1}{2\beta^2 N^2}\frac{\partial}{\partial\lambda_k^2} + \frac{1}{6}\lambda_k^3 - \frac{1}{2\alpha'}\lambda_k^2\right)\right]\Delta(\lambda)\,. \tag{1.23}$$

The $U(N)$ invariance of the ground state also implies that the ground state is *symmetric* under exchange of eigenvalues (since $S_N \subset U(N)$). On the other hand, $\Delta(\lambda)$ is *antisymmetric* under exchange of two eigenvalues; thus, we can redefine our ground state wavefunctions to absorb this factor so that the *new* ground states are *antisymmetric* and the Hamiltonian becomes

$$H' = \beta N \sum_k \left(-\frac{1}{2\beta^2 N^2}\frac{\partial}{\partial\lambda_k^2} + \frac{1}{6}\lambda_k^3 - \frac{1}{2\alpha'}\lambda_k^2\right)\,. \tag{1.24}$$

Since the ground state wavefunctions are antisymmetric, we wind up with a theory described by N noninteracting (free) fermions, each moving in the potential $\frac{1}{6}\lambda^3 - \frac{1}{2\alpha'}\lambda^2$ (see figure 1.2). Thus, we have a "fermi sea" filled with N fermions. The energy spacing is roughly $\frac{1}{\beta\sqrt{\alpha'}}$, so the "height" of the fermi sea above the minimum of the potential ε_F is proportional to $\frac{N}{\beta}$. There is a critical point of the potential at $\lambda = 0$ with energy ε_c; to the left of ε_c, the potential falls to $-\infty$. Call the number of levels between the critical point and the fermi sea μ', which is given by

$$\mu' \sim \beta(\varepsilon_c - \varepsilon_F) \sim \beta(\kappa_c^2 - \kappa'^2) \sim \beta a^2 \mu\,. \tag{1.25}$$

In the double-scaling limit, we keep μ' and μ fixed to recover the continuum limit. We see now that this corresponds to keeping the number of energy levels between the peak and the fermi sea constant.

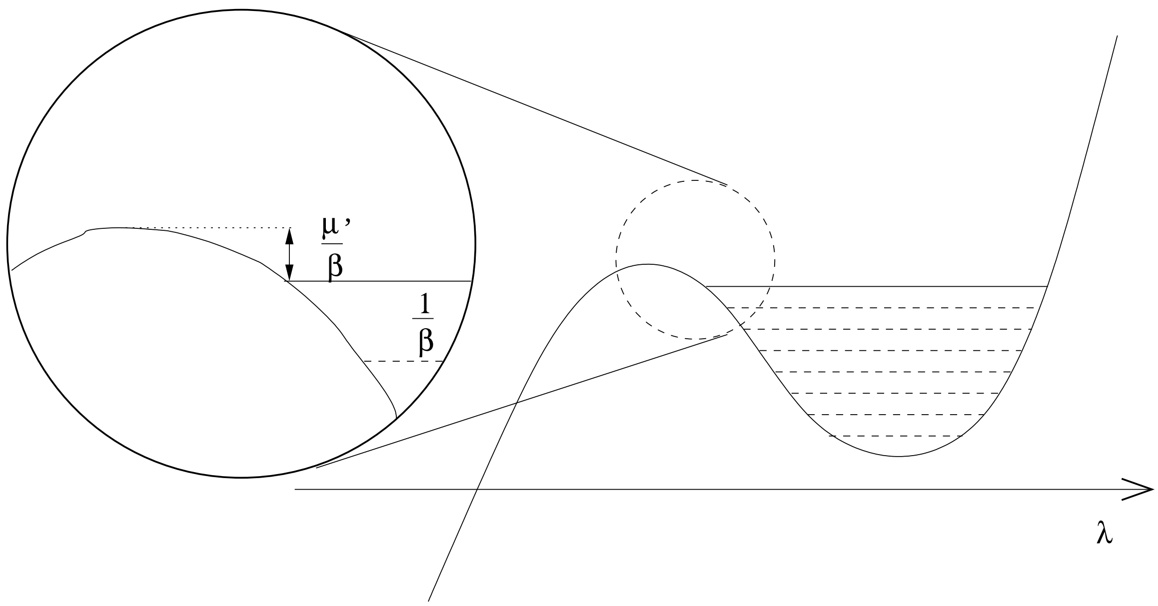

Figure 1.2: Free fermion potential and energy spacings of fermi sea.

The double scaling limit also involves taking $N \to \infty$, which corresponds to weakly coupled string theory since $g_s \sim 1/N$. In the free fermion picture, this corresponds to taking the number of fermions to infinity, giving us a classically incompressible fluid that corresponds, through this long chain of relationships, to string theory backgrounds. Thus, we can study complicated backgrounds of $1+1$-dimensional string theory by studying configurations of classical, incompressible fluids, which is what we do in chapter 2.

1.3 Gauged Linear Sigma Models

In chapter 5, we will use a different set of tools to study string theory from a worldsheet perspective. In particular, we will use the technology of gauged linear sigma models (GLSMs) to construct a nonlinear sigma model (NLSM) on a non-Kähler manifold. Gauged linear sigma models were first introduced in [124] as a way of studying properties of conformal field theories. The basic idea is that studying CFTs can be hard, especially when we don't have explicit expressions for the fields appearing in the action (1.1) as is typically the case with Calabi-Yau compactifications; on the other hand, supersymmetric gauge theories in $1+1$-dimensions are relatively simple. In particular, if the gauge theory only has gauge interactions, then the theory becomes free in the UV since the gauge parameter has dimensions of mass. What this means is any calculation that is protected by supersymmetry from RG flow can be done in the free UV theory and applied to the interacting IR theory (in fact, we can broaden the models by including a superpotential, which modifies the story in a calculable way). Thus, the game is to find a GLSM that flows in the IR to a conformal fixed point that is the CFT we wish to study.

Using the conventions in appendix C.1, we start with the action

$$\begin{aligned} S_0 \;=\; & -\frac{1}{4\pi}\int \mathrm{d}^2y\,\mathrm{d}^2\theta\bigg\{ i\overline{\Phi}{}_0^i e^{Q_i^a V_a}\mathcal{D}_-\Big(e^{Q_i^a V_a}\Phi_0^i\Big) + \frac{1}{2}\overline{\Gamma}{}_0^m e^{2q_m^a V_a}\Gamma_0^m + \frac{1}{8e_a^2}\overline{\Upsilon}_a\Upsilon_a\bigg\} \\ & +\frac{it^a}{4}\int \mathrm{d}^2y\,d\theta^+\;\Upsilon_a|_{\bar\theta^+=0} + \text{h.c.} \end{aligned} \tag{1.26}$$

where $t^a = r^a + i\theta^a$ is the complexified Fayet-Iliopoulos (FI) parameter, $i = 1,\dots,N$, $m = 1,\dots,r$, and $a = 1,\dots,s$. S_0 has a symmetry under $\Phi_0^i \to e^{-iQ_i^a\Lambda_a}\Phi_0^i$, $\Gamma_0^m \to e^{-iq_m^a\Lambda_a}\Gamma_0^m$, and the gauge field transforms as in appendix C.1. Λ_a are a set of chiral superfields, whose lowest components are complex scalars, call them $i\ln(z_a)$. Then the gauge transformation acts on the lowest components of Φ_0^i by

$$\phi^i \to z_a^{Q_i^a}\phi^i \qquad \text{(no sum)}, \tag{1.27}$$

which for $s=1$ is exactly the identification one makes to obtain a weighted projective space from $\mathbb{C}^N\backslash\{0\}$. So the complex scalars ϕ^i become homogeneous coordinates on a weighted projective space when $s=1$, or more generally for some toric variety when $s>1$.

This can be made more precise by analyzing the component expression of the action in Wess-Zumino (WZ) gauge where the full gauge parameter is reduced to the real part of the lowest component of Λ_a, thus leaving us with a $U(1)^s$ gauge symmetry. The gauge multiplet contains a real auxiliary field D_a that can be integrated out to yield a potential for the scalar fields

$$U_a(\phi) \propto \frac{e_a^2}{2}\Big(\sum_i Q_i^a|\phi^i|^2 - r^a\Big)^2. \tag{1.28}$$

The supersymmetric vacua must have a vanishing scalar potential, which for $s=1$, $Q_i^a>0$, and $r^1>0$, implies that the ϕ^i parameterize a $2N-1$-dimensional ellipsoid. When we gauge fix the $U(1)$ symmetry, we take a $U(1)$ quotient of this ellipsoid which yields a weighted projective space $W\mathbb{P}^{N-1}_{\vec{Q}}$, as expected. Another way to look at this is that $\sum_a U_a(\phi)$ provides a mass term for s combinations of the ϕ^i, leaving us with $N-s$ massless scalars which survive to the IR theory. From now on, let us analyze the $s=1$ case in order to simplify the exposition.

The analysis of the scalar components was more involved in WZ gauge than before, but the analysis for the fermions ψ_+^i and γ_-^m is beautiful in this picture: we need only look at the Yukawa couplings

$$\mathcal{L}_{Yuk} \propto iQ_i^a \phi^i \bar{\psi}_+^i \bar{\lambda}_{-a} + h.c. \tag{1.29}$$

Because the scalars ϕ^i get a VEV (1.28), we see that the combination of fermions given by ψ_+, where

$$\psi_+^i = \phi^i \psi_+ \, , \tag{1.30}$$

gets a mass term against the gaugino λ_{-a} so that only $N-1$ right-moving fermions remain massless, the same number as massless scalars. In particular, we can encode this information in an exact sequence

$$\begin{array}{ccccccccc} & & \psi_+ & \longmapsto & \left(\phi^1\psi_+, \ldots, \phi^N\psi_+\right) & & & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \bigoplus_i \mathcal{O}(Q_i) & \longrightarrow & T & \longrightarrow & 0 \\ & & & & \left(\psi_+^1, \ldots, \psi_+^N\right) & \longmapsto & \sum_i Q_i \bar{\phi}^i \psi_+^i & & \end{array} \tag{1.31}$$

where $\mathcal{O}(n)$ are line bundles over $W\mathbb{P}^{N-1}_{\vec{Q}}$ with $c_1(\mathcal{O}(n)) \propto n$ and T is the sheaf of which the massless right-moving fermions are (pullbacks of) sections. In fact, this exact sequence makes clear that

$$T = TW\mathbb{P}^{N-1}_{\vec{Q}}. \tag{1.32}$$

The left-moving fermions γ_-^m have no Yukawa couplings in S_0 (when $E^m(\Phi) = 0$), so they simply transform as (pullbacks of) sections of $\oplus_m \mathcal{O}(q_m)$. Finally, we note that the gauge coupling e_a has dimensions of mass, meaning that it blows up in the IR. This freezes out the dynamics of the gauge multiplet, allowing us to integrate it out. Thus, there is no gauge field in the IR theory and we are left with a theory where the ϕ^i are homogeneous coordinates on $W\mathbb{P}^{N-1}_{\vec{Q}}$, their superpartners are pullbacks of sections of the tangent bundle $TW\mathbb{P}^{N-1}_{\vec{Q}}$, and the left-moving fermions transform as pullbacks of sections of the vector bundle $\oplus_m \mathcal{O}(q_m)$. This is precisely the structure of the *nonlinear sigma model* on $W\mathbb{P}^{N-1}_{\vec{Q}}$. In fact, if we integrate out the gauge field at tree level we will generate an effective spacetime metric that is the analog of the Fubini-Study metric for $W\mathbb{P}^{N-1}_{\vec{Q}}$. Of course, in this case there will be no conformal fixed point because the NLSM on a weighted projective space is not conformal.

We can modify the construction by adding a superpotential as in appendix C.1. Geometrically, the superpotential will involve a quasi-homogeneous polynomial of the Φ^i which will be set to zero for supersymmetric vacua, cutting out a hypersurface of the $W\mathbb{P}^{N-1}_{\vec{Q}}$ (more generally, we can cut out an intersection of hypersurfaces from a toric variety). As before, the right-moving fermions will transform as pullbacks of sections of the tangent bundle to the hypersurface (intersection). Similarly, we can use a superpotential to create more complicated gauge bundles, of which the left-moving fermions will be pullbacks of sections. These modifications are explained briefly in appendix C.1.1.

There is a limitation to these models, however, which is that they do not yield nonzero H-flux. In chapter 5, we will introduce a class of "torsion linear sigma models" that include *nonzero* H-flux, thus yielding backgrounds with nontrivial anomaly cancelation (1.10).

Chapter 2

Collective Field Description of Matrix Cosmologies

2.1 Introduction

The soluble string theory in 1+1 dimensions is a rich toy model for the study of nonperturbative phenomena often not accessible to analysis in higher dimensional theories. Of such phenomena, one class are the processes with a nontrivial time evolution. Examples include tachyon condensation, creation and evaporation of black holes, and cosmological evolution.

An important step towards the study of time-dependent phenomena in the $c = 1$ matrix model for the 2d string was the description of D0-brane decay, or open string tachyon condensation [93, 104]. The matrix model provides an exact picture of the time evolution as the classical motion of a single matrix eigenvalue; its predictions were compared to worldsheet string analysis and found to agree.

Classical collective motions of the entire Fermi sea, as opposed to a motion of a single eigenvalue, were described for example in [106, 9]. These describe nontrivial time dependent backgrounds for the 2d string theory and were interpreted as closed string tachyon condensation in [91]. Another class of time-dependent solutions—droplets of large but finite number of eigenvalues, corresponding to closed universe cosmologies—was proposed in [91]. Since these classical time-dependent solutions of the matrix model correspond to large motions of the Fermi surface, small fluctuations about the Fermi surface carry important information about propagation of stringy spacetime fields. As we will review below, these small fluctuations can be described in the Das-Jevicki collective field approach by a 2d effective field theory, whose action generically contains a nontrivial, time-varying metric.

A step toward understanding these time-dependent solutions was taken by Alexandrov in [7], where coordinates were found in which the metric was made trivial. However, the method presented there does not extend to compact Fermi droplets. The main purpose of this note is to extend the construction of Alexandrov coordinates to arbitrary Fermi surfaces, including compact cases.

To this end, we study the Alexandrov coordinates in some detail. In section 2.2, we briefly review the collective field description of small fluctuations about a time-dependent Fermi surface. In section 2.3, we explicitly construct the Alexandrov coordinates for an arbitrary solution. In section 2.4, we analyze a special class of backgrounds (including some compact cases) for which the entire, interacting action is static. The collective field action for these solutions is shown to take a standard form with a time-independent coupling constant. In section 2.5, we construct the Alexandrov coordinates for a finite collection of fermion eigenvalues: a compact droplet cosmology. We briefly discuss the possibility of formulating a spacetime string theory interpretation of such a configuration. Finally, in appendix A, we analyze the interaction term in the effective action and show by an explicit example that it is not always possible to make it static.

2.2 Notation and Alexandrov Coordinates

In the double scaling limit, matrix quantum mechanics is defined by the action

$$S = \frac{1}{2} \int dt \ \mathrm{Tr} \left(\dot{M}(t)^2 + M(t)^2 \right) \ , \tag{2.1}$$

where M is a Hermitian matrix whose size in this limit is taken to infinity. As is well known (for reviews, see for example [113, 92]), upon quantization the singlet sector of the matrix quantum mechanics is described by an infinite number of free (noninteracting), nonrelativistic fermions representing the eigenvalues of M. The fermions inherit the same potential as the matrix M, and hence the single variable Hamiltonian is

$$H = \frac{1}{2}(p^2 - x^2) \ . \tag{2.2}$$

Since the number of fermions is large, the classical limit of the theory is that of an incompressible Fermi liquid moving in phase space (x, p) under the equations of motion given by the Hamiltonian (2.2). We will restrict our analysis in this note to situations where the Fermi surface (the boundary of the Fermi sea) can be given by its upper and lower branch, which we will denote with $p_\pm(x, t)$, see figure 2.1. It is

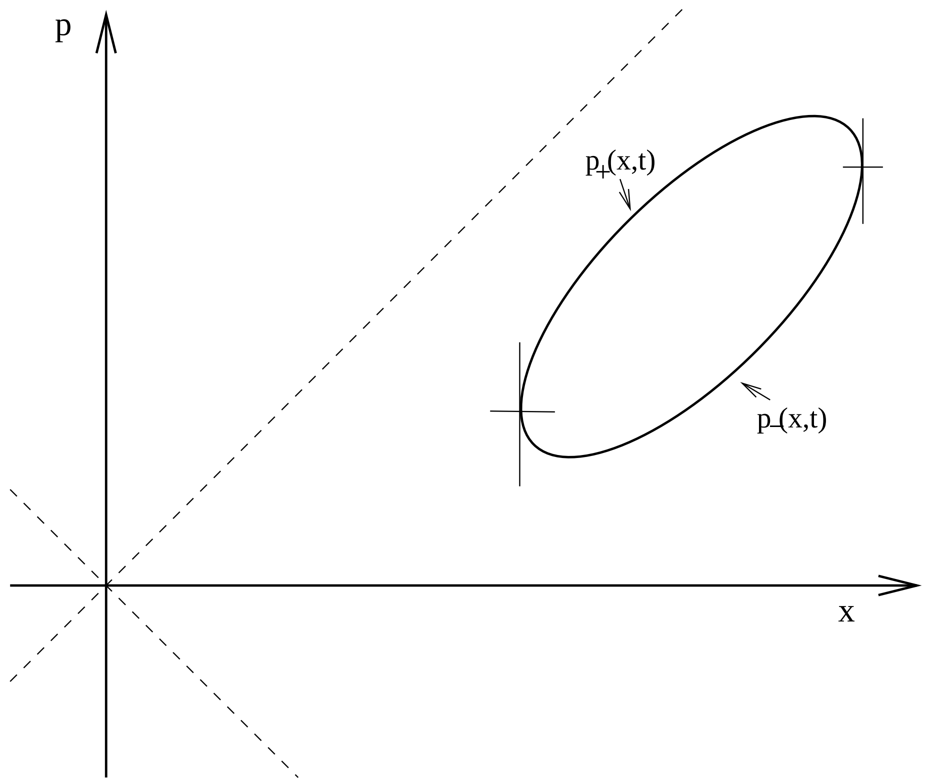

Figure 2.1: A compact Fermi surface in phase space. The upper and lower branches of the surface are labelled, and vertical points where they meet (and the collective theory becomes strongly coupled) are marked.

easy to show that $p_\pm(x,t)$ satisfy

$$\partial_t p_\pm + p_\pm \partial_x p_\pm = x \ . \tag{2.3}$$

One way to directly connect the classical limit of the matrix quantum mechanics with the collective description of fermion motion is via a procedure developed by Das and Jevicki [38]. Define a field $\varphi(x,t)$ by

$$\varphi(x,t) = \frac{1}{\pi}\mathrm{Tr}[\delta(x - M(t))] \tag{2.4}$$

so that $\varphi(x,t)$ is the density of eigenvalues at point x and time t. In the fermion description, we have the relation

$$\varphi = \frac{p_+ - p_-}{2} \ . \tag{2.5}$$

The action for the collective field is [38]

$$S = \int \frac{dt\,dx}{2\pi} \left\{ \frac{Z^2}{\varphi} - \frac{1}{3}\varphi^3 + (x^2 - 2\mu)\varphi \right\} \ , \tag{2.6}$$

where $Z = \int dx \partial_t \varphi$, so the equation of motion is

$$\partial_t \left(\frac{Z}{\varphi} \right) - \frac{Z}{\varphi} \partial_x \left(\frac{Z}{\varphi} \right) = \varphi \partial_x \varphi - x \ . \tag{2.7}$$

Furthermore, we have the relation [8]

$$\frac{Z}{\varphi} = -\frac{p_+ + p_-}{2} \ , \tag{2.8}$$

which allows us to verify that (2.7) is consistent with (2.3).

We want to consider a fixed solution $\varphi_0(x,t)$ of (2.7) and study the effective action for small fluctuations about this solution. In the string theory dual to the matrix model, this corresponds to studying the small fluctuations about a string background given by the solution $\varphi_0(x,t)$. Let $\partial_x \eta(x,t)$ denote the small fluctuations

$$\varphi = \varphi_0 + \sqrt{\pi} \partial_x \eta \tag{2.9}$$

and let $Z_0 = \int dx \partial_t \varphi_0$.

Rewriting the action and grouping terms in powers of η we find (noticing that terms linear in η vanish by the equations of motion)

$$\begin{aligned} S &= \int \frac{dt\,dx}{2\pi} \left\{ \frac{(Z_0 + \sqrt{\pi}\partial_t \eta)^2}{\varphi_0 + \sqrt{\pi}\partial_x \eta} - \frac{1}{3}(\varphi_0 + \sqrt{\pi}\partial_x \eta)^3 + (x^2 - 2\mu)(\varphi_0 + \sqrt{\pi}\partial_x \eta) \right\} \\ &= S_{(0)} + S_{(2)} + S_{int} \end{aligned} \tag{2.10}$$

where $S_{(0)}$ has no η-dependence,

$$S_{(2)} = \frac{1}{2}\int \frac{dt\ dx}{\varphi_0}\left\{\left(\partial_t\eta - \frac{Z_0}{\varphi_0}\partial_x\eta\right)^2 - \varphi_0^2(\partial_x\eta)^2\right\}\ , \tag{2.11}$$

and

$$\begin{aligned} S_{int} \quad &= \quad \frac{1}{2}\int \frac{dt\ dx}{\varphi_0}\left\{-\frac{\sqrt{\pi}}{3}\varphi_0(\partial_x\eta)^3\right. \\ &\qquad \left. + \left(\partial_t\eta - \frac{Z_0}{\varphi_0}\partial_x\eta\right)^2\sum_{n=1}^{\infty}(-\sqrt{\pi})^n\left(\frac{\partial_x\eta}{\varphi_0}\right)^n\right\}\ . \end{aligned} \tag{2.12}$$

In [7], it is proposed that coordinates (τ, σ) exist in which $S_{(2)}$ takes a standard form of a kinetic term for a field in a flat metric

$$S_{(2)} = \int d\tau^+ d\tau^-\ \partial_{\tau^-}\eta\,\partial_{\tau^+}\eta\ , \tag{2.13}$$

where $\tau^{\pm}(x,t) = \tau \pm \sigma$ are the lightcone coordinates. We shall refer to the coordinates (τ, σ) as the *Alexandrov coordinates*. In [7], these coordinates were constructed from a specific form of the solution φ_0. In the next section, we prove (at least locally) their existence for all φ_0.

2.3 Alexandrov Coordinates – Existence

It is quite simple to show, using the equations of motion for the two branches of the solution (2.3), that the action (2.11) takes on the form in (2.13) as long as the coordinates $\tau^{\pm}$ satisfy

$$(\partial_t + p_{\pm}\partial_x)\tau^{\pm} = 0\ . \tag{2.14}$$

Equation (2.14) can easily be solved (at least locally). The exact form of the solution depends on whether the slope of the solution $p_{\pm}$ is steeper or shallower than 1. The regions where $\alpha(x,t) \equiv \partial_x p_\pm$ satisfies $|\alpha| > 1$ will be referred to as the steep regions, and those were $|\alpha| < 1$ will be referred to as the shallow regions. In the steep regions, we have that

$$\tau^\pm = t - \coth^{-1}\left(\partial_x p_\pm\right) \ , \tag{2.15}$$

and in the shallow regions we get

$$\tau^\pm = t - \tanh^{-1}\left(\partial_x p_\pm\right) \ . \tag{2.16}$$

The solution above is not unique—a conformal change of coordinates does not change the form of the quadratic part of the action (2.13), so any change of coordinates of the form

$$\tau'^\pm = \tau'^\pm(\tau^\pm) \tag{2.17}$$

will provide another solution to equation (2.14). For example, the following is also a good solution

$$\tau^\pm = \frac{\tanh t - \partial_x p_\pm}{1 - \partial_x p_\pm \tanh t} \tag{2.18}$$

as is

$$\tau^\pm = \frac{\coth t - \partial_x p_\pm}{1 - \partial_x p_\pm \coth t} \ . \tag{2.19}$$

Note that if p_+ and p_- are flat on some overlapping region ($\partial_x p_\pm = 0$), then these coordinates will be degenerate. However, we can easily parameterize these flat regions in a nondegenerate way so that the metric is still flat.

The solutions (2.15) and (2.16) are valid locally on steep and shallow coordinate patches respectively (modulus the degenerate case mentioned above). To create a single coordinate system, we can 'glue together' the various steep, shallow, and flat patches by using the freedom of conformal coordinate changes. While our expressions guarantee the existence of Alexandrov coordinates on each patch and while there are no obvious obstacles to the 'gluing' procedure, constructing the coordinates in this way would be very cumbersome, even in cases where the resulting coordinate systems are simple. Instead, in all the examples given in this chapter, the Alexandrov coordinates are constructed by the procedure given in [7] (but see the comment at the end of section 2.5).

Another issue is that, as was shown in [7], the resulting coordinates often have boundaries (this will also be seen in section 2.5). The boundaries come in two categories. The first are timelike boundaries corresponding to the end(s) of the Fermi sea; the boundary conditions on those can be determined from the conservation of fermion number [38]. The second class of boundaries contains boundaries which are either spacelike or timelike, do not have a clear interpretation, and for which appropriate boundary conditions are not known. We will return to the issues of boundaries in the discussion in section 2.5.

We will close this section with a simple example as an illustration. Consider a moving hyperbolic Fermi surface given parametrically by [91]

$$\begin{aligned} x &= \sqrt{2\mu}\cosh\sigma + \lambda e^t \\ p &= \sqrt{2\mu}\sinh\sigma + \lambda e^t \ . \end{aligned} \tag{2.20}$$

In this case, we have $\varphi_0 = \sqrt{(x-\lambda e^t)^2 - 2\mu} = \sqrt{2\mu}\sinh\sigma$. The Alexandrov coordinates are given simply by σ in the parametrization above and by $\tau = t$. It is a simple matter to check that the action takes the form

$$\begin{aligned} S &= \int d\tau d\sigma \left\{ \frac{1}{2}((\partial_\tau \eta)^2 - (\partial_\sigma \eta)^2) - \frac{\sqrt{\pi}}{6\varphi_0^2}(3(\partial_\tau \eta)^2(\partial_\sigma \eta) + (\partial_\sigma \eta)^3) \right. \\ & \left. + \frac{(\partial_\tau \eta)^2}{2} \sum_{n=2}^{\infty} \left(-\frac{\sqrt{\pi}(\partial_\sigma \eta)}{\varphi_0^2} \right)^n \right\} . \end{aligned} \tag{2.21}$$

Note that the coupling diverges at the point $\sigma = 0$ which corresponds to the edge of the Fermi sea, and that it does not depend on τ.

2.4 Alexandrov Coordinates – Special Case

In this section, we study a class of solutions (of which an example appeared at the end of the previous section) for which the Alexandrov coordinates can be written as

$$\sigma = \sigma(x,t) , \qquad \tau = \tau(t) . \tag{2.22}$$

We shall see that this leads to a very restricted class of solutions, but a class which includes both infinite and finite (compact) Fermi seas. Thus, it encompasses the two generic types of dynamic solutions.

With the coordinate ansatz above, we have

$$\begin{aligned} dt\,dx &= \frac{d\tau\, d\sigma}{|\partial_x \sigma\, \partial_t \tau|} \\ \partial_x &= \partial_x \sigma\, \partial_\sigma \\ \partial_t &= \partial_t \tau\, \partial_\tau + \partial_t \sigma\, \partial_\sigma . \end{aligned} \tag{2.23}$$

Demanding that the kinetic term take the standard flat form

$$S_{(2)} = \int d\tau\, d\sigma \left[\frac{1}{2} (\partial_\tau \eta)^2 - \frac{1}{2} (\partial_\sigma \eta)^2 \right] \tag{2.24}$$

leads to the requirements that

$$\partial_t \sigma = \frac{Z_0}{\varphi_0} \partial_x \sigma \qquad \text{and} \qquad \left| \frac{\partial_t \tau}{\partial_x \sigma} \right| = \varphi_0 \ . \tag{2.25}$$

These constraints can be solved explicitly, provided that the solution is only vertical at endpoints ($\varphi_0 = 0$). Since τ depends only on t, we find that $\partial_t \tau = (\partial_\tau t)^{-1}$, $\partial_x \sigma = (\partial_\sigma x)^{-1}$, $\partial_x \tau = \partial_\sigma t = 0$,

$$\partial_\tau x = -\frac{\partial_t \sigma}{(\partial_x \sigma)(\partial_t \tau)} \ , \qquad \text{and} \qquad \partial_t \sigma = -\frac{\partial_\tau x}{(\partial_\sigma x)(\partial_\tau t)} \ . \tag{2.26}$$

Using the first equation in (2.25) we find

$$\partial_x Z_0 = -\frac{1}{\partial_\tau t} \left[\varphi_0 \partial_\tau \ln(\partial_\sigma x) + \frac{\partial_\tau x}{\partial_\sigma x} \partial_\sigma \varphi_0 \right] \ , \tag{2.27}$$

which is equal to

$$\partial_x Z_0 = \partial_t \varphi_0 = \frac{1}{\partial_\tau t} \left[\partial_\tau \varphi_0 - \frac{\partial_\tau x}{\partial_\sigma x} \partial_\sigma \varphi_0 \right] \ . \tag{2.28}$$

Comparing these two expressions, we obtain a differential equation for φ_0

$$\partial_\tau \ln(\varphi_0) = -\partial_\tau \ln(\partial_\sigma x) \tag{2.29}$$

whose solution is clearly of the form $\varphi_0 = f(\sigma)^2 (\partial_\sigma x)^{-1}$. This we can rewrite, using the second equation in (2.25), as

$$\varphi_0(\sigma, \tau) = f(\sigma) \sqrt{g(\tau)} \ , \tag{2.30}$$

where $g(\tau) = (\partial_\tau t)^{-1}$ and we assume $f(\sigma) > 0$. We also have $\partial_x \sigma = \sqrt{g}/f$.

Now we can use the equation of motion (2.7) to find the forms of f and g. Using (2.25), notice that

$$\partial_\tau = (\partial_\tau t) \partial_t + (\partial_\tau x) \partial_x = (\partial_\tau t) \left(\partial_t - \frac{Z_0}{\varphi_0} \partial_x \right) \ , \tag{2.31}$$

so the equation of motion (2.7) implies

$$\partial_\sigma \partial_\tau \left(\frac{Z}{\varphi} \right) = \partial_\sigma (\varphi \partial_x \varphi) - \frac{1}{\partial_x \sigma} \ . \tag{2.32}$$

Substituting the explicit form of φ_0 in terms of f and g into this equation, we obtain

$$2g\partial_\tau^2 g - (\partial_\tau g)^2 + 4 - 4g^2 \frac{\partial_\sigma^2 f}{f} = 0 \ . \tag{2.33}$$

Since g only depends on τ, and f only on σ, we see that $\partial_\sigma^2 f = -\alpha f$ where α is a constant.

Consider first the situation when α is positive. Then

$$f(\sigma) = f_1 \sin(\sqrt{\alpha}(\sigma - \sigma_1)) \ , \tag{2.34}$$

where f_1 and σ_1 are real numbers. To ensure that $\varphi_0 \geq 0$, we must restrict $f_1 > 0$ and $\sigma_1 \leq \sigma \leq \sigma_1 + \frac{\pi}{\sqrt{\alpha}}$. Requiring g to be real yields

$$g(\tau) = \frac{1}{\sqrt{\alpha}} \left[\sqrt{c^2+1} \, \cos(2\sqrt{\alpha}(\tau - \tau_1)) + c \right] \ , \tag{2.35}$$

where c and τ_1 are real constants of integration. If α is negative and $|c| \geq 1$, we have

$$\begin{aligned} f(\sigma) &= f_1 \sinh(\sqrt{|\alpha|}\sigma) + f_2 \cosh(\sqrt{|\alpha|}\sigma) \ \text{ and} \\ g(\tau) &= \frac{1}{\sqrt{|\alpha|}} \left[\sqrt{c^2-1} \, \cosh(2\sqrt{|\alpha|}(\tau - \tau_1)) + c \right] \ , \end{aligned} \tag{2.36}$$

while for $|c| < 1$

$$g(\tau) = \frac{1}{\sqrt{|\alpha|}} \left[\sqrt{1-c^2} \, \sinh(2\sqrt{|\alpha|}(\tau - \tau_1)) + c \right] \ . \tag{2.37}$$

Notice that positivity of $f(\sigma)$ restricts the choice of f_1 and f_2 while positivity of $g(\tau)$ in some of these cases restricts the range of τ to a finite or semi-infinite interval.

Let $F(\sigma) = \int d\sigma f(\sigma)$ so that $x(\sigma,\tau) = (F(\sigma) + k(\tau))/\sqrt{g(\tau)}$ for some function $k(\tau)$. We can also show that Z_0/φ_0 is of the form

$$\frac{Z_0}{\varphi_0} = h(\tau) - \frac{\partial_\tau g}{2\sqrt{g}} F \ . \tag{2.38}$$

The functions $h(\tau)$ and $k(\tau)$ can be computed using the equation of motion. Computing $p_\pm$ from (2.5) and (2.8), we get the following relationship

$$\alpha g^2 \left(x - \frac{k(\tau)}{\sqrt{g(\tau)}} \right)^2 + \left(p + x \frac{\partial_\tau g(\tau)}{2} + h(\tau) \right)^2 = f_1^2 g(\tau) \tag{2.39}$$

which we recognize as an ellipse (a hyperbola) if α is positive (negative). Notice that, from equation (2.35), the compact (elliptical) solutions correspond to a finite range of τ.

The interaction terms (2.12) simplify under our assumption to

$$S_{int} = \int d\tau\, d\sigma \left[\frac{1}{6} \Lambda (\partial_\sigma \eta)^3 + \frac{1}{2} (\partial_\tau \eta)^2 \sum_{n=1}^{\infty} \Lambda^n (\partial_\sigma \eta)^n \right] \ , \tag{2.40}$$

where the effective coupling constant is

$$\Lambda = -\frac{\sqrt{\pi}}{\varphi_0} \partial_x \sigma = -\frac{\sqrt{\pi}}{f(\sigma)^2} \ . \tag{2.41}$$

So we find that the coupling constant is time-independent for this class of solutions. We note that the moving-hyperbola solution (2.21) falls into this class.

As long as $|\Lambda\, \partial_\sigma \eta| < 1$, we can sum the series to get

$$S_{int} = \int d\tau\, d\sigma \left[\frac{1}{6} \Lambda \left(\partial_\sigma \eta\right)^3 + \frac{1}{2} \left(\partial_\tau \eta\right)^2 \left(\frac{\Lambda \partial_\sigma \eta}{1 - \Lambda \partial_\sigma \eta} \right) \right] \ . \tag{2.42}$$

The first interaction term diverges as $\varphi_0 \to 0$, which occurs when the width of the Fermi sea goes to zero. This corresponds to strong coupling at the tip of the static hyperbolic Fermi surface. The second interaction term diverges as $|\Lambda\, \partial_\sigma \eta| \to 1$. We have

$$\Lambda\, \partial_\sigma \eta = -\frac{\sqrt{\pi}}{\varphi_0} \partial_x \eta = \frac{\varphi_0 - \varphi}{\varphi_0}\ , \tag{2.43}$$

so the breakdown happens when the excitations become comparable to the width of the Fermi sea (as can also been seen directly from (2.10)). In this case, the Fermi sea may pinch and split into two, so we would not expect to be able to neglect interactions between the upper and lower Fermi surface. Thus, the collective theory becomes strongly coupled exactly in the places one would expect it to from general considerations.

We have demonstrated that, under the restriction (2.22), the action takes a universal, static form (2.40). The natural question to ask is whether such a universal form of the action might exist for all solutions. As a partial answer to this question, in appendix A we analyze explicitly an example which does not fall into the class of solutions studied in this section. We show that, even with the freedom of conformal change of coordinates, it is not always possible to make the interaction term static in Alexandrov coordinates.

2.5 Fermi Droplet Cosmology

In this last section, we construct an explicit example of the class of solutions discussed above—a droplet solution in which only a finite region of phase space is filled (so that the Fermi surface is a closed curve). Such solutions are believed to give rise to time dependent backgrounds in the spacetime picture [91], although no precise correspondence has been found so far.

In the simplest case, the Fermi surface is a circle in phase space with radius R and center $(p, x) = (0, x_0)$ at time $t = 0$. Notice that we must demand $x_0 > \sqrt{2}R$ in order for the surface not to cross the diagonals $p = \pm x$ (otherwise, some of the fermions will spill over the potential barrier as the droplet bounces off it).

It is not difficult to write down the evolution of this Fermi surface

$$e^{-2t}(x + p - x_0 e^t)^2 + e^{2t}(x - p - x_0 e^{-t})^2 = 2R^2 \ . \tag{2.44}$$

Solving for p we find

$$\varphi_0 = \frac{\sqrt{R^2 \cosh 2t - (x - x_0 \cosh t)^2}}{\cosh 2t} \ . \tag{2.45}$$

A sensible σ-coordinate is an angle parameterizing the upper surface, running from 0 to π between the points where $\varphi_0 = 0$. These are given by

$$x = x_0 \cosh t \pm R\sqrt{\cosh 2t} \ , \tag{2.46}$$

so the simplest guess for an Alexandrov coordinate (which we call θ to stress its angular nature) is such that

$$x = x_0 \cosh t - R \cos\theta \sqrt{\cosh 2t} \ . \tag{2.47}$$

Using the second condition in (2.25), we find

$$\partial_t \tau = \frac{1}{\cosh 2t} \ , \tag{2.48}$$

which gives

$$\tau = \tan^{-1}(\tanh t) \ . \tag{2.49}$$

Thus, τ runs over the finite range $-\pi/4 \leq \tau \leq \pi/4$. In these new coordinates, we find

$$x = \frac{1}{\sqrt{\cos 2\tau}}(x_0 \cos\tau - R\cos\theta) \ , \qquad \varphi_0 = R\sqrt{\cos 2\tau}\sin\theta \ . \tag{2.50}$$

It can be checked that these coordinates do fulfill the first condition in (2.25) as well.

We see that

$$\Lambda = -\frac{\sqrt{\pi}}{\varphi_0}\partial_x\theta = -\frac{\sqrt{\pi}}{R^2\sin^2\theta} \ , \tag{2.51}$$

and

$$g(\tau) = \cos 2\tau \ , \qquad f(\theta) = R\sin\theta \ , \tag{2.52}$$

and the action (2.10) simplifies to

$$\begin{aligned} S \ &= \ \int d\tau\ d\theta\ \left\{ \frac{1}{2}[(\partial_\tau\eta)^2 - (\partial_\theta\eta)^2] - \frac{\sqrt{\pi}}{6R^2\sin^2\theta}(\partial_\theta\eta)^3 \right. \\ & \left. +\frac{1}{2}(\partial_\tau\eta)^2\sum_{n=1}^{\infty}\left(-\frac{\sqrt{\pi}}{R^2\sin^2\theta}\partial_\theta\eta\right)^n \right\} \ . \end{aligned} \tag{2.53}$$

As anticipated, the theory is strongly coupled at the endpoints of the droplet where $\varphi_0 \to 0$. Note that the coordinates are smooth across the steep/shallow divide[1].

As an aside, consider a modification to the droplet discussed above. At time $t = 0$, replace the regions $\pi/4 < \theta < 3\pi/4$ and $5\pi/4 < \theta < 7\pi/4$ by straight lines so that the droplet takes the form of a rectangle with semi-circular ends. A straightforward computation leads to the conclusion that one can find global coordinates which yield a flat kinetic term in the action. As one might expect, time is still compact as it was in the elliptical case, indicating that the compactness is not merely an accident occurring only for this particular shape.

While the droplets are amusing objects in matrix theory, it would be more interesting and satisfying if they had a clear spacetime interpretation in string theory. The collective field description, which we have constructed here, suggests that they have

[1]It is possible to explicitly reach the θ-coordinate from the generally applicable forms (2.15), (2.16) by using appropriate conformal transformations on each patch, but the computation is complicated.

an interpretation as some closed string backgrounds. The massless scalar fluctuations should correspond to some string field, the analog of the tachyon in $c = 1$ Liouville string. The strongly coupled regions at each end of the droplet should correspond to 'tachyon walls'—strongly coupled regions of large tachyon VEV. If such a closed string, worldsheet description could be found, it would provide an example of an open-closed string, finite N duality between a time-dependent finite universe and the matrix quantum mechanics of the $D0$ branes making up the droplet.

Unfortunately, it is not clear how to construct such a spacetime interpretation. The natural time τ is compact, corresponding to the fact that in the fermion time, t, fluctuations of a compact Fermi surface become frozen in the past and future. Also, examining the interaction term, we notice that the coupling Λ is bounded from below so that the theory does not approach a free theory in any region (though the coupling can be made arbitrarily small by taking R large). This makes it unlikely that it will be possible to define an S-matrix. In addition, in the standard $c = 1$ story, the matrix-to-spacetime dictionary is complicated by the presence of leg pole factors, additional phases needed to match the matrix model S-matrix to its string worldsheet counterpart. Supposedly such a complication would appear for the droplet cosmologies as well, but there is no obvious candidate for what it might be. Therefore, it seems unlikely that a spacetime analysis of tachyon scattering can be carried out as has been done in the case of the standard, static Fermi sea as well as the moving hyperbola solution (2.20) [90, 37].

Another way to view the complication introduced by the finite extent of the time τ is that in Alexandrov coordinates there appear boundaries (in this case, spacelike boundaries at $\tau = \pm\pi/4$). What the boundary condition on these should be is not clear. The appearance of boundaries is not unique to compact Fermi surfaces; boundaries of this type, both timelike and spacelike, have appeared in the analysis of noncompact Fermi surfaces in [7].

Perhaps it is possible to find a solution to the effective spacetime theory which would mimic the properties of the droplet cosmology outlined above. This intriguing question is left for future research.

Chapter 3

Falling D0-Branes in 2D Superstring Theory

3.1 Introduction

In both bosonic and supersymmetric Liouville Field Theory (LFT), there exist static D0 and D1-branes. In particular, the static D0-branes—the so-called ZZ branes—sit in the strong coupling region $\phi \to +\infty$ [50]. In the bosonic system with Euclidean time, Lukyanov, Vitchev, and Zamolodchikov, showed the existence of a time-dependent boundary state, the paperclip brane, that breaks into two hairpin-shaped branes in the UV region [101]. They derived the wave function of the boundary state from the classical shape of the brane in the spacetime. Under the Wick-rotation from Euclidean time into Minkowski time, the hairpin brane is reinterpreted as the falling D0-brane.

The falling D0-brane in the supersymmetric system was first considered by Kutasov [95]. He studied the classical dynamics of the falling D0-brane in the vicinity of a stack of NS5-branes that produce a linear dilaton background. In his treatment, the radial position (along the Liouville direction) of the D0-brane is a dynamical field living on its worldvolume, and so the corresponding DBI action gives the classical trajectory of the D-brane in this background:

$$e^{-\frac{Q\phi}{2}} = \frac{\tau_p}{E} \cosh \frac{Qt}{2} \ , \tag{3.1}$$

where Q is the background charge of the linear dilaton, and τ_p and E are the tension and energy of the D0-brane, respectively.

In [101], they considered free bosonic string theory with a linear dilaton and boundary conditions on the bosonic fields. As was noted in [109], although [101] considered no Liouville potential, they required their boundary state to carry the $\mathcal{W}$-symmetry, which is defined as the operators commuting with two screening charges. These screening charges are essentially just Liouville potentials. In fact, in [102] it was shown that the linear dilaton theory with the particular boundary conditions considered in [101] is dual to a linear dilaton theory with a boundary Liouville potential. In $\mathcal{N} = 2$ SLFT, which has Euclidean time, a type of time-dependent boundary state solution was derived in [49]; the Wick-rotation was then carried out in [109] to study the wave function of the falling D0-brane in the $\mathcal{N} = 2$ SLFT system and was found to reproduce the trajectory (3.1) in the classical limit.

In the $\mathcal{N} = 2$ case, the $\mathcal{W}$-algebra is then replaced by the $\mathcal{N} = 2$ SCA, leading to the suggestion that this is the supersymmeterized version of the hairpin brane [109]. While the hairpin construction was shown to live in a theory with a boundary Liouville potential [102], their theory contained no bulk Liouville potential. Since the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories typically contain both bulk and boundary Liouville potentials, to claim a supersymmeterized version of the hairpin brane we will make a comparison (at the end of section 3.3.3) in the limit that the bulk cosmological constant is turned off.

We show that in $\mathcal{N} = 1$, 2D superstring theory with a linear dilaton background—which we will use interchangeably with $\hat{c}_m = 1$ $\mathcal{N} = 1$ SLFT—there exists a similar, time-dependent boundary state corresponding to the falling D0-brane. The naive argument for the existence of the falling D0-brane is as follows. As is well known, the mass of the D0-brane is inversely related to the string coupling as

$$m = \frac{1}{g_s} = e^{-\phi} \,, \tag{3.2}$$

so the mass of the D0-brane decreases as it runs along the Liouville direction from the weak coupling region ($\phi \to -\infty$) to the strong coupling region ($\phi \to +\infty$). Thus, if we set a D0-brane free at the weak coupling region, it will roll along the Liouville direction towards the strong coupling region until it is reflected back by the boundary Liouville potential (this point will be expanded upon at the end of section 3.3.3). This is the falling D0-brane solution which can be described by a time-dependent closed string boundary state of the $\mathcal{N} = 1$, 2D superstring.

In the bosonic case, the hairpin brane satisfies symmetries in addition to those of the action (conformal symmetry). The additional symmetry is known as the $\mathcal{W}$-symmetry and is generated by higher spin currents [101]. The hairpin brane is then constructed from the integral equations that are defined by the $\mathcal{W}$-symmetry. In the $\mathcal{N} = 1$, 2D superstring, it should be possible to use the supersymmeterized version of the $\mathcal{W}$-symmetry to go through a similar construction and find a falling D0-brane.

However, we will argue that it can also be obtained by adapting the falling D0-brane solution in $\mathcal{N}=2$ SLFT [109], [49], to the $\mathcal{N}=1$, 2D superstring.

In section 3.2, we briefly review properties of $\mathcal{N}=1$ SLFT, including the construction of boundary states corresponding to ZZ-branes and FZZT-branes using the modular bootstrap approach. In section 3.3, we review properties of $\mathcal{N}=2$ SLFT as well as the construction of the boundary state corresponding to the falling D0-brane. Finally, in section 3.4 we argue that we may slightly modify the $\mathcal{N}=2$ SLFT falling D0-brane boundary state to obtain the solution in $\mathcal{N}=1$, 2D superstring theory, and we discuss the number of falling D0-branes in the Type 0A and 0B projections. It would be interesting to understand these falling D0-branes in the context of matrix models, but this is beyond the scope of this chapter.

3.2 $\mathcal{N}=1$, 2D Superstring Theory and its Boundary States

3.2.1 $\hat{c}_m = 1$ $\mathcal{N}=1$ SLFT

The $\mathcal{N}=1$ super Liouville theory can be obtained by the quantization of a two dimensional supergravity theory [114]. After eliminating the auxiliary field by its equation of motion, adding $\hat{c}_m = 1$ matter, and setting $\alpha' = 2$, the free part of the action is

$$S_0 = \frac{1}{2\pi}\int d^2z \left[\delta_{\mu\nu}\left(\partial X^\mu \bar{\partial} X^\nu + \psi^\mu \bar{\partial}\psi^\nu + \tilde{\psi}^\mu \partial \tilde{\psi}^\nu\right) + \frac{Q}{4} R X^1\right] , \tag{3.3}$$

where $\mu,\ \nu = 1,\ 2$. Since we are considering 2D superstring theory below, we will write $\phi = X^1$ and $Y = X^2$ as is common in the literature. The $\mathcal{N}=1$ SLFT also includes a potential term

$$S_{int}^{\mathcal{N}=1} = 2i\mu b^2 \int d^2z \ : e^{b\phi} :: \left(\psi^1\tilde{\psi}^1 + 2\pi\mu e^{b\phi}\right) : \ , \tag{3.4}$$

where we must have $Q = b + \frac{1}{b}$ for conformal invariance (note that the normal ordering is crucial for this result and comes from the elimination of the auxiliary field). In the case of Neumann boundary conditions (FZZT brane) we can also have a boundary term preserving the superconformal invariance

$$S_B = \int_{\partial\Sigma} \left[\frac{QK}{4\pi}\phi + \mu_B b\gamma\psi^1 e^{b\phi/2}\right] \ , \tag{3.5}$$

where K is the boundary curvature scalar, μ_B is the boundary cosmological constant, and γ is a "boundary fermion" normalized so that $\gamma^2 = 1$ [108].

The stress energy tensor and superconformal current are

$$\begin{aligned} T &= -\frac{1}{2}\partial Y\partial Y - \frac{1}{2}\partial\phi\partial\phi + \frac{Q}{2}\partial^2\phi - \frac{1}{2}\delta_{\mu\nu}\psi^\mu\partial\psi^\nu \\ G &= i(\psi^1\partial\phi + \psi^2\partial Y - Q\partial\psi^1) \ , \end{aligned} \tag{3.6}$$

which produce the $\mathcal{N}=1$ superconformal algebra (SCA)

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r,-s} \\ [L_m, G_r] &= \frac{m-2r}{2}G_{m+r} \ , \end{aligned} \tag{3.7}$$

where $c = \frac{3}{2}\hat{c} = \frac{3}{2}(1 + 2Q^2 + 1)$, and r and s take integer (half-integer) values in the R (NS) sector. For a critical string theory, we must set $Q = 2$, corresponding to $b = 1$.

The primary fields in the NS sector are $N_{p,\omega} =: e^{(\frac{Q}{2}+ip)\phi+i\omega Y}:$ with weights $h^{\rm NS}_{p,\omega} = \frac{1}{2}(\frac{Q^2}{4}+p^2+\omega^2)$, while in the R sector they are $R^{\pm}_{p,\omega} = \sigma^{\pm} N_{p,\omega}$ with weights $h^{\rm R}_{p,\omega} = h^{\rm NS}_{p,\omega} + \frac{1}{16}$. (If we bosonize the complex fermion $\psi^{\pm} = \frac{1}{\sqrt{2}}(\psi^2 \pm i\psi^1) =: e^{\pm iH}:$, then $\sigma^{\pm}$ is given by $\sigma^{\pm} =: e^{\pm iH/2}:$.) The open string character is

$$\chi^{\sigma,\pm}_{p,\omega}(\tau) = {\rm Tr}_{\mathcal{H}^{\sigma}_{p,\omega}}[q^{L_0 - c/24}(\pm 1)^F] \; , \tag{3.8}$$

where $q \equiv e^{2\pi i\tau}$ and we trace over the descendants of $N_{p,\omega}$ or $R_{p,\omega}$ for $\sigma = $ NS, R, respectively. For non-degenerate representations, the open string characters (which result from a trace over a corresponding primary state and its descendants) are [4], [108],

$$\begin{aligned}
\chi^{\rm NS,+}_{p,\omega}(\tau) &= q^{\frac{1}{2}(p^2+\omega^2)}\frac{\theta_{00}(\tau,0)}{\eta(\tau)^3} \\
\chi^{\rm NS,-}_{p,\omega}(\tau) &= q^{\frac{1}{2}(p^2+\omega^2)}\frac{\theta_{01}(\tau,0)}{\eta(\tau)^3} \\
\chi^{\rm R,+}_{p,\omega}(\tau) &= q^{\frac{1}{2}(p^2+\omega^2)}\frac{\theta_{10}(\tau,0)}{\eta(\tau)^3} \\
\chi^{\rm R,-}_{p,\omega}(\tau) &= 0 \; .
\end{aligned} \tag{3.9}$$

3.2.2 Open/Closed Duality: Boundary States

As is well known, we can realize boundary conditions for an open string as constraints on states in the closed string spectrum [42], [41], [60], and [111]. For example, Neumann and Dirichlet boundary conditions in the open string are realized classically as $\partial X(y) \mp \bar{\partial} X(y) = 0$ and $\psi(y) \mp \eta\tilde{\psi}(y) = 0$, where $\eta = \pm 1$, $y \in \mathbf{R}$, and we have taken the boundary to lie along the real axis (the upper sign is for Neumann boundary conditions and the lower for Dirichlet). To tranform from the open channel to the closed channel, we must perform the coordinate transformation $z \to z$ and $\bar{z} \to \bar{z}^{-1}$, resulting in the conditions $\partial X(y) \pm y^{-2}\bar{\partial} X(y^{-1}) = 0$ and $\psi(y) \pm i\eta y^{-1}\tilde{\psi}(y^{-1}) = 0$. When we quantize the theory, these become constraints on closed string boundary states:

$$\begin{aligned} \text{Neumann:} \qquad & (\alpha_m + \tilde{\alpha}_{-m})|B,\eta\rangle = (\psi_r + i\eta\tilde{\psi}_{-r})|B,\eta\rangle = 0 \\ \text{Dirichlet:} \qquad & (\alpha_m - \tilde{\alpha}_{-m})|B,\eta\rangle = (\psi_r - i\eta\tilde{\psi}_{-r})|B,\eta\rangle = 0 \ . \end{aligned} \tag{3.10}$$

In general, for a state in the closed string Hilbert space to be a boundary state, it must satisfy two conditions. First, the state must satisfy contraints coming from the requirement that the corresponding boundary vertex operator preserve the symmetries of the original theory. In the case of a simple bosonic theory, this amounts to requiring conformal invariance which, in the language of boundary states, translates to the constraints

$$(L_m - \tilde{L}_{-m})|B\rangle = 0 \ . \tag{3.11}$$

Second, the state must satisfy constraints coming from the open/closed duality of a cylinder diagram. If an open string satisfies some boundary conditions α and β on its left and right ends, respectively, then the corresponding closed string boundary states must satisfy:

$$\langle B,\alpha|q_c^{\frac{1}{2}H_c}|B,\beta\rangle = \mathrm{Tr}_{\mathcal{H}_{\alpha\beta}}[q_o^{H_o}] \ , \tag{3.12}$$

where H_c and H_o are the closed and open string Hamiltonians, $q_c = e^{2\pi i\tau_c}$ and $q_o = e^{2\pi i\tau_o}$, and the trace on the right is taken over the open string spectrum that satisfies the specified boundary conditions, $\mathcal{H}_{\alpha\beta}$. The open and closed string moduli are related through worldsheet duality by a modular transformation, $\tau_c = -\frac{1}{\tau_o}$.

3.2.3 Ishibashi and Cardy States: The Modular Bootstrap

In $\hat{c}_m = 1$ $\mathcal{N} = 1$ SLFT, a closed string boundary state must satisfy [4], [108],

$$\begin{aligned}(L_m - \tilde{L}_{-m})|B,\alpha;\eta,\sigma\rangle &= 0 \\ (G_r - i\eta\tilde{G}_{-r})|B,\alpha;\eta,\sigma\rangle &= 0 \ , \end{aligned} \tag{3.13}$$

where α labels an open string conformal family, $\sigma =$ NS, R (really NS-NS and R-R, but we will commonly use this short-hand), and $\eta = \pm$ gives the spin structure of the boundary states. Additionally, the boundary states must satisfy the open/closed duality requirement (3.12). These states are commonly referred to as the Cardy states.

To find the Cardy states, it is convenient to use an orthonormal basis of states satisfying (3.13). The so-called Ishibashi states $|i;\eta,\sigma\rangle\rangle$ of the theory form such a basis [86] and are defined to satisfy the additional constraints

$$\langle\langle i;\eta,\sigma|q_c^{\frac{1}{2}H_c}|j;\eta',\sigma'\rangle\rangle = \delta_{ij}\delta_{\sigma\sigma'}\chi_i^{\sigma,\eta\eta'}(\tau_c) \equiv \delta_{ij}\delta_{\sigma\sigma'}\mathrm{Tr}_{\mathcal{H}_i^\sigma}[q_c^{H_o}(\eta\eta')^F] \ , \tag{3.14}$$

where $\mathcal{H}_i^\sigma$ is spanned by the conformal family corresponding to the 'i' representation of the constraint algebra, and the χ_i are the characters. As a point of clarification, note that in an Ishibashi state i, η, and σ, denote the representation of a *closed* string conformal family, the boundary condition on a *closed* string state, and the *closed* string sector (NS-NS or R-R), respectively. In a Cardy state, η and σ have the same meaning while α labels an *open* string conformal family. This statement will become more clear in section 3.2.4.

These constraints imply that the Ishibashi states are constructed as [108], [59],

$$\begin{aligned} |i;\eta,\mathrm{NS}\rangle\rangle &= |i;\mathrm{NS}\rangle_L |i;\mathrm{NS}\rangle_R + \text{descendants} \\ |i;\eta,\mathrm{R}\rangle\rangle &= a|i;\mathrm{R}^+\rangle_L |i;\mathrm{R}^+\rangle_R - i\eta a|i;\mathrm{R}^-\rangle_L |i;\mathrm{R}^-\rangle_R \\ &\quad + b|i;\mathrm{R}^-\rangle_L |i;\mathrm{R}^+\rangle_R - i\eta b|i;\mathrm{R}^+\rangle_L |i;\mathrm{R}^-\rangle_R + \text{descendants} \ , \end{aligned} \tag{3.15}$$

where the coefficients a and b are determined by the constraint equations (note that this implies the coefficients of descendants in both sectors will have some η dependence). In fact, when we take the Type 0A projection we will have $a = 0$ and when we take the Type 0B projection we will have $b = 0$. Now we may use these Ishibashi states to represent the Cardy states schematically as

$$|B,\alpha;\eta,\sigma\rangle = \sum_i \Psi_\alpha(i;\eta,\sigma)|i;\eta,\sigma\rangle\rangle \ , \tag{3.16}$$

where α and i may range over some combination of a continuous and discrete spectrum. Cardy showed [25] that the trace in (3.12) may also be represented as

$$\mathrm{Tr}_{\mathcal{H}^\sigma_{\alpha\beta}}[q_o^{H_o}(\pm 1)^F] = \sum_i n^{i,\sigma}_{\alpha\beta}\chi_i^{\sigma,\pm}(q_o) \ , \tag{3.17}$$

where the $n^{i,\sigma}_{\alpha\beta}$ are non-negative integers representing the multiplicty of $\mathcal{H}^\sigma_i$ in $\mathcal{H}^\sigma_{\alpha\beta}$ (Cardy's condition). Using (3.12), (3.16), (3.17), and the modular transformations of the open string characters, we may determine the 'wave functions' $\Psi_\alpha(i;\eta,\sigma)$ (actually, there is an extra freedom that is fixed by noting that these wave functions are one-point functions on the disk and have specific transformation properties under reflection [59]). This is what is known as the modular bootstrap construction.

3.2.4 ZZ and FZZT Boundary States

As an example, let us demonstrate how the modular bootstrap is applied to determine the boundary states corresponding to the ZZ brane and the FZZT brane. Since the stress tensor and the superconformal current of the $\hat{c}_m = 1$ $\mathcal{N} = 1$ SLFT are simply a sum of the corresponding currents of the $\hat{c}_m = 1$ theory and the $\mathcal{N} = 1$ SLFT, the tensor product of an Ishibashi state from each theory will be an Ishibashi state of the combined theory. The ZZ and FZZT boundary states have been constructed for the $\mathcal{N} = 1$ SLFT [4], [59], and so a trivial modifaction allows us to write them in our theory.

The open string spectrum corresponding to the descendants of the vacuum state is given by an open string stretching between two vacuum Cardy states, while the spectrum corresponding to an excited state is given by an open string stretching between the corresponding excited Cardy state and a vacuum Cardy state:

$$\begin{aligned} \chi_{\text{vac}}^{\tilde{\sigma},\widetilde{\eta\eta'}}(\tau_o) &= \langle \text{vac}; \eta, \sigma | q_c^{\frac{1}{2}H_c} | \text{vac}; \eta', \sigma \rangle \\ \chi_{p,\omega}^{\tilde{\sigma},\widetilde{\eta\eta'}}(\tau_o) &= \langle \text{vac}; \eta, \sigma | q_c^{\frac{1}{2}H_c} | p, \omega; \eta', \sigma \rangle \ , \end{aligned} \tag{3.18}$$

where $\nu(\tilde{\sigma}) = \frac{1}{2}|\eta - \eta'|$; $\nu(\sigma) = 0, 1$, corresponds to NS and R, respectively; and $\widetilde{\eta\eta'} = e^{i\pi\nu(\sigma)}$.

Our goal is to determine the wave function of the Cardy states expressed as a linear combination of the Ishibashi states, which we denote by

$$|B, p, \omega; \eta, \sigma\rangle = \int_{-\infty}^{\infty} dp'\, d\omega'\; \Psi_{p,\omega}(p', \omega'; \eta, \sigma) |p', \omega'; \eta, \sigma\rangle\rangle \ . \tag{3.19}$$

To apply this to the vacuum, note that the vacuum state has zero weight and so, in the NS sector, has momentum $p = -\frac{i}{2}(\frac{1}{b} + b)$, $\omega = 0$. Recall that this corresponds to a (1,1) degenerate representation since every conformal family built from a primary of momentum $p = -\frac{i}{2}(\frac{m}{b} + nb)$ is degenerate at level mn. This means that the open string character corresponding to the vacuum representation is

$$\chi_{\rm vac}^{\rm NS,+}(\tau) = [q^{-(\frac{1}{b}+b)^2/8} - q^{-(\frac{1}{b}-b)^2/8}]\frac{\theta_{00}(\tau,0)}{\eta(\tau)^3} , \tag{3.20}$$

and the others are obtained similarly. Then by inserting the expansion of the Cardy states into (3.18) and using the normalization of the Ishibashi states (3.14), the above open/closed duality equations are rewritten as

$$\begin{aligned} \chi_{\rm vac}^{\tilde\sigma,\widetilde{\eta\eta'}}(\tau_o) &= \int_{-\infty}^{\infty} dp'\ d\omega'\ \Psi_{\rm vac}^{(\sigma)*}(p',\omega';\eta)\Psi_{\rm vac}^{(\sigma)}(p'\omega';\eta')\chi_{p',\omega'}^{\sigma,\eta\eta'}(\tau_c) \\ \chi_{p,\omega}^{\tilde\sigma,\widetilde{\eta\eta'}}(\tau_o) &= \int_{-\infty}^{\infty} dp'\ d\omega'\ \Psi_{\rm vac}^{(\sigma)*}(p',\omega';\eta)\Psi_{p,\omega}^{(\sigma)}(p'\omega';\eta')\chi_{p',\omega'}^{\sigma,\eta\eta'}(\tau_c) . \end{aligned} \tag{3.21}$$

The wave functions of the vacuum boundary states are simply one-point functions on the disk which must transform in specific ways under reflection [59]. The transformation properties under reflection, combined with the modular transformations of the open string characters (given by the S-transformation matrix) determine the wave functions of the vacuum boundary states to be

$$\begin{aligned} \Psi_{\rm vac}^{\rm (NS)}(p,\omega;\eta) &= \frac{\pi(\mu\pi\gamma(\frac{bQ}{2}))^{-ip/b}}{ip\Gamma(-ipb)\Gamma(-i\frac{p}{b})}\Psi_{\omega'=0}^{({\rm NS}),\hat c_m=1}(\omega;\eta) \\ \Psi_{\rm vac}^{\rm (R)}(p,\omega;+) &= \frac{\pi(\mu\pi\gamma(\frac{bQ}{2}))^{-ip/b}}{\Gamma(\frac{1}{2}-ipb)\Gamma(\frac{1}{2}-i\frac{p}{b})}\Psi_{\omega'=0}^{({\rm R}),\hat c_m=1}(\omega;+) , \end{aligned} \tag{3.22}$$

where $\Psi_{\omega'}^{(\sigma),\hat c_m=1}$ denotes the wave function for the $\hat c_m = 1$ matter boundary state whose properties do not concern us here. (Note that there is no R-sector $(1,1)$ boundary state with $\eta = -$ for the same reason that the open string character in this sector is zero [59], [47].) These vacuum boundary states, the ZZ branes, correspond to static Euclidean D0-branes that sit in the strong coupling region, $\phi \to +\infty$.

The behavior of the fourier transform of (3.22) is most transparent by utilizing the product representation of the gamma functions. If this product is cutoff after N terms, one will be taking the fourier transform of a function of the form $P_N(p)e^{ia_N(p)}$, where $P_N(p)$ is a polynomial in p and a_N tends to infinity as N tends to infinity. Such a fourier transform will give a sum of delta functions and derivatives of delta functions located at a_N. Thus, the fourier transform of the wave functions in (21) will be localized at infinity as claimed.

All the excited states are non-degenerate representations with continuous momentum p'. The modular transformation of the open string character of the continuous representation together with the solution of the ZZ boundary state gives the excited boundary state (FZZT brane)

$$\begin{aligned}
\Psi^{(\mathrm{NS})}_{p',\omega'}(p,\omega;\eta) &= -\frac{\cos(2\pi pp')}{2\pi}\left(\mu\pi\gamma\left(\tfrac{bQ}{2}\right)\right)^{-ip/b} ip\Gamma(ipb)\Gamma\left(i\tfrac{p}{b}\right)\Psi^{(\mathrm{NS}),\hat{c}_m=1}_{\omega'}(\omega;\eta)\\
\Psi^{(\mathrm{R})}_{p',\omega'}(p,\omega;+) &= \frac{\cos(2\pi pp')}{2\pi}\left(\mu\pi\gamma\left(\tfrac{bQ}{2}\right)\right)^{-ip/b}\Gamma\left(\tfrac{1}{2}+ipb\right)\\
&\quad\times\Gamma\left(\frac{1}{2}+i\frac{p}{b}\right)\Psi^{(\mathrm{R}),\hat{c}_m=1}_{\omega'}(\omega;+)\ . \qquad (3.23)
\end{aligned}$$

Their pole structures show that they are Euclidean D1-branes, extended in the Liouville direction.

3.2.5 An Argument for Additional Symmetry

As we saw above, the characters for $\hat{c}_m = 1$ $\mathcal{N} = 1$ SLFT are simply a product of the individual characters for $\hat{c}_m = 1$ matter and $\mathcal{N} = 1$ SLFT. This led us to a trivial modification of the ZZ and FZZT boundary states of $\mathcal{N} = 1$ SLFT that resulted in static branes, as we saw in section 3.2.4. In fact, we were destined to realize this result because we restricted ourselves to the subset of $\hat{c}_m = 1$ $\mathcal{N} = 1$ Ishibashi states that were simply a tensor product of the Ishibashi states of the separate theories. We thus implicitly required our boundary states to satisfy an additional symmetry (namely, that they separately satisfy $\mathcal{N} = 1$ boundary conditions for each direction). It was because of this additional symmetry we imposed, combined with the fact the characters decouple, that the wave functions did not mix the two directions.

If we want to find a falling D0-brane, we clearly cannot impose the restrictions mentioned above. Naturally, the time direction should satisfy 'Neumann-like' boundary conditions while the Liouville direction should satisfy 'Dirichlet-like' conditions. However, what conditions to impose are not obvious. In the bosonic case of the hairpin brane [101], the authors had to impose the $\mathcal{W}$-symmetry to get the time dependence necessary to obtain a falling D0-brane with the desired trajectory.

It turns out that in the $\mathcal{N} = 2$ SLFT, which has the same matter content as $\hat{c}_m = 1$ $\mathcal{N} = 1$ SLFT, there are additional symmetries that naturally follow from the action. Applying these additional constraints to $\mathcal{N} = 1$ Ishibashi states, one finds boundary states with trajectories that match that of the falling D0-brane (3.1). We will show that in $\mathcal{N} = 1$, 2D superstring theory with linear dilaton background (which is equivalent to $\hat{c}_m = 1$ $\mathcal{N} = 1$ SLFT), there exists a type of falling D0-brane that has the additional $\mathcal{N} = 2$ SCA symmetry and can be obtained by a slight modification of the falling D0-brane solution of the $\mathcal{N} = 2$ SLFT.

3.3 $\mathcal{N}=2$ SLFT and its Boundary States

3.3.1 $\mathcal{N}=2$ SLFT

The $\mathcal{N}=2$ SLFT theory has the same free action as $\hat{c}_m=1$ $\mathcal{N}=1$ SLFT given by (3.3), but has different interaction terms. In fact, there are two types of interaction terms that are consistent with the $\mathcal{N}=2$ superconformal symmetry. The chiral interaction terms are

$$\begin{aligned} S_c^{\mathcal{N}=2} &= 2\mu b^2 \int d^2z \Big(\frac{\pi}{2}\mu : e^{b(\phi+iY)} :: e^{b(\phi-iY)} : +(\psi^1\tilde{\psi}^1-\psi^2\tilde{\psi}^2)e^{b\phi}\cos bY \\ &\quad -(\psi^2\tilde{\psi}^1+\psi^1\tilde{\psi}^2)e^{b\phi}\sin bY\Big)\ , \end{aligned} \tag{3.24}$$

while the non-chiral interaction terms are

$$S_{nc}^{\mathcal{N}=2} = \mu' \int d^2z \left(\partial\phi - i\partial Y + \frac{i}{b}\psi^1\psi^2\right)\left(\bar{\partial}\phi + i\bar{\partial}Y + \frac{i}{b}\tilde{\psi}^1\tilde{\psi}^2\right)e^{\frac{1}{b}\phi}\ . \tag{3.25}$$

In this theory, the background charge does not get renormalized, so we have $Q=\frac{1}{b}$ instead of $Q=\frac{1}{b}+b$ as we had in the $\mathcal{N}=1$ SLFT and the bosonic LFT. Only the non-chiral interaction preserves the $\mathcal{N}=2$ supersymmetry after a Wick rotation of the Euclidean time, Y. Additionally, there is a boundary action (as in the $\mathcal{N}=1$) case with Liouville potentials multiplied by a boundary cosmological constant μ_B (the exact form is not particularly illuminating, the interested reader is referred to [3]).

Since the free part of the action is the same as for $\hat{c}_m=1$ $\mathcal{N}=1$ SLFT, T and G—which we will call G^1 for this section—are the same as before. However, since this is an $\mathcal{N}=2$ theory, we have a current G^2 corresponding to the second supercharge. Both the chiral and non-chiral interaction terms are invariant under a combined shift in Euclidean time and rotation between the two fermions, leaving us with an additional $U(1)$ current, J:

$$\begin{aligned}
T &= -\frac{1}{2}\partial\phi\partial\phi - \frac{1}{2}\partial Y\partial Y - \frac{1}{2}\delta_{\mu\nu}\psi^\mu\partial\psi^\nu + \frac{Q}{2}\partial^2\phi \\
G^1 &= i\left(\psi^1\partial\phi + \psi^2\partial Y - Q\partial\psi^1\right) \\
G^2 &= -i\left(\psi^2\partial\phi - \psi^1\partial Y - Q\partial\psi^2\right) \\
J &= i\left(\psi^1\psi^2 + Q\partial Y\right) \ .
\end{aligned} \tag{3.26}$$

It will be convenient to define $G^\pm \equiv \frac{1}{\sqrt{2}}(G^1 \pm iG^2)$, which allows us to write the $\mathcal{N}=2$ SCA as

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n} \ , \\
[L_m, G_r^\pm] &= \left(\frac{m}{2}-r\right)G_{m+r}^\pm \ , \qquad [J_m, G_r^\pm] = \pm G_{m+r}^\pm \ , \\
\{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{12}(4r^2-1)\delta_{r,-s} \ , \qquad \{G_r^\pm, G_s^\pm\} = 0 \ , \\
[L_m, J_n] &= -nJ_{m+n} \ , \qquad [J_m, J_n] = \frac{c}{3}m\delta_{m,-n} \ ,
\end{aligned} \tag{3.27}$$

with central charge $c = 3\hat{c} = 3(1+Q^2)$, so again $Q=2$ corresponds to a critical string theory, but in this case we must have $b=\frac{1}{2}$. The primary fields are the same as in section 3.2.1, with corresponding $U(1)$ charges

$$\begin{aligned}
j_{p,\omega}^{\mathrm{NS}} &= Q\omega \\
j_{p,\omega}^{\mathrm{R},\pm} &= j_{p,\omega}^{\mathrm{NS}} \pm \frac{1}{2} \ .
\end{aligned} \tag{3.28}$$

The open string character is

$$\chi_\xi^{\sigma,\pm}(\tau,\nu) = \mathrm{Tr}_{\mathcal{H}_\xi^\sigma}[q^{L_0-c/24}y^{J_0}(\pm 1)^F] \ , \tag{3.29}$$

where $q = e^{2\pi i\tau}$, $y = e^{2\pi i\nu}$, and ξ denotes the 'ξ' representation of the constraint algebra. The characters of $\mathcal{N}=2$ SCA representations split into three classes using the fermionic operator $G_{-\frac{1}{2}}^\pm$ [49], [108]:

- **Class 1 (Graviton):** The graviton representation is defined by the open string primaries satisfying $G^{\pm}_{-\frac{1}{2}}|\text{graviton}\rangle = 0$, which implies that $L_{-1}|\text{graviton}\rangle = 0$. This constrains the momentum to $p = -\frac{i}{2b}$, $\omega = 0$, which implies $h = 0$. Thus, the graviton representation corresponds to the unique vacuum state (the identity operator). After eliminating this state, the character becomes

$$\chi^{\text{NS},+}_{\text{vac}}(\tau,\nu) = q^{-\frac{1}{8b^2}} \frac{1-q}{(1+y\sqrt{q})(1+y^{-1}\sqrt{q})} \frac{\theta_{00}(\tau,\nu)}{\eta(\tau)^3} \ . \tag{3.30}$$

(Note that in all three classes, $\chi^{\text{NS},-}_{\xi}$ is obtained by replacing θ_{00} with θ_{01}, $\chi^{\text{R},+}_{\xi}$ by replacing both θ_{00} with θ_{10} and $j = Q\omega$ with $j = Q\omega \pm \frac{1}{2}$ (chiral/anti-chiral), and $\chi^{\text{R},-}_{\xi} = 0$.)

- **Class 2 (Massive):** The massive representation is defined by $G^{\pm}_{-\frac{1}{2}}|\text{massive}\rangle \neq 0$. For generic p and ω, this representation is non-degenerate. The NS character is obtained by summing over all the descendants while using $j = Q\omega$:

$$\chi^{\text{NS},+}_{[p,\omega]}(\tau,\nu) = q^{\frac{1}{2}(p^2+\omega^2)} y^{Q\omega} \frac{\theta_{00}(\tau,\nu)}{\eta(\tau)^3} \ . \tag{3.31}$$

- **Class 3 (Massless):** The massless representation is defined by $G^{+}_{-\frac{1}{2}}|\text{chiral}\rangle = 0$ or $G^{-}_{-\frac{1}{2}}|\text{anti-chiral}\rangle = 0$ for the chiral or anti-chiral representations, respectively. This implies that the momentum must satisfy $\frac{Q}{2} + ip = \pm\omega$, respectively. The character for the chiral representation is obtained by eliminating the contribution from the $G^{+}_{-\frac{1}{2}}$ mode:

$$\chi^{\text{NS},+}_{\omega}(\tau,\nu) = q^{-\frac{1}{8b^2}} \frac{(y\sqrt{q})^{Q\omega}}{1+y\sqrt{q}} \frac{\theta_{00}(\tau,\nu)}{\eta(\tau)^3} \ . \tag{3.32}$$

3.3.2 $\mathcal{N}=2$ Ishibashi States and Cardy States

An $\mathcal{N}=2$ boundary state is constructed as in the $\mathcal{N}=1$ system. Naturally, an $\mathcal{N}=2$ boundary state must satisfy the $\mathcal{N}=1$ conditions

$$(L_m - \tilde{L}_{-m})|B;\eta,\sigma\rangle = (G^1_r - i\eta\tilde{G}^1_{-r})|B;\eta,\sigma\rangle = 0\ . \tag{3.33}$$

Additionally, an $\mathcal{N}=2$ boundary state will satisfy one of two different conditions. An A-Type boundary state will satisfy

$$(J_m - \tilde{J}_{-m})|B;\eta,\sigma\rangle = (G^\pm_r - i\eta\tilde{G}^\mp_{-r})|B;\eta,\sigma\rangle = 0\ , \tag{3.34}$$

while a B-Type state will satisfy

$$(J_m + \tilde{J}_{-m})|B;\eta,\sigma\rangle = (G^\pm_r - i\eta\tilde{G}^\pm_{-r})|B;\eta,\sigma\rangle = 0\ . \tag{3.35}$$

B-Type conditions correspond to 'Neumann-like' boundary conditions on Euclidean time while A-Type conditions correspond to 'Dirichlet-like' boundary conditions. Since we are interested in studying D0-branes, we will focus on the B-Type states for the rest of this chapter.

If we denote the R-sector primary states as $|h,j;R^\pm\rangle_L$, where $j=Q\omega$ as in (3.28) and $\pm$ denotes the spin structure, then $J_0|h,j;R^\pm\rangle_L = (j\pm\frac{1}{2})|h,j;R^\pm\rangle_L$ (and similarly in the right-moving sector). We can check from (3.15) that the B-Type, R-R sector Ishibashi states can be constructed schematically as

$$|h,j;\eta,\mathrm{R}\rangle\rangle \propto |h,j;\mathrm{R}^-\rangle_L|h,-j;\mathrm{R}^+\rangle_R - i\eta|h,j;\mathrm{R}^+\rangle_L|h,-j;\mathrm{R}^-\rangle_R + \text{descendants}\ , \tag{3.36}$$

Since $\psi^+_0|h,j;\mathrm{R}^-\rangle_L = |h,j;\mathrm{R}^+\rangle_L$, it is clear that $(-1)^{F+\tilde{F}}=-1$ on the primary states in (3.36). Furthermore, from the commutation relations $\{J_0, G^\pm_0\} = \pm G^\pm_0$, we can see that the constraint $(J_0 + \tilde{J}_0) = 0$ implies that all descendants in (3.36) must have an equal number of fermionic raising operators on the left-moving and right-moving sides, modulo 2. Therefore, B-Type, R-R sector Ishibashi states will be projected out by the Type 0B GSO projection and so are only present in Type 0A—note that a similar argument implies that A-Type, R-R sector Ishibashi states are only present in Type 0B. Thus, B-Type states will yield stable D0-branes in Type 0A and unstable D0-branes in Type 0B.

B-Type Ishibashi states are constructed to form an orthonormal basis for states satisfying (3.33) and (3.35), and must also satisfy

$$\begin{array}{ll} \textbf{Class 1} & \langle\langle \text{vac}; \eta, \sigma | q_c^{\frac{1}{2}H_c} y_c^{\frac{1}{2}(J_0 - \tilde{J}_0)} | \text{vac}; \eta, \sigma \rangle\rangle = \chi_{\text{vac}}^{\sigma, \eta\eta'}(\tau_c, \nu_c) \\ \textbf{Class 2} & \langle\langle p, \omega; \eta, \sigma | q_c^{\frac{1}{2}H_c} y_c^{\frac{1}{2}(J_0 - \tilde{J}_0)} | p', \omega'; \eta, \sigma \rangle\rangle = \delta(p - p')\delta(\omega - \omega')\chi_{[p,\omega]}^{\sigma, \eta\eta'}(\tau_c, \nu_c) \\ \textbf{Class 3} & \langle\langle \omega; \eta, \sigma | q_c^{\frac{1}{2}H_c} y_c^{\frac{1}{2}(J_0 - \tilde{J}_0)} | \omega'; \eta, \sigma \rangle\rangle = \delta(\omega - \omega')\chi_{\omega}^{\sigma, \eta\eta'}(\tau_c, \nu_c) \ , \end{array} \tag{3.37}$$

while all other correlators between Ishibashi states vanish. The open and closed parameters are related by the modular transformation $\tau_o = -\frac{1}{\tau_c}$ and $\nu_o = \frac{\nu_c}{\tau_c}$. The B-Type Cardy states are then constructed as a linear combination of the B-Type Ishibashi states such that the Cardy states satisfy

$$\begin{aligned} \langle B, O; \eta, \sigma | q_c^{\frac{1}{2}H_c} y_c^{\frac{1}{2}(J_0 - \tilde{J}_0)} | B, \xi; \eta', \sigma \rangle &= e^{i\pi\tilde{c}\frac{\nu_o^2}{\tau_o}} \chi_{\xi}^{\tilde{\sigma}, \widetilde{\eta\eta'}}(\tau_o, \nu_o) \\ \langle B, O; \eta, \sigma | q_c^{\frac{1}{2}H_c} y_c^{\frac{1}{2}(J_0 - \tilde{J}_0)} | B, O; \eta', \sigma \rangle &= e^{i\pi\tilde{c}\frac{\nu_o^2}{\tau_o}} \chi_{\text{vac}}^{\tilde{\sigma}, \widetilde{\eta\eta'}}(\tau_o, \nu_o) \ , \end{aligned} \tag{3.38}$$

where $\chi_{\xi}^{\sigma,\pm}(\tau, \nu)$ is the open string character of the ξ representation of the constraint algebra, O represents the graviton state, and $\tilde{\sigma}$ and $\widetilde{\eta\eta'}$ are defined as in equation (3.18).

3.3.3 Falling Euclidean D0-Brane in $\mathcal{N}=2$ SLFT

As in $\hat{c}_m = 1$ $\mathcal{N}=1$ SLFT, the modular transformation of the Class 1 (Graviton) representation (identity operator) gives the wave function of the vacuum boundary state. Then the modular transformation of the Class 2 (Massive) non-degenerate representation of the open string produces the wave function of the excited boundary state which corresponds to the FZZT brane (Falling Euclidean D0-brane) solution [49], [109], [5], [6]:

$$\begin{aligned}\Psi_{[p',\omega']}(p,\omega;\eta,\sigma) \;=\; & \sqrt{2}Q\tilde{\mu}^{-ipQ}e^{-2\pi i\omega\omega'}\cos(2\pi pp') \\ & \times\frac{\Gamma(-iQp)\Gamma\left(1-i\frac{2p}{Q}\right)}{\Gamma\left(\frac{1}{2}-i\frac{p}{Q}+\frac{\omega}{Q}-\frac{\nu(\sigma)}{2}\right)\Gamma\left(\frac{1}{2}-i\frac{p}{Q}-\frac{\omega}{Q}+\frac{\nu(\sigma)}{2}\right)}\ ,\end{aligned} \tag{3.39}$$

where $\tilde{\mu}$ is the renormalized bulk cosmological constant (it is, in fact, proportional to μ [108] and we will henceforth drop the distinction between the two as it just corresponds to a finite, constant shift of the dilaton in the position space picture). Note also that this wave function has no dependence on η.

The Fourier transform of the momentum space wave function into the position space wave function is

$$\tilde{\Psi}^{(\mathrm{NS})}_{[p',\omega']}(\phi,Y) \equiv \int_{-\infty}^{\infty}\frac{dp\,d\omega}{(2\pi)^2}\,e^{-ip(\phi+Q\ln\mu)}e^{-i\omega Y}\,\Psi^{(\mathrm{NS})}_{[p',\omega']}(p,\omega)\ . \tag{3.40}$$

We can construct solutions where p' and ω' are nonzero from the solution in which they are both zero [109]

$$\tilde{\Psi}^{(\mathrm{NS})}_{[0,0]}(\phi,Y) = \frac{\sqrt{2}}{\mu\pi Q(2\cos\frac{QY}{2})^{\frac{2}{Q^2}+1}}\cdot\exp\left[-\frac{\phi}{Q}-\frac{e^{-\frac{\phi}{Q}}}{\mu(2\cos\frac{QY}{2})^{\frac{2}{Q^2}}}\right]\ . \tag{3.41}$$

Then it is simple to see that

$$\tilde{\Psi}^{(\mathrm{NS})}_{[p',\omega']}(\phi,Y) = \frac{1}{2}\tilde{\Psi}^{(\mathrm{NS})}_{[0,0]}(\phi-2\pi p',Y+2\pi\omega') + \frac{1}{2}\tilde{\Psi}^{(\mathrm{NS})}_{[0,0]}(\phi+2\pi p',Y+2\pi\omega')\ . \tag{3.42}$$

Actually, p' is not an independent variable but is instead related to the bulk and boundary cosmological constants through [5]

$$\left(\mu_B^2 + \frac{\mu^2}{4Q^4\mu_B^2}\right) = \frac{\mu}{32\pi Q}\cosh\left(2\pi p'/Q\right) \tag{3.43}$$

(at least for $\omega' = 0$). Since [101] and [102] contain a theory with no bulk cosmological constant, we should take the limit $\mu \to 0$ to make a comparison with the hairpin brane. In this limit, $p' \to \pm\infty$, which is irrelevant since (3.42) contains a sum of both signs of p', so let us take $p' \to +\infty$ in which case

$$e^{2\pi p'/Q} \underset{\mu\to 0}{\longrightarrow} \frac{64\pi Q\mu_B^2}{\mu} \ . \tag{3.44}$$

One then finds

$$\lim_{\mu\to 0}\tilde{\Psi}_{[p',0]}(\phi, Y) = \frac{\sqrt{2}}{64\pi^2 Q^2\mu_B^2(2\cos\frac{QY}{2})^{\frac{2}{Q^2}+1}} \cdot \exp\left[-\frac{\phi}{Q} - \frac{e^{-\frac{\phi}{Q}}}{64\pi Q\mu_B^2(2\cos\frac{QY}{2})^{\frac{2}{Q^2}}}\right] \ . \tag{3.45}$$

The classical shape of this falling Euclidean D0-brane is given by the peak of its wave function

$$e^{-\frac{Q\phi}{2}} = 128\pi Q\mu_B^2\cos\frac{QY}{2} \ , \tag{3.46}$$

reproducing the trajectory of the hairpin brane [101] for appropriate shift of the Liouville direction. This supports the suggestion ([95] and [109]) that this is the supersymmetric extension of the hairpin brane. Note that if we had instead considered the limit $\mu_B \to 0$, we would have found that the wave function vanishes. Thus, the boundary Liouville potential is necessary for the existence of these boundary states.

The wave function $\tilde{\Psi}_{[0,0]}(\phi, Y)$ is also peaked along the trajectory

$$e^{-\frac{Q\phi}{2}} = 2\cos\frac{QY}{2} \tag{3.47}$$

and we will simply refer to this wave function for the rest of our discussions. Since we are only interested in bulk one-point functions, limits can always be taken if one wishes to look at the case with vanishing bulk cosmological constant.

3.3.4 Falling D0-brane in $\mathcal{N} = 2$ SLFT

For the wave function $\tilde{\Psi}_{[0,0]}(\phi, Y)$, the Wick-rotation from the Euclidean time Y into the Minkowski time t, together with a shift in the Liouville direction $\phi \to \phi - \frac{2}{Q}\ln\tilde{r} - Q\ln\mu$, produces the classical trajectory of the falling D0-brane in $\mathcal{N} = 2$ SLFT [109]:

$$\tilde{r}e^{-\frac{Q\phi}{2}} = 2\cosh\frac{Qt}{2} \ , \tag{3.48}$$

which matches with (3.1) once we set $\tilde{r} = \frac{2E}{\tau_p}$.

Therefore, the falling D0-brane wave function in position space is [109]

$$\tilde{\Psi}^{(\mathrm{NS})}_{[0,0]}(\phi, t) = \frac{\sqrt{2}}{\mu\pi Q(2\cosh\frac{Qt}{2})^{\frac{2}{Q^2}+1}} \cdot \exp\left[-\frac{\phi - \frac{2}{Q}\ln\tilde{r}}{Q} - \frac{e^{-\frac{\phi - \frac{2}{Q}\ln\tilde{r}}{Q}}}{\mu(2\cosh\frac{Qt}{2})^{\frac{2}{Q^2}}}\right] \ . \tag{3.49}$$

Then the Fourier transform to momentum space yields

$$\Psi^{(\mathrm{NS})}_{[0,0]}(p, q) = \frac{-i\sqrt{2}Q\mu^{-ipQ}e^{i\frac{2p}{Q}\ln\tilde{r}}\sinh(\frac{2\pi p}{Q})}{\cosh(\frac{2\pi p}{Q}) + \cosh(\frac{2\pi q}{Q})} \cdot \frac{\Gamma(-iQp)\Gamma(1 - i\frac{2p}{Q})}{\Gamma(\frac{1}{2} - i\frac{p}{Q} + i\frac{q}{Q})\Gamma(\frac{1}{2} - i\frac{p}{Q} - i\frac{q}{Q})} \ , \tag{3.50}$$

and a half spectral flow gives the R-sector wave function

$$\Psi^{(\mathrm{R})}_{[0,0]}(p, q) = \frac{-i\sqrt{2}Q\mu^{-ipQ}e^{i\frac{2p}{Q}\ln\tilde{r}}\sinh(\frac{2\pi p}{Q})}{\cosh(\frac{2\pi p}{Q}) - \cosh(\frac{2\pi q}{Q})} \cdot \frac{\Gamma(-iQp)\Gamma(1 - i\frac{2p}{Q})}{\Gamma(1 - i\frac{p}{Q} + i\frac{q}{Q})\Gamma(-i\frac{p}{Q} - i\frac{q}{Q})} \ . \tag{3.51}$$

3.4 Falling D0-brane in $\mathcal{N} = 1$, 2D Superstring Theory

3.4.1 Using $\mathcal{N} = 2$ SLFT to Study Boundary States in 2D Superstring Theory

We propose that the $\mathcal{N} = 2$ SLFT boundary states may be used to study falling D0-branes in the $\mathcal{N} = 1$, 2D superstring with linear dilaton background (which is equivalent to $\hat{c}_m = 1$ $\mathcal{N} = 1$ SLFT theory). Notice that the field content of both theories is the same, as are the stress tensor and the first supercharge. Additionally, as is apparent from the constraints on an $\mathcal{N} = 2$ boundary state, any $\mathcal{N} = 2$ boundary state also satisfies the $\mathcal{N} = 1$ constraints (3.13). So an $\mathcal{N} = 2$ SLFT boundary state is also a boundary state of the $\mathcal{N} = 1$, 2D superstring, with the $\mathcal{N} = 2$ boundary interaction term.

However, this alone is not enough; if we want to use the $\mathcal{N} = 2$ Ishibashi states, we must also be able to construct a Cardy state from them that will generate the $\hat{c}_m = 1$ $\mathcal{N} = 1$ open string character. In fact, we can do this. In (3.31), ν is the 'modulus' of the $U(1)$ charge and appears nontrivially in the functional form of the character. But if we set $\nu = 0$ ($y = 1$), the $U(1)$ charge acts trivially on the $\mathcal{N} = 2$ states and we can see that the character of the $\mathcal{N} = 2$ Class 2 (Massive) representation (3.31) is equivalent to the $\hat{c}_m = 1$ $\mathcal{N} = 1$ character (3.8). Such a state was found in [49], [109], [5], [6], and presented in sections 3.3.3 and 3.3.4. The momentum space wave functions are the same as in (3.50) and (3.51):

$$\begin{aligned}
\Psi^{(\mathrm{NS})}_{[0,0]}(p,q) &= \frac{-i\sqrt{2}Q\mu^{-ipQ}e^{i\frac{2p}{Q}\ln\tilde{r}}\sinh(\frac{2\pi p}{Q})}{\cosh(\frac{2\pi p}{Q})+\cosh(\frac{2\pi q}{Q})}\cdot\frac{\Gamma(-iQp)\Gamma(1-i\frac{2p}{Q})}{\Gamma(\frac{1}{2}-i\frac{p}{Q}+i\frac{q}{Q})\Gamma(\frac{1}{2}-i\frac{p}{Q}-i\frac{q}{Q})} \\
\Psi^{(\mathrm{R})}_{[0,0]}(p,q) &= \frac{-i\sqrt{2}Q\mu^{-ipQ}e^{i\frac{2p}{Q}\ln\tilde{r}}\sinh(\frac{2\pi p}{Q})}{\cosh(\frac{2\pi p}{Q})-\cosh(\frac{2\pi q}{Q})}\cdot\frac{\Gamma(-iQp)\Gamma(1-i\frac{2p}{Q})}{\Gamma(1-i\frac{p}{Q}+i\frac{q}{Q})\Gamma(-i\frac{p}{Q}-i\frac{q}{Q})}
\end{aligned} \quad (3.52)$$

This is not a surprising result. Recall that in section 3.2.5, we argued that in $\hat{c}_m = 1$ $\mathcal{N} = 1$ SLFT we could find a falling boundary state with the Liouville and time directions coupled nontrivially by assuming additional symmetries that coupled the two directions. This additional symmetry is a symmetry only of the boundary state and not of the theory as a whole. The coupling of the Liouville and time directions is then achieved by imposing this additional symmetry on the Hilbert space of the original $\hat{c}_m = 1$ $\mathcal{N} = 1$ SLFT boundary states.

Note that it is also possible to derive this falling D0-brane in the $\mathcal{N} = 1$, 2D superstring by directly solving the constraint equations satisfied by the boundary states, similar to the derivation of the bosonic hairpin brane [101]. The equations are constrained by the $\mathcal{W}$-symmetry in the bosonic case, while by the $\mathcal{N} = 2$ SCA in the $\mathcal{N} = 1$, 2D superstring.

3.4.2 Number of D0-branes after GSO projection

In $\mathcal{N} = 1$, 2D superstring theory, there are two distinct types of boundary states in each of the NS-NS and R-R sectors, corresponding to the different boundary conditions for world sheet fermions ($\eta = \pm$). Therefore, the Type 0, non-chiral GSO projection produces four types of stable D0-branes (two branes and two anti-branes) in the Type 0A theory, and two unstable D0-branes in the Type 0B theory. In the

Type 0A theory, the D0$^{\pm}$-branes are sourced by two different R-R gauge fields, $C_1^{(\pm)}$ [122].

The D-brane boundary states in the Type 0A theories are given by the non-chiral GSO projection $\frac{1\pm(-1)^{F+\tilde{F}}}{2}$, with the upper sign for the NS-NS sector and the lower sign for the R-R sector. As explained in section 3.3.2, the D0-branes of our theory correspond to the B-Type boundary states. Since the B-Type, R-R Ishibashi states survive the Type 0A GSO projection, the D0-branes in Type 0A will be stable. We can represent them schematically as [60], [122],

$$\begin{aligned} |D0;+\rangle &= |B0;+,\mathrm{NS}\rangle + |B0;+,\mathrm{R}\rangle \\ |D0;-\rangle &= |B0;-,\mathrm{NS}\rangle + |B0;-,\mathrm{R}\rangle \\ |\overline{D}0;+\rangle &= |B0;+,\mathrm{NS}\rangle - |B0;+,\mathrm{R}\rangle \\ |\overline{D}0;-\rangle &= |B0;-,\mathrm{NS}\rangle - |B0;-,\mathrm{R}\rangle\ . \end{aligned} \tag{3.53}$$

On the other hand, the D-brane boundary states in the Type 0B theories are given by the non-chiral GSO projection $\frac{1+(-1)^{F+\tilde{F}}}{2}$ for both the NS-NS and R-R sectors. In this case, the B-Type, R-R Ishibashi states are projected out by the Type 0B projection. This leaves us with two unstable D0-branes represented schematically as

$$\begin{aligned} |\widehat{D0};+\rangle &= |B0;+,\mathrm{NS}\rangle \\ |\widehat{D0};-\rangle &= |B0;-,\mathrm{NS}\rangle\ . \end{aligned} \tag{3.54}$$

Note that the sign, $\eta = \pm$, really does denote different D0-branes since, by Cardy's condition (3.12) and (3.18), these states yield different spectra. It would be interesting to see how states with different values of η distinguish themselves from each other in the context of matrix models.

3.5 Discussion and Summary

Recall that in the static case, the D0-brane (ZZ brane) and the D1-brane (FZZT brane) boundary states were derived from the degenerate and the non-degenerate representation of the open string character, respectively. So it may seem a little puzzling that the falling D0-brane is constructed from the non-degenerate representation instead of the degenerate one.

In the static case, the difference between the D0-brane and the D1-brane is that the D0-brane is localized at the same point in the Liouville direction for all time, while the D1-brane is extended. The open strings ending on this D0-brane can only take on a fixed (imaginary) value of Liouville momentum, while the open strings ending on the D1-brane can take on any value of Liouville momentum.

In the case of the falling D0-brane, if we partition the two-dimensional spacetime into spacelike hypersurfaces, the falling D0-brane is again localized in the Liouville direction along each hypersurface. However, the Liouville position from one hypersurface to the next is not the same (the falling D0-brane is, not surprisingly, *moving*), and so open strings ending on the falling D0-brane can take on any value of Liouville momentum. This is the reason that we must use the non-degenerate representation of the open string characters to define the falling D0-brane boundary state.

To summarize, we have shown that a falling D0-brane boundary state in $\mathcal{N} = 1$, 2D superstring theory can be obtained by adapting the falling D0-brane boundary state solution in $\mathcal{N} = 2$ SLFT [109]. In particular, there exist four types of stable, falling D0-branes (two branes and two anti-branes) in Type 0A theory and two types of unstable, falling D0-branes in Type 0B theory. As is well known, Type 0, $\mathcal{N} = 1$, 2D superstring theory has a dual description in the language of matrix models. An interesting question would be to understand these falling D0-branes in the context of the dual matrix model.

Chapter 4

Towards the Massless Spectrum of Non-Kähler Heterotic Compactifications

4.1 Introduction

Heterotic string theory has long been known to have great promise for reproducing the standard model; unfortunately, amidst the excitement of branes, string dualities, and flux compactifications of type II and M-theory, it has been partially forgotten. Compactifications of heterotic string theory that preserve $\mathcal{N} = 1$ supersymmetry in four dimensions and with vanishing background flux were first studied in [24]; this was expanded to an analysis including nonzero H-flux by Strominger in [118].

In recent years, studies of compactifications of type II theories with various nonzero flux backgrounds have become common. One of the most compelling reasons to study flux compactifications is that flux-free compactifications on Calabi-Yau manifolds lead to large moduli spaces corresponding to unconstrained scalar fields in the low-energy, four-dimensional description. This is phenomenologically unsatisfying since it leaves us with a continuously infinite number of vacua. On the other hand, it is well known that flux compactifications typically lift this degeneracy. In fact, fluxes could conceivably be used to break supersymmetry or lift the cosmological constant to some positive value *à la* KKLT [88].

How flux compactifications of heterotic string theory achieve such noble goals is not yet well-understood from the spacetime/geometric perspective. The difference between heterotic and type II theories is twofold: for one, we have the freedom to choose a gauge bundle which need not be the tangent bundle; second, there are no R-R fluxes to turn on, just the NS-NS 3-form flux H. The main difficulty in including H-flux is that it is encoded in the geometry of the internal manifold as torsion,[1] meaning we have to deal with non-Kähler, though still complex, compactifications [118], [85].

Giving up the Kähler condition destroys many nice results on Kähler geometry that we are accustomed to using. For example, it is no longer generically true that de Rham cohomology is related to Dolbeault cohomology

$$H^m_{dR}(K;\mathbb{R})^{\mathbb{C}} \ncong \bigoplus_{p+q=m} H^{p,q}_{\bar\partial}(K;\mathbb{C}). \tag{4.1}$$

We also have that

$$H^{p,q}_{\bar\partial}(K;\mathbb{C}) \ncong H^{q,p}_{\bar\partial}(K;\mathbb{C}). \tag{4.2}$$

Another loss is that the Levi-Civita connection no longer annihilates the complex

[1] See [100] for a nice discussion on torsion.

structure; instead, any connection annihilating the complex structure must contain torsion. The Levi-Civita connection is the one commonly found in supergravity actions, and so one must be more careful when working with a non-Kähler compactification (see appendix B.2.2, for example). Other important properties of Kähler manifolds include the Lefschetz decomposition and the Hodge-Riemann bilinear relations [73].

There are many other nice properties and theorems special to Kähler manifolds, and these play no small role in the dearth of studies of supersymmetric heterotic compactifications with H-flux. Nevertheless, in recent years various groups have taken up this challenge for many reasons [39, 16, 18, 12, 13, 15, 17, 77, 40, 53, 103], one of the most compelling reasons being the potential to lift moduli of the flux-free compactifications.

One might think that studying heterotic theory is moot since we expect any such study would be dual to some flux compactification of the better-understood type II or M-theory, but we should not forget that the use of duality is often that a difficult problem in one theory can be a simple one in the dual theory. Even more compelling is that heterotic theory actually has a microscopic description in terms of $(0, 2)$ conformal field theories, while in type II compactifications such a description does not yet exist. If we fail to study flux compactifications of the heterotic theory we could be missing a wonderful opportunity, especially considering how natural heterotic theories seem when we are interested in reproducing properties of the standard model.

This chapter considers properties of the massless spectrum of compactifications of heterotic supergravity using the construction recently developed by Fu and Yau [55].

In their paper, Fu and Yau constructed gauge bundles over a sub-class of non-Kähler 3-folds studied by Goldstein and Prokushkin [70] and proved the existence of solutions to Strominger's system. We will refer to this construction as the FSY geometry and to the underlying 3-fold as a GP manifold. For a physical discussion and explicit examples of FSY geometries, see [14].

In section 4.2, we review the constraints that Strominger derived on compactifications of heterotic supergravity preserving $\mathcal{N} = 1$ supersymmetry in four dimensions [118]. In section 4.3, we review the constructions of Goldstein and Prokushkin and of Fu and Yau. We then analyze the volume of the GP manifold in the FSY geometry as well as the volume of the fibers. In section 4.4, we discuss the applicability of the supergravity approximation, compute the massless fields arising from the gaugino upon compactification, and find an explicit result for the choice of a trivial gauge bundle in terms of Hodge numbers of the GP manifold. We compute the Hodge diamond of the GP manifold in section 4.5 and then discuss our results and future directions in section 4.6.

4.2 Superstrings with Torsion

In [118], Strominger examined heterotic compactifications on warped product manifolds. His compactification assumed a maximally symmetric four-dimensional spacetime $\mathcal{M}_4$ and internal six-dimensional manifold K with metric

$$g^0_{MN}(x,y) = e^{-D(y)/2} \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{mn}(y) \end{pmatrix}, \tag{4.3}$$

where x^μ are coordinates on a patch of $\mathcal{M}_4$, y^m are coordinates on a patch of K (as we will soon see, K must be complex and we will refer to the coordinates on a patch of K as $\{z^a, \bar{z}^{\bar{a}}\}$), and capital Roman indices $M, N, \ldots$, are used for the full ten-dimensional spacetime. To ensure a supersymmetric configuration, the supersymmetry variations of the fields must vanish. This is trivially true for variations of the bosonic fields if we assume no fermionic condensates (see [103] for an example with condensates), so to preserve supersymmetry the variations of the fermionic fields must vanish.

After converting to string frame and applying some other simplifications, these variations yield the constraints[2]

$$\begin{aligned} \nabla_M \epsilon - \frac{3}{8} H_M \epsilon = 0, \\ (\nabla\!\!\!/ \phi)\epsilon - \frac{1}{4} H \epsilon = 0, \\ F_{MN}\Gamma^{MN}\epsilon = 0, \end{aligned} \tag{4.4}$$

where $H \equiv H_{MNP}\Gamma^{MNP}$ and $H_M \equiv H_{MNP}\Gamma^{NP}$. Additionally, as required by anomaly cancellation, there is a modified Bianchi identity for the 3-form field strength H

$$\frac{3}{2} dH = \frac{\alpha'}{2}\left(\mathrm{tr} R \wedge R - \frac{1}{30}\mathrm{Tr} F \wedge F\right), \tag{4.5}$$

where tr is a trace over the vector representation of $O(1,9)$ and Tr is a trace over the adjoint represention of either $SO(32)$ or $E_8 \times E_8$ (if we choose $SO(32)$, we can write this as simply a trace over the vector representation without the factor of $\frac{1}{30}$ [24]). Also, R refers to the Ricci 2-form of the Hermitian connection.[3] Finally, the warp factor is forced to be equal to the dilaton $D(y) = \phi(y)$.

[2]Throughout this note, we use the conventions: $H^{AS} = \frac{3}{2} H^{us}$, $\phi^{AS} = -\frac{1}{4}\phi^{us}$, where AS refers to the conventions in [118].

[3]This connection has nonzero connection coefficients $\Gamma^a{}_{bc} = g^{a\bar{a}} g_{\bar{a}b,c}$ and $\Gamma^{\bar{a}}{}_{\bar{b}\bar{c}} = g^{\bar{a}a} g_{a\bar{b},\bar{c}}$.

The spinor ϵ is a ten-dimensional, Majorana-Weyl spinor, so we can decompose it as the sum of tensor products of four and six dimensional Weyl spinors, say $\epsilon = \epsilon_4 \otimes \eta + c.c.$. Furthermore, the assumption of maximal symmetry in $\mathcal{M}_4$ requires that F and H have no components tangent to $\mathcal{M}_4$ and that they, as well as the dilaton ϕ, only depend on the internal manifold K. These facts simplify the constraints to

$$\begin{aligned} &\nabla_\mu \epsilon_4 = 0, \qquad\qquad \nabla_m \eta - \tfrac{3}{8} H_m \eta = 0, \\ &(\nabla_m \phi)\gamma^m \eta - \tfrac{1}{4} H \eta = 0, \qquad F_{mn}\gamma^{mn}\eta = 0. \end{aligned} \tag{4.6}$$

Thus, η is covariantly constant with respect to a metric-compatible connection with torsion $\frac{3}{2}H$, the Strominger connection.[4] There is an ambiguity in the Bianchi identity (4.5) in the choice of connection appearing in R. In [55] and [14], the Hermitian connection is used and hence it is the one we mention below (4.5). However, the connection with torsion $-\frac{3}{2}H$, referred to as the "minus" connection, is sometimes used [85], [17]. The ambiguity arises from a field redefinition in the effective action picture or from a choice of regularization scheme in the sigma model picture [115].

Strominger showed in [118] that η could be used to construct an almost complex structure for K ($J_m{}^n \equiv i\eta^\dagger \gamma_m{}^n \eta$) that is H-covariantly constant

$$\nabla_m J_n{}^p + \frac{3}{2} H^p{}_{ms} J_n{}^s - \frac{3}{2} H^s{}_{mn} J_s{}^p = 0 \tag{4.7}$$

and has vanishing Nijenhuis tensor. Thus, J is integrable and is a complex structure for K; in fact, the metric g_{mn} (4.3) is Hermitian with respect to J. The supersymmetry constraints (4.4) also imply the existence of a nowhere-vanishing, holomorphic

[4] This is sometimes called the "H-connection" or the "plus" connection.

(3, 0)-form $\Omega_{abc} = e^{-2\phi}\eta^\dagger\gamma_{abc}\eta^*$, thus implying the vanishing of the first Chern class. These conditions are equivalent to the existence of an $SU(3)$-structure.

In sum, Strominger recast (4.4) into the geometrical form:

1. (K, g) must be a complex Hermitian manifold with vanishing first Chern class.
2. Using J to denote the fundamental form in addition to the complex structure, we have
$$\frac{3}{2}H = \frac{i}{2}(\bar{\partial} - \partial)J; \tag{4.8}$$
3. $d^\dagger J = i(\bar{\partial} - \partial)\ln ||\Omega||;$[5]
4. F is a (1, 1)-form and must satisfy $J^{a\bar{b}}F_{a\bar{b}} = 0$;
5. finally, the modified Bianchi identity (4.5) must be satisfied.[6]

These we refer to as "Strominger's system", which is the system of constraints that the FSY geometry was designed to solve.

4.3 The GP Manifold and FSY Geometry

4.3.1 A Review

In their paper [70], Goldstein and Prokushkin gave an explicit construction of all complex $(n + 1)$-folds that can be realized as principal holomorphic T^2 bundles over a complex n-fold. In particular, they showed that if the base 2-fold was Calabi-Yau,

[5]Note that there was a sign error in [118] that was corrected in [119].

[6]For the rest of the chapter, we will work in units where $\alpha' = 1$.

one could use this to construct a complex Hermitian 3-fold satisfying constraints 1-3 of Strominger's system. The remaining task for a heterotic solution was to construct a gauge bundle satisfying constraints 4-5.

Unlike the Calabi-Yau case, in non-Kähler compactifications one cannot embed the Strominger-connection in the gauge connection because the curvature form will have $(0,2)$ and $(2,0)$ components, so the choice of gauge bundle satisfying Strominger's system becomes much more complicated. Fu and Yau undertook the difficult task of constructing just such a gauge bundle and were able to prove the existence of solutions to Strominger's system [55].[7] We briefly review these constructions here.

Let M be a complex Hermitian 2-fold and choose

$$\frac{\omega_P}{2\pi}, \frac{\omega_Q}{2\pi} \in H^2(M;\mathbb{Z}) \cap \Lambda^{1,1}T^*M \tag{4.9}$$

(actually, Goldstein and Prokushkin only required that $\omega_P + i\omega_Q$ have no $(0,2)$-component, but Fu and Yau used the restriction that we have stated). Being elements of integer cohomology, there are two unit-circle bundles over M, say S^1_P and S^1_Q, whose curvature 2-forms are ω_P and ω_Q, respectively. Together, these form a T^2 bundle over M which we will refer to as K, $K \xrightarrow{\pi} M$.

Given this setup, Goldstein and Prokushkin showed that if M admits a non-vanishing, holomorphic $(2,0)$-form, then K admits a non-vanishing, holomorphic $(3,0)$-form. Furthermore, they showed that if ω_P or ω_Q are nontrivial in cohomology on M, then K admits *no* Kähler metric. They were able to construct the non-

[7] In their original paper [54], Fu and Yau proved the existence of a solution to the system of equations considered in section 4.2 but with opposite sign for (4.5). In [55], they have solved the system of equations from section 4.2, which are the solutions relevant to heterotic compactifications. The sign difference dates to a sign error in [118]. Fu and Yau have also considered a wider class of gauge bundles in the more recent paper.

vanishing holomorphic $(3,0)$-form and a Hermitian metric on K simply from data on M. In particular, for the choice $M = K3$, they were able to compute the Betti numbers of K, as well as $h^{0,1}$ and $h^{1,0}$.

The curvature 2-form ω_P determines a non-unique connection ∇ on S^1_P (and similarly for ω_Q and S^1_Q). A connection determines a split of TK into a vertical and horizontal subbundle—the horizontal subbundle is composed of the elements of TK that are annihilated by the connection 1-form, the vertical subbundle is then, roughly speaking, the elements of TK tangent to the fibers. Over an open subset $U \subset M$, we have a local trivialization of K and we can use unit-norm sections, ξ of S^1_P and ζ of S^1_Q, to define local coordinates for $z \in U \times T^2$ by

$$z = (p, e^{ix}\xi(p), e^{iy}\zeta(p)), \tag{4.10}$$

where $p = \pi(z) \in U$. The sections ξ and ζ also define connection 1-forms via

$$\nabla\xi = i\alpha \otimes \xi \qquad \text{and} \qquad \nabla\zeta = i\beta \otimes \zeta, \tag{4.11}$$

where $\omega_P = d\alpha$ and $\omega_Q = d\beta$ on U, and α and β are necessarily real to preserve the unit-norms of ξ and ζ.

The complex structure is given on the fibers by $\partial_x \to \partial_y$ and $\partial_y \to -\partial_x$ while on the horizontal distribution it is induced by projection onto M (actually, this just gives an almost complex structure, but Goldstein and Prokushkin proved that it is integrable [70]). Given a Hermitian 2-form ω_M on M, the 2-form

$$\omega_u = \pi^*\left(e^u\omega_M\right) + (dx + \pi^*\alpha) \wedge (dy + \pi^*\beta), \tag{4.12}$$

where u is some smooth function on M, is a Hermitian 2-form on K with respect to this complex structure. The connection 1-form

$$\rho = (dx + \pi^*\alpha) + i(dy + \pi^*\beta) \tag{4.13}$$

annihilates elements of the horizontal distribution of TK while reducing to $dx + idy$ along the fibers. These data define the complex Hermitian 3-fold (K, ω_u), which we call the GP manifold [70].

Fu and Yau undertook the more difficult problem of proving the existence of gauge bundles over the GP manifold with Hermitian-Yang-Mills connections satisfying the Bianchi identity (4.5). They took the Hermitian form (4.12) and converted the Bianchi identity into a differential equation for the function u. Under the assumption

$$\left(\int_{K3} e^{-4u}\frac{\omega_{K3}^2}{2}\right)^{1/4} \ll 1 = \int_{K3}\frac{\omega_{K3}^2}{2}, \tag{4.14}$$

they then specialized to a $K3$ base and showed that there exists a solution u to the Bianchi identity for *any* compatible choice of gauge bundle V and curvatures ω_P and ω_Q[8] such that the gauge bundle V over K is the pullback of a stable, degree 0 bundle E over $K3$, $V = \pi^*E$ [55]; this is what we call the FSY geometry. In [55] and [14], it was shown that no such solution exists for a T^4 base. This is in agreement with arguments from string duality ruling out the Iwasawa manifold as a solution to the heterotic supersymmetry constraints [63].

[8] See equation (4.21) for an explanation.

4.3.2 The Volume

Of the Total Space

From here on, we will take the base M to be $K3$. In this chapter, we will be concerning ourselves with questions relating to supergravity compactifications on the FSY geometry. We must therefore understand the curvature scales involved in the construction. One issue that we can address is the overall volume of K

$$\mathrm{Vol}(K) \propto \int_K \omega_u^3. \tag{4.15}$$

Let $\{U_\alpha\}$ be a good open cover of M with subordinate partition of unity $\{\rho_\alpha\}$. Then $\{V_\alpha := \pi^{-1}(U_\alpha)\}$ is a good open cover of K with induced subordinate partition of unity $\tilde{\rho}_\alpha$ that is constant along the fibers, so $\tilde{\rho}_\alpha \pi^* = \pi^* \rho_\alpha$. Furthermore, we have the local trivialization $V_\alpha \overset{\phi_\alpha^{-1}}{\cong} U_\alpha \times T^2$, so we have

$$\int_K \omega_u^3 = \sum_\alpha \int_{U_\alpha \times T^2} \phi_\alpha^* (\tilde{\rho}_\alpha \omega_u^3). \tag{4.16}$$

In fact, we may choose the ϕ_α so that the vertical subbundle of $TV_\alpha \cong TU_\alpha \times TT^2$ is mapped isomorphically to $U_\alpha \times TT^2$ and similarly for the horizontal subbundle of TV_α to $TU_\alpha \times T^2$. We also have the inclusion $\iota_\alpha : \; T^2 \hookrightarrow U_\alpha \times T^2$ which induces the trivial projection $\iota_\alpha^* : \; \Omega^*(U_\alpha \times T^2) \to \Omega^*(T^2)$. Now, ω_u^3 is a sum of terms of the form $\pi^*\gamma \wedge \lambda$, where $\gamma \in \Omega^*(U_\alpha)$ and λ annihilates elements of the horizontal subbundle of TV_α. Then we find

$$\int_{U_\alpha \times T^2} \phi_\alpha^*(\pi^*\gamma \wedge \lambda) = \left(\int_{U_\alpha} \gamma \right) \left(\int_{T^2} \iota^*(\phi_\alpha^*(\lambda)) \right). \tag{4.17}$$

We find for the volume of K

$$\int_K \omega_u^3 = \sum_\alpha \left(\int_{U_\alpha} \rho_\alpha e^{2u} \omega_M^2 \right) \left(\int_{T^2} dx \wedge dy \right) \propto \int_M e^{2u} \omega_M^2 \gg 1, \tag{4.18}$$

which follows from (4.14), so K is large enough for the supergravity approximation to be valid. However, we should also check the volume of the T^2 fibers.

Of the Fibers

In the GP manifold, the T^2 fibers are taken to have size $4\pi^2$. This is simply because the coordinates x and y were defined such that they have periodicity 2π (4.10). If we rescale both by an integer N, the form defining the horizontal distribution becomes

$$\rho_N = \left(dx + \frac{\pi^*\alpha}{N}\right) + i\left(dy + \frac{\pi^*\beta}{N}\right). \tag{4.19}$$

This normalization of ρ_N preserves the property that $\frac{i}{2}\rho_N \wedge \bar{\rho}_N$ restricted to the fibers is just $dx \wedge dy$, so the Hermitian form (4.12) keeps the same form with ρ replaced by ρ_N.

In the Bianchi identity (4.5), the only place in which ω_P and ω_Q enter is through the Hermitian form and so rescaling the T^2 is, as far as the Bianchi identity is concerned, equivalent to keeping the volume of the T^2 fixed and instead rescaling ω_P and ω_Q each by N. More generally, we find that these two setups produce the same solution for the function u:

$$\begin{array}{ccc} \text{Vol}(T^2) = 4\pi^2 & & \text{Vol}(T^2) = 4\pi^2 NM \\ & \longleftrightarrow & \\ \text{Curvatures: } N\omega_P, M\omega_Q & & \text{Curvatures: } \omega_P, \omega_Q \end{array} \tag{4.20}$$

where $N, M \in \mathbb{Z}^+$.

It would seem, then, that we are free to rescale the T^2 to be arbitrarily large. However, there is a constraint pointed out in [55] and [14] which restricts ω_P and ω_Q quite heavily. The constraint comes from integrating the Bianchi identity and is

$$24 - Ch_2(E) = \int_{K3} \left(\left|\left|\frac{\omega_P}{2\pi}\right|\right|^2 + \left|\left|\frac{\omega_Q}{2\pi}\right|\right|^2 \right) \frac{\omega_M^2}{2}, \tag{4.21}$$

where $Ch_2(E)$ is the integral of the second Chern character ($\text{tr}F^2$) over $K3$, and similarly 24 comes from integrating $\text{tr}R^2_{K3}$ which yields the Euler characteristic of $K3$.

Furthermore, $\int_{K3} \left|\left|\frac{\omega_P}{2\pi}\right|\right|^2 \frac{\omega_M^2}{2}$ can be computed using the intersection form on $K3$ and is known to be a positive, even integer. It therefore seems we cannot make the volume of the T^2 significantly larger than the string scale. However, this statement is not quite right; as Goldstein and Prokushkin note [70], even if ω_P or ω_Q (but not both) is trivial in cohomology, the 3-fold is still non-Kähler. There is nothing to prevent us from considering models in which one of the circle bundles, say S^1_P, is trivial. In these cases, since $\int_{K3} \left|\left|\frac{\omega_P}{2\pi}\right|\right|^2 \frac{\omega_M^2}{2} = 0$, we are free to rescale that circle to our heart's content. We are then left with just one circle that cannot be made much larger than the string scale. The correct statement, then, is that we cannot make both circles arbitrarily large.

4.4 Heterotic Supergravity

We see that the volume of the $K3$ is large but the volume of the T^2 fibers is generically of order the string scale. This is a problem for a simple KK reduction of the ten-dimensional supergravity, but not because of curvature scales; rather, it is because we know that nonzero winding and momentum modes of the string become light when the T^2 is of order α'. This can be simply remedied by including these new light degrees of freedom in the dimensionally-reduced action. We will leave this for future work and just work with the compactification of the ten-dimensional effective action below. Note also that large curvatures associated with the gauge bundle can become problematic for a supergravity approximation, but we see from (4.21) that the curvature is bounded above since the right-hand side of the equation is non-negative.

4.4.1 Linearized EOM's

The string-frame action is (see appendix B.1):

$$\begin{aligned}
L^{(S)}_{Het} &= -\tfrac{1}{2}e^{-2\phi}\sqrt{-G}\Big[\tfrac{1}{\kappa^2}R - \tfrac{4}{\kappa^2}D_M\phi D^M\phi + \tfrac{1}{2\kappa^2}\mathrm{Tr}(F^2) + \tfrac{3}{4\kappa^2}H^2 \\
&+ \bar{\psi}_M\Gamma^{MNP}D_N\psi_P + \bar{\psi}_M\Gamma^{MP}\Gamma^N\psi_P D_N\phi + \bar{\lambda}\Gamma^M D_M\lambda \\
&- \bar{\lambda}\Gamma^M\lambda D_M\phi + \mathrm{Tr}\big(\bar{\chi}\Gamma^M D_M\chi - \bar{\chi}\Gamma^M\chi D_M\phi\big) \\
&+ \frac{1}{2}\mathrm{Tr}\big(\bar{\chi}\Gamma^M\Gamma^{NP}(\psi_M + \frac{\sqrt{2}}{12}\Gamma_M\lambda)F_{NP}\big) + \tfrac{1}{\sqrt{2}}\bar{\psi}_M\Gamma^N\Gamma^M\lambda D_N\phi \\
&- \frac{1}{8}\mathrm{Tr}\big(\bar{\chi}\Gamma^{MNP}\chi\big)H_{MNP} - \frac{1}{8}\Big(\bar{\psi}_M\Gamma^{MNPQR}\psi_R + 6\bar{\psi}^N\Gamma^P\psi^Q \\
&- \sqrt{2}\bar{\psi}_M\Gamma^{NPQ}\Gamma^M\lambda\Big)H_{NPQ} + (\text{Fermions})^4.
\end{aligned} \tag{4.22}$$

Decomposing $\chi = \chi^I T^I$ and $F_{MN} = F^I_{MN}T^I$, where T^I are generators of the gauge group satisfying $\mathrm{Tr}(T^I T^J) = \delta^{IJ}$, we find the linearized equations of motion for the fermions:

Gaugino:

$$\begin{aligned}
0 &= 2\Gamma^M D_M\chi^I - 2\Gamma^M\chi^I D_M\phi + \frac{1}{2}\Gamma^M\Gamma^{NP}\psi_M F^I_{NP} + \frac{\sqrt{2}}{3}\Gamma^{NP}\lambda F^I_{NP} \\
&\quad -\frac{1}{4}\Gamma^{MNP}\chi^I H_{MNP}
\end{aligned} \tag{4.23}$$

Dilatino:

$$\begin{aligned}
0 &= 2\Gamma^M D_M\lambda - 2\Gamma^M\lambda D_M\phi - \frac{\sqrt{2}}{3}\Gamma^{NP}\chi^I F^I_{NP} + \frac{1}{\sqrt{2}}\Gamma^M\Gamma^N\psi_M D_N\phi \\
&\quad -\frac{\sqrt{2}}{8}\Gamma^M\Gamma^{NPQ}\psi_M H_{NPQ}
\end{aligned} \tag{4.24}$$

Gravitino:

$$\begin{aligned} 0 &= 2\Gamma^{MNP}D_N\psi_P + 2\Gamma^{MP}\Gamma^N\psi_P D_N\phi + \frac{1}{2}\Gamma^{NP}\Gamma^M\chi^I F^I_{NP} \\ &+\frac{1}{\sqrt{2}}\Gamma^N\Gamma^M\lambda D_N\phi - \frac{1}{4}\Gamma^{MNPQR}\psi_R H_{NPQ} - \frac{3}{2}\Gamma_N\psi_P H^{MNP} \\ &+\frac{\sqrt{2}}{8}\Gamma^{NPQ}\Gamma^M\lambda H_{NPQ}. \end{aligned} \tag{4.25}$$

Strominger's solution [118], and also the solution in the preceding paper [24], trivially satisfied these equations of motion by assuming no fermionic condensates. We are interested in the four-dimensional effective theory, in particular the massless spectrum, arising from compactifications on the FSY geometry. Since we know we will have supersymmetry in the four-dimensional theory, we can just look for variations of the fermionic fields satisfying the equations of motion while holding the bosonic fields fixed; the massless bosonic fields will then simply be superpartners of the massless fermionic fields.

There is, of course, a limitation in our methodology. We have ignored higher order α' corrections and a superpotential that is expected to be generated (see [18], for example)—these should lift some of the massless fields. We expect that this method will ultimately provide an upper bound to the number of massless fields, so let us proceed with it.

In ten dimensions, $\mathcal{N} = 1$ supersymmetry implies that we have one Majorana-Weyl spinor supercharge. We can decompose a ten-dimensional Majorana-Weyl spinor as

$$\epsilon_-^{(10)} = \epsilon_-^{(4)} \otimes \epsilon_+^{(6)} + \epsilon_+^{(4)} \otimes \epsilon_-^{(6)} \tag{4.26}$$

where $\epsilon_\pm^{(4)}$ are four-dimensional, charge conjugate Weyl spinors, while $\epsilon_\pm^{(6)}$ are six-dimensional, charge conjugate Weyl spinors. We take as our convection for the ten-dimensional gamma matrices:

$$\Gamma^\mu = \gamma^\mu \otimes \mathbf{1} \qquad \text{and} \qquad \Gamma^m = \gamma_5 \otimes \gamma^m, \tag{4.27}$$

where γ^μ are the four-dimensional gamma matrices, γ^m the six-dimensional ones, and γ_5 is the four-dimensional chirality operator. The six-dimensional, H-covariantly constant spinor η (4.6) and its charge conjugate η^* then satisfy $\gamma^{\bar{a}}\eta = 0 = \gamma^a\eta^*$, which in fact implies that η has positive chirality and η^* has negative chirality.

Note that the set $\{\eta, \gamma^a\eta, \gamma^{ab}\eta, \gamma^{abc}\eta\}$ spans the space of six-dimensional spinors. The last one, $\gamma^{abc}\eta$, is the only one annihilated by all the γ^a's, so it should be proportional to η^*. In particular,

$$\eta^* = \frac{1}{\sqrt{48}} e^{2\phi}\Omega_{abc}\gamma^{abc}\eta \tag{4.28}$$

up to an overall phase. It is also true that η^* is H-covariantly constant, which follows by noting that $e^{2\phi}\Omega_{abc}$ is H-covariantly constant.

4.4.2 Counting the Massless Gauginos

We can use (4.28) and the basis $\{\eta, \gamma^a\eta, \gamma^{ab}\eta, \gamma^{abc}\eta\}$ to write the most general Ansatz for the variation of the gaugino as

$$\delta\chi = \epsilon_- \otimes \left(C\eta + C_{ab}\gamma^{ab}\eta\right) - \epsilon_+ \otimes \left(\bar{C}\eta^* + \bar{C}_{\bar{a}\bar{b}}\gamma^{\bar{a}\bar{b}}\eta^*\right), \tag{4.29}$$

where $C, C_{ab} \in \Omega^*(K; V)$ are forms valued in some representation V of the gauge group. This is the most general form since χ must be a ten-dimensional Majorana-Weyl spinor. See appendix A of [103] for details on this.

When we choose a gauge bundle with structure group G and embed it in $E_8 \times E_8$ or $SO(32)$, the adjoint will decompose into a sum of products of representations of the smaller groups. One of these terms will transform as an adjoint of G and we will ignore variations of this term since it is the one that couples to the other fermions. This simplifies our lives by allowing us to consider the variation of the other portions of the gaugino independent from the other fermions. This implies the gaugino equation of motion takes the form

$$0 = 2\left(D_a C + 4D^b C_{ba} - 4\partial^b\phi C_{ba}\right)\gamma^a\eta + 2(D_a C_{bc})\gamma^{abc}\eta, \tag{4.30}$$

where we recall that D is the Levi-Civita connection plus the gauge connection—see appendix B.3 for the derivation of (4.30).

By rescaling the C's by $e^{\phi/2}$, the equations take the form

$$\begin{aligned} 0 &= D_a C + \frac{1}{2}\partial_a\phi C + 4D^b C_{ba} - 2\partial^b\phi C_{ba} \\ 0 &= D_{[a}C_{bc]} + \frac{1}{2}\partial_{[a}\phi C_{bc]}, \end{aligned} \tag{4.31}$$

or by writing $C_{(0)} = C$ and $C_{(2)} = \frac{1}{2}C_{ab}dz^a \wedge dz^b$ we can recast these equations as

$$0 = \mathcal{D}C_{(0)} + 4\mathcal{D}^\dagger C_{(2)} \qquad \text{and} \qquad 0 = \mathcal{D}C_{(2)}. \tag{4.32}$$

This defines the differential operator $\mathcal{D} : \ \Omega^{p,q}(K;V) \to \Omega^{p+1,q}(K;V)$, while the adjoint is defined via the inner product

$$(\alpha, \beta) := \int_K \alpha^\dagger \wedge \beta, \tag{4.33}$$

where $\alpha^\dagger = \bar{\alpha}^T$ takes values in the dual vector bundle to V.

$\mathcal{D}^2$ is just the $(2,0)$ part the curvature 2-form of the gauge bundle, which is required to be a $(1,1)$-form by supersymmetry, so $\mathcal{D}^2 = 0$ and similarly $\mathcal{D}^{\dagger 2} = 0$. We then find that

$$0 = \Delta_{\mathcal{D}} C_{(0)} \qquad \text{and} \qquad 0 = \Delta_{\mathcal{D}} C_{(2)} \tag{4.34}$$

so that the C's are $\mathcal{D}$-harmonic forms. The space of solutions to (4.31) is then spanned by these forms, reducing the question of counting massless gaugino modes to the question of computing the dimensions of the $\mathcal{D}$-cohomology groups $H^{0,0}_{\mathcal{D}}(K;V)$ and $H^{2,0}_{\mathcal{D}}(K;V)$. Furthermore, since the dilaton ϕ is continous over the compact manifold K, the cohomology of $e^{-a\phi}\mathcal{D}e^{a\phi}$ (for constant a) is the same as that of $\mathcal{D}$, meaning we can rescale the C's to consider equations of the form $\mathcal{D}C + a(d\phi)\wedge C = 0$ for any a. In particular, we can choose a to eliminate the above ϕ dependence so that $\mathcal{D}$ will become a standard twisted Dolbeault operator.

These cohomologies are rather abstract and we cannot simplify things as in the Calabi-Yau case by embedding the Strominger connection in the gauge connection. The reason for this is that the field strength must be a $(1,1)$-form by the supersymmetry constraints, but the Ricci 2-form will not be purely $(1,1)$ for the Strominger connection. However, we do have the simpler option of choosing the bundle to be trivial, as explained in [14].[9] A trivial line bundle is semi-stable, not stable; however, this is not a problem since the main usage of stability appears in the Donaldson-Uhlenbeck-Yau theorem, which proves the existence of a connection that yields a $(1,1)$ curvature form satisfying $F_{a\bar{b}}g^{a\bar{b}} = 0$. In the case of a trivial bundle, these conditions are obviously met and so semi-stable will suffice.

For the choice of trivial line bundle, the twisted cohomology problem reduces to

[9]There is also an example of a nontrivial gauge bundle presented in [14], along with a proof that all stable bundles over the GP manifold, K, satisfying Strominger's system must be a bundle pulled back from the $K3$ tensored with a line bundle over K.

that of the ∂ operator, or if we consider the complex conjugate equations, the solutions are given by the usual Dolbeault cohomology groups $H^{0,0}_{\bar{\partial}}(K;\mathbb{C})$ and $H^{0,2}_{\bar{\partial}}(K;\mathbb{C})$. As we will see in the next section, $h^{0,0} = 1 = h^{0,2}$, so for this choice of gauge group we get two massless fermions transforming in the adjoint of $E_8 \times E_8$ or $SO(32)$.

4.4.3 A Quick Check

As a check on this method, let us consider what happens in the Calabi-Yau case using standard embedding. Under $E_8 \times E_8 \to SU(3) \times E_6 \times E_8$, the adjoint $(\mathbf{248}, \mathbf{1}) + (\mathbf{1}, \mathbf{248})$ decomposes as

$$(\mathbf{1}, \mathbf{78}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{248}) + (\mathbf{8}, \mathbf{1}, \mathbf{1}) + (\mathbf{3}, \mathbf{27}, \mathbf{1}) + (\bar{\mathbf{3}}, \overline{\mathbf{27}}, \mathbf{1}). \tag{4.35}$$

As mentioned earlier, we will focus on the variations of the gaugino other than the adjoint of $SU(3)$, $(\mathbf{8}, \mathbf{1}, \mathbf{1})$. First, the adjoint of $E_6 \times E_8$, $(\mathbf{1}, \mathbf{78}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{248})$, are scalars as far as the $SU(3)$ connection is concerned, so we get $h^{0,0}(CY_3) + h^{0,2}(CY_3) = 1$ fermion transforming in the adjoint of $E_6 \times E_8$.

For the $(\mathbf{3}, \mathbf{27}, \mathbf{1})$, the equations take the form

$$0 = D_a C^d + 4 D^b C_{ba}{}^d, \qquad 0 = D_{[a} C_{bc]}{}^d, \tag{4.36}$$

where we have suppressed indices for $E_6 \times E_8$. We can use the metric to lower the holomorphic index d to an antiholomorphic index, thus leaving us with

$$h^{1,2}(CY_3) = h^{1,0}(CY_3) + h^{1,2}(CY_3) \tag{4.37}$$

fermions transforming in the $\mathbf{27}$ of E_6. Finally for the $(\bar{\mathbf{3}}, \overline{\mathbf{27}}, \mathbf{1})$, we have an equation similar to (4.36) except with an antiholomorphic index $\bar{a}$ in place of d. Lowering $\bar{a}$

to a holomorphic index using the metric is not useful since it will not be antisymmeterized with the other indices. However, we can contract $\bar{a}$ with one index from the covariantly-constant, antiholomorphic $(0,3)$-form $\bar{\Omega}$, yielding $h^{2,0}(CY_3)+h^{2,2}(CY_3) = h^{1,1}(CY_3)$ fermions transforming in the $\overline{\mathbf{27}}$ of E_6. These results are in agreement with the well-known counting of massless modes arising from the connection 1-form, A_M. See, for example, [112].

4.5 Computing the Hodge Diamond

Goldstein and Prokushkin [70] explained a method for computing the Hodge numbers and used it to compute $h^{0,1}$ and $h^{1,0}$. They showed that the Dolbeault cohomology groups $H^{p,q}_{\bar{\partial}}(K)$ are left invariant by the actions of ∂_x and ∂_y, which literally means that if we move an element of $H^{p,q}_{\bar{\partial}}(K)$ around a fiber $\pi^{-1}(p)$, $p \in K3$, the form should remain constant. This means we can express all forms as a sum of wedge products of ρ, $\bar{\rho}$, and forms pulled back from the $K3$.

We should note, before proceeding, that we will restrict attention to the case where ω_P and ω_Q are anti-selfdual $(1,1)$-forms. The construction in [70] only requires that $\omega_P + i\omega_Q$ have no $(0,2)$-component and that they each have anti-selfdual $(1,1)$-components. Fu and Yau restrict to GP manifolds where ω_P and ω_Q are $(1,1)$-forms [55], so the computations that follow hold for the case considered by Fu and Yau but do not encompass all manifolds constructed by Goldstein and Prokushkin.

Let us review the computation of $h^{1,0}$ from [70] for illustration first. If $\xi \in H^{1,0}_{\bar{\partial}}(K)$, then

$$\xi = (\pi^* s)\rho + \pi^* s^{1,0}, \tag{4.38}$$

where s is a function on $K3$ and $s^{1,0}$ is a $(1,0)$-form on $K3$. Recall that $\bar{\partial}\rho = \pi^*(\omega_P + i\omega_Q)$ and $\bar{\partial}\bar{\rho} = 0$. Thus, $0 = \bar{\partial}\xi = \pi^*(\bar{\partial}s)\wedge\rho + \pi^*\left(s(\omega_P + i\omega_Q) + \bar{\partial}s^{1,0}\right)$. The first term tells us $\bar{\partial}s = 0$, which means that s must be a constant. The term pulled back from $K3$ tells us that $s(\omega_P + i\omega_Q) = -\bar{\partial}s^{1,0}$, but since $\omega_P + i\omega_Q$ is assumed to be nontrivial in the Dolbeault cohomology of $K3$, then the only solution to this is $s = 0$ and $\bar{\partial}s^{1,0} = 0$. Thus, we have the result $h^{1,0}(K) = h^{1,0}(K3) = 0$. Similarly, Goldstein and Prokushkin found $h^{0,1}(K) = h^{0,1}(K3) + 1 = 1$.

Note that the Hodge theorem implies that even for non-Kähler manifolds [73]

$$H^{p,q}_{\bar{\partial}}(K) = H^{n-p,n-q}_{\bar{\partial}}(K)\,, \tag{4.39}$$

however $H^{p,q}_{\bar{\partial}}(K) \neq H^{q,p}_{\bar{\partial}}(K)$. In any event, we only have to compute some of the Hodge numbers. If $\xi \in H^{1,1}_{\bar{\partial}}(K)$, then

$$\xi = (\pi^*s)\rho\wedge\bar{\rho} + \rho\wedge\pi^*s^{0,1} + \bar{\rho}\wedge\pi^*s^{1,0} + \pi^*s^{1,1}. \tag{4.40}$$

Requiring $\bar{\partial}\xi = 0$ implies

$$\bar{\partial}s = 0, \qquad s(\omega_P + i\omega_Q) - \bar{\partial}s^{1,0} = 0, \qquad \bar{\partial}s^{0,1} = 0,$$
$$\text{and} \quad (\omega_P + i\omega_Q)\wedge s^{0,1} + \bar{\partial}s^{1,1} = 0. \tag{4.41}$$

As above, we find: $s = 0$; $\bar{\partial}s^{1,0} = 0$, which then implies $s^{1,0} = 0$ ($h^{1,0}(K3) = 0$); and $s^{0,1} = \bar{\partial}t$ ($h^{0,1}(K3) = 0$), where t is a function on $K3$, which then implies $s^{1,1} = t^{1,1} - t(\omega_P + i\omega_Q)$, where $\bar{\partial}t^{1,1} = 0$. So we have

$$\xi = \rho\wedge\bar{\partial}\pi^*t + \pi^*\left(t^{1,1} - t(\omega_P + i\omega_Q)\right) = \pi^*t^{1,1} - \bar{\partial}\left((\pi^*t)\rho\right). \tag{4.42}$$

This last term is exact, and $\pi^*(t^{1,1} + \bar{\partial}u^{1,0}) = \pi^*t^{1,1} + \bar{\partial}\pi^*u^{1,0}$, so we find $h^{1,1}(K) = h^{1,1}(K3) = 20$.

Now take $\xi \in H^{2,0}_{\bar\partial}(K)$, so we have

$$\xi = \rho \wedge \pi^* s^{1,0} + \pi^* s^{2,0}. \tag{4.43}$$

$\bar\partial\xi = 0$ implies $\bar\partial s^{1,0} = 0$, so $s^{1,0} = 0$, which in turn implies that $\bar\partial s^{2,0} = 0$, so $s^{2,0} = c\Omega^{2,0}_{K3}$, where c is a constant and $\Omega^{2,0}_{K3}$ is the nowhere-vanishing, holomorphic $(2,0)$-form on $K3$. Thus, $h^{2,0}(K) = 1$.

If $\xi \in H^{0,2}_{\bar\partial}(K)$, then

$$\xi = \bar\rho \wedge \pi^* s^{0,1} + \pi^* s^{0,2}. \tag{4.44}$$

Then $\bar\partial\xi = 0$ implies that $\bar\partial s^{0,1} = \bar\partial s^{0,2} = 0$. Shifting $s^{0,1}$ or $s^{0,2}$ by a $\bar\partial$-exact form just shifts ξ by a $\bar\partial$-exact form, so $h^{0,2}(K) = h^{0,1}(K3) + h^{0,2}(K3) = 1$.

Finally, suppose $\xi \in H^{1,2}_{\bar\partial}(K)$, then

$$\xi = \rho \wedge \bar\rho \wedge \pi^* s^{0,1} + \bar\rho \wedge \pi^* s^{1,1} + \rho \wedge \pi^* s^{0,2} + \pi^* s^{1,2}. \tag{4.45}$$

Requiring $\bar\partial\xi = 0$ implies

$$\bar\partial s^{0,1} = 0, \qquad (\omega_P + i\omega_Q) \wedge s^{0,1} - \bar\partial s^{1,1} = 0, \qquad \bar\partial s^{0,2} = 0,$$
$$\text{and} \quad (\omega_P + i\omega_Q) \wedge s^{0,2} + \bar\partial s^{1,2} = 0; \tag{4.46}$$

however, these last two equations are trivially true since $K3$ is a complex 2-fold. These translate into: $s^{0,1} = \bar\partial t$; $s^{1,1} = t^{1,1} + t(\omega_P + i\omega_Q)$, where $\bar\partial t^{1,1} = 0$; $s^{1,2} = \bar\partial u^{1,1}$ $(h^{1,2}(K3) = 0)$; and $s^{0,2} = c\bar\Omega^{0,2}_{K3} + \bar\partial t^{0,1}$, where $\bar\Omega^{0,2}_{K3}$ is the complex conjugate of the holomorphic $(2,0)$-form on $K3$ and c is a constant. So we have now

$$\begin{aligned} \xi &= \rho\wedge\bar\rho\wedge\bar\partial\pi^*t + \bar\rho\wedge\pi^*t^{1,1} + \pi^*(\omega_P + i\omega_Q)\wedge\bar\rho(\pi^*t) \\ &\quad + c\rho\wedge\pi^*\bar\Omega^{0,2}_{K3} + \rho\wedge\bar\partial\pi^*t^{0,1} + \bar\partial\pi^*u^{1,1} \\ &= \bar\partial\left((\pi^*t)\rho\wedge\bar\rho\right) + \bar\rho\wedge\pi^*t^{1,1} + c\rho\wedge\pi^*\bar\Omega^{0,2}_{K3} - \bar\partial\left(\rho\wedge\pi^*t^{0,1}\right) \\ &\quad + \pi^*\left((\omega_P + i\omega_Q)\wedge t^{0,1}\right) + \bar\partial\pi^*u^{1,1} \\ &\cong \bar\rho\wedge\pi^*t^{1,1} + c\rho\wedge\pi^*\bar\Omega^{0,2}_{K3} + \pi^*\left((\omega_P + i\omega_Q)\wedge t^{0,1}\right), \end{aligned} \tag{4.47}$$

where c is constant, $\bar\partial t^{1,1} = 0$, and '$\cong$' means equal up to $\bar\partial$-exact terms (which also identifies ξ under $t^{1,1} \to t^{1,1} + \bar\partial u^{1,0}$). Notice that this last term is necessarily $\bar\partial$-closed, but since $h^{1,2}(K3) = 0$, it is also $\bar\partial$-exact and thus we have

$$\xi \cong \bar\rho\wedge\pi^*t^{1,1} + c\rho\wedge\pi^*\bar\Omega^{0,2}_{K3}, \tag{4.48}$$

which implies $h^{1,2}(K) = h^{1,1}(K3) + h^{0,2}(K3) = 21$.

Finally, we know from [118] that $h^{3,0}(K) = 1$ and we can use this to fill out the Hodge diamond:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 1 & & \\ & 1 & & 20 & & 1 & \\ 1 & & 21 & & 21 & & 1 \\ & 1 & & 20 & & 1 & \\ & & 1 & & 0 & & \\ & & & 1 & & & \end{array} \tag{4.49}$$

We do not expect that for a non-Kähler manifold the Hodge numbers will add up to the Betti numbers, and indeed they do not. The Betti numbers were computed in [70] using the Gysin sequence. For the sake of comparison they are

	$\omega_Q \neq n\omega_P$	$\omega_Q = n\omega_P$
$b_0(K)$	1	1
$b_1(K)$	0	1
$b_2(K)$	20	21
$b_3(K)$	42	42

(4.50)

for any $n \in \mathbb{Z}$. We hope that this cohomology calculation will be useful when an index is found to count the number of moduli fields, but we leave this for future work.

4.6 Discussion

In [24] and [118], the supersymmetry constraints were satisfied in part by assuming no fermionic condensates. In order to get the first image of the massless spectrum from compactification on the FSY geometry, we have counted the solutions of the variations of the gaugino that satisfy the linearized equations of motion from heterotic supergravity. We found they are given by the cohomology of forms valued in a vector bundle using the gauge connection. The ability to choose a trivial bundle relies on $c_2(TK) = 0$, which is true for the GP manifold. Taking the gauge bundle to be trivial allowed us to relate the twisted Dolbeault cohomology to the ordinary Dolbeault cohomology of the GP manifold, which we then computed.

This counting is far from a full treatment of the effective action or even the massless spectrum resulting from compactification on the FSY geometry. First of all, one must include the new light modes arising from toroidal compactifications. After including these new fields, one way to get an upper bound on the number of massless fields would be to count the solutions of the variations of all the fermions that satisfy the linearized equations of motion. Unfortunately, these are complicated, coupled differential equations and perhaps no simpler to solve than other potential methods for addressing aspects of the effective action, such as trying to understand the moduli space of FSY geometries.[10] The reason we expect this to give only an upper bound is because we have ignored quartic fermionic terms in the action, higher order α' corrections, and a conjectured superpotential (see [18], for example), all of which we expect to impose additional constraints and decrease the number of true massless fields.

There are many exciting topics yet to be explored in the realm of torsional compactifications of the heterotic string, the four-dimensional effective action being one of the ultimate goals. The moduli space would also be interesting since we expect that the inclusion of flux in the heterotic theory will lift most of the moduli that we have in Calabi-Yau compactifications. Understanding the moduli space could then help in understanding the four-dimensional effective action. It would also be interesting to be able to compare the effective action for these heterotic compactifications to the type IIB dual that was studied in [52]. Since the FSY geometry is the first smooth construction satisfying the $\mathcal{N} = 1$ supersymmetry constraints derived from the supergravity approximation,[11] the time is ripe for studying flux compactifications

[10]For example, in [17] the authors explored variations of the supersymmetry constraints under a set of simplifying assumptions.

[11]An orbifold limit of a torsional T^2 bundle over $K3$ was considered in [39] by duality chasing.

of the heterotic string; we hope the reader has gained some interest in studying these partially-forgotten topics!

Chapter 5

Linear Models for Flux Vacua

5.1 Introduction

It is a beautiful and frustrating fact of life that Calabi-Yaus have interesting moduli spaces. On the one hand, the topology and geometry of their moduli spaces govern the low-energy physics of string theory compactified on a Calabi-Yau, so understanding their structure teaches us about four-dimensional stringy physics. On the other, the resulting massless scalars are a phenomenological disaster.

Dodging this bullet has proven surprisingly difficult. At the level of type II supergravity, beautiful work of KKLT and others[1] demonstrates that a judicious choice of fluxes and branes wrapping suitable cycles in a fiducial Calabi-Yau can generate a scalar potential which fixes all moduli of the underlying CY. However, since these type II flux vacua necessarily involve RR fluxes and other effects which are not amenable

[1]See in particular [105, 76, 89, 88] for foundational work, and [48] for a complete review and further references.

to worldsheet analysis, it is difficult to construct a microscopic description for them, and a sufficiently hard-nosed physicist could rationally wonder whether these vacua, in fact, exist.

Duality provides a powerful hint. For a large class of flux vacua, such as the KST models of [89], there exists [16] a duality frame involving a heterotic compactification on a non-Kähler manifold of $SU(3)$-structure with non-trivial gauge and NS-NS 3-form flux, $H \neq 0$, all of which is in principle amenable to worldsheet analysis. A microscopic description of heterotic flux vacua would thus provide a microscopic description of the dual KST vacua.

Of course, there are excellent reasons that most work has focused on Kähler compactifications, which necessarily have $H = 0$. In particular, only for Kähler manifolds does Yau's Theorem ensure the existence of solutions to the tree-level supergravity equations; the beautiful results of Gross & Witten [74] and Nemachamsky & Sen [110] then ensure that these classical solutions extend smoothly to solutions of the exact string-corrected equations. When $H \neq 0$, the story is much more complicated, due in part to the absence of effective computational tools analogous to Hodge theory or special geometry for non-Kähler manifolds, and in part to the tremendous analytic complexity of the Bianchi identity,

$$dH = \alpha' \left(\mathrm{tr} R \wedge R - \mathrm{Tr} F \wedge F \right). \tag{5.1}$$

Indeed, twenty years passed between Strominger's geometric statement of the supersymmetry constraints [118] and the proof by Fu and Yau of the existence of a class of solutions to these leading-order equations [55]. Moreover, since the Bianchi identity scales inhomogeneously with the global conformal mode, any solution has total volume-modulus fixed near the string scale, so such compactifications can *not* be described by conventional, weakly-coupled NLSMs. Whether these Fu-Yau solutions, like Calabi-Yaus, can be smoothly extended to solutions of the exact string equations has thus remained very much unclear.

The purpose of this chapter is to develop tools with which to study heterotic compactifications with non-vanishing H, *i.e.* holomorphic vector bundles over non-Kähler manifolds with intrinsic torsion satisfying (5.1). Motivated by Fu and Yau, we focus on torus bundles over Kähler bases, $T^m \to X \to S$, with gauge bundle $\mathcal{V}_X$ and NS-NS flux H turned on over the total space X. When $m = 2$ and $S = K3$, this is precisely the Fu-Yau compactification.[2]

Our strategy closely parallels the familiar gauged linear sigma model (GLSM) approach to Calabi-Yau compactifications [124]: we build a massive 2d gauge theory which flows in the IR to an interacting CFT with all the properties that we expect of a Fu-Yau compactification. In the Calabi-Yau case, the GLSM flows to a NLSM whose large-radius limit is the chosen Calabi-Yau. This is not possible in the Fu-Yau case as no large-radius limit exists; however, the classical moduli space of the one-loop effective potential of our GLSM will precisely reproduce the Fu-Yau geometry. We thus take the CFT to which our torsion linear sigma model (TLSM) flows to provide a microscopic definition of the Fu-Yau compactification.

A central ingredient in these models is a two-dimensional implementation of the Green-Schwarz mechanism. The $ch_2(T_S) - ch_2(\mathcal{V}_S)$ anomaly of a $(0,2)$ nonlinear

[2] While we refer to these geometries as *Fu-Yau* geometries, it should be emphasized that Strominger's elaboration of the precise equations to be solved was crucial to the eventual construction of solutions by Fu and Yau, as well as to earlier studies of the underlying manifolds by Goldstein and Prokushkin [70].

sigma model on $\mathcal{V}_S \to S$ is contained[3] in the gauge anomaly of $(0,2)$ GLSMs for S. In compactifications with intrinsic torsion, this sum does not vanish even in cohomology. To restore gauge invariance, we introduce a novel $(0,2)$ multiplet containing a doublet of axions whose gauge variation precisely cancels the gauge anomaly. The one-loop geometry of the resulting model is easily seen to be a T^2 fibration X over the Calabi-Yau S – a Fu-Yau geometry – with the anomaly cancellation conditions of the TLSM reproducing the conditions for the existence of a solution to the Bianchi identity.

Crucial to our construction is a manifest $(0,2)$ supersymmetry with non-anomalous R-current and a non-anomalous left-moving $U(1)$. These ensure the perturbative non-renormalization of the superpotential and are necessary for the existence of a chiral GSO projection. The worry, as usual in a $(0,2)$ theory, is that worldsheet instantons may generate a non-perturbative superpotential [43, 44]. The power of a gauged linear description is that the moduli space of worldsheet instantons is embedded within the moduli space of gauge theory instantons, which is manifestly compact; without a direction along which to get an IR divergence, it is thus impossible to generate the poles required for the generation of a spacetime superpotential[117, 11]. Such arguments have been used to rigorously forbid the existence of non-perturbative superpotentials for $(0,2)$ gauged linear sigma models of Calabi-Yau geometries; while some technical details differ so that we cannot present a direct proof, these results appear to extend unproblematically to our torsion linear models.

Along the way we will construct a number of $(2,2)$ TLSMs for generalized Kähler geometries, including non-compact models built out of chiral and twisted chiral mul-

[3] In fact, this is a somewhat subtle story, as we shall elaborate below.

tiplets and more intricate models in which we gauge chiral currents built out of semi-chiral multiplets. While not our main interest in this chapter, these models provide useful guidance in our construction of $(0,2)$ models with torsion and are worth studying on their own merits.

This chapter is a brief introduction to the structure of torsion linear sigma models, focusing on a few basic results and proofs-of-principle. A self-contained follow-up including examples and details omitted below is in preparation.

5.2 Torsion in $(2,2)$ GLSMs

While a number of $(2,2)$ gauged linear sigma models with non-trivial NS-NS flux have been studied in the literature – most notably the (4,4) H-monopole GLSM [123] – the structure of general models has received relatively little attention. In this section, we will review the incorporation of NS-NS flux into $(2,2)$ models, emphasizing features which will generalize to the more complicated $(0,2)$ examples studied below. A more complete discussion of the rich structure of general $(2,2)$ torsion will be addressed elsewhere.

Let us start with a standard $(2,2)$ GLSM for some toric variety V built out of chiral and vector supermultiplets. The IR geometry of such models is necessarily Kähler. What we seek is a way to introduce non-trivial $H = dB \neq 0$ into a standard $(2,2)$ GLSM. Since H is an obstruction to Kählerity, we are also looking for a construction of non-Kähler geometries via $(2,2)$ GLSMs. It has long been known that sigma models built entirely out of chiral multiplets are necessarily Kähler [62], so we would seem to need to introduce non-chiral multiplets. However, since a $(2,2)$ gauge field minimally coupled to chiral multiplets cannot be minimally coupled to twisted chirals while preserving $(2,2)$, there would seem to be a no-go argument forbidding minimally-coupled GLSMs for non-Kähler geometries with non-vanishing H.

As is often the case, this no-go statement tells us exactly where to go. Recall that B appears in the GLSM through the imaginary parts of the complexified FI parameters $t^a = r^a + i\theta^a$ appearing in the twisted chiral superpotential,

$$-\frac{1}{2\sqrt{2}}\int d^2\tilde{\theta}\; t^a\Sigma_a + \text{h.c.} \;=\; -\,r^a D_a + 2\theta^a v_{+-a}.$$

More precisely, t^a are the restriction of the complexified Kahler class $\mathcal{J} = J + iB$ to the hyperplane classes $H_a \in H^2(V)$ corresponding to the gauge fields Σ_a, *i.e.* $B = \theta^a H_a$. To get $H \neq 0$ we must promote some of the θ^a, say m of them, to dynamical fields. Note that this *adds* dimensions to the geometry, so we are no longer working with a sigma model on V, but with a geometry with local product structure $V \times (S^1)^m$.

For the moment, consider promoting a single FI parameter to a dynamical field. Since the FI parameter appears in the twisted chiral superpotential, $(2,2)$ supersymmetry requires that it be promoted to a twisted chiral multiplet Y with action

$$\begin{aligned} -\frac{N^a}{2\sqrt{2}}\int d^2\tilde{\theta}\; Y\Sigma_a + \text{h.c.} - \frac{1}{8}\int d^4\theta\; k^2(Y+\overline{Y})^2 \\ = -k^2[(\partial r)^2 + (\partial\theta)^2] \;-N^a\,[rD_a - 2\theta\, v_{+-a}] + \ldots \end{aligned} \tag{5.2}$$

where $k \in \mathbb{R}$ and $y = r + i\theta \in \mathbb{C}^*$ is the scalar component of Y. The geometry is thus a complex manifold with local product structure, $W \sim V \times \mathbb{C}^*$, and NS-NS potential $B = \theta N^a H_a$ on the total space W that is no longer closed,

$$H = d\theta \wedge N^a H_a \neq 0.$$

The resulting IR geometry is non-Kähler, evading the no-go statement above by coupling the gauge supermultiplet minimally to chirals and *axially* to twisted chirals. Note that the resultant H-flux has two legs along V and one along the S^1 coordinatized by θ. Note, too, that this is precisely the form of the relevant couplings in the (4,4) H-monopole GLSM.

In some sense, what we have done by promoting the FI parameter/Kähler modulus t to a dynamical field Y is to take the variety V and construct a new variety W as a fibration of V over a complex line in the Kähler moduli space of V. This should give us pause; the moduli space includes points where the original variety V goes singular, so this fibration is degenerate. How do we know that the total space of the fibration is, in fact, smooth?

Consider, for example, the resolved conifold $F = xy - wz - r = 0$ in $\mathbb{C}^4$, and let W be the fibration of the conifold over the complex line r. The point $r = 0$ is a very singular point – even the CFT is singular – and it is natural to wonder if W is singular at $r = 0$. In fact, it is straightforward to see that W is completely well behaved at $r = 0$. Like V, W is the vanishing locus of F, now viewed as a function on $\mathbb{C}^5$. However, since $\partial_r F = -1$, F is strictly transverse, so the hypersurface $W = F^{-1}(0)$ is everywhere smooth. By virtue of the linear nature of the axial coupling, a similar result can be argued to obtain for all $(2,2)$ models in which the FI parameter is promoted to a dynamical field.

T-dualizing the dynamical FI parameter is revealing. Consider a GLSM with gauge group $U(1)^s$, $(N+s)$ chirals Φ_I, and m axially coupled twisted chirals, Y_l, with

Lagrangian,

$$\mathcal{L} = \int d^4\theta \left[-\frac{1}{4e_a^2}\overline{\Sigma}_a\Sigma_a + \frac{1}{4}\overline{\Phi}_I e^{2Q_I^a V_a}\Phi_I - \frac{1}{8}k_l^2(\overline{Y}_l + Y_l)^2 \right] - \frac{1}{2\sqrt{2}}\int d^2\tilde{\theta}\, M_l^a Y_l \Sigma_a + \text{h.c.}$$

Dualizing all the twisted chirals Y_l into chiral multiplets $\mathcal{P}_l$ results in a simple model,

$$\tilde{\mathcal{L}} = \int d^4\theta \left[-\frac{1}{4e_a^2}\overline{\Sigma}_a\Sigma_a + \frac{1}{4}\overline{\Phi}_I e^{2Q_I^a V_a}\Phi_I + \frac{1}{8k_l^2}(\overline{\mathcal{P}}_l + \mathcal{P}_l + 2M_l^a V_a)^2 \right].$$

All matter fields are now chiral, so the classical moduli space is automatically Kähler. But with which Kähler metric, on which space? We can clearly eat the imaginary component of all m fields $\mathcal{P}_l$ to make m of the gauge fields massive (provided $s \geq m$), and use their real components to solve m of the D-terms. However, integrating out the massive vectors and scalars deforms the Kähler potential for the $(N+s)$ Φ_Is. The surviving $(s-m)$ gauge fields then effect a Kähler quotient of $\mathbb{C}^{N+s}$, but now starting with a deformed Kähler structure. The IR geometry is thus an $N+m$ dimensional variety whose topology is controlled by the charges Q_I^a of the Φ_I under the surviving gauge fields but with deformed Kähler structure[82]. This can be used to construct GLSMs for, say, squashed spheres. T-dualizing with this squashed metric then gives non-trivial B, which was what we found above.

It is fun to note in passing that we could just as well have dualized the *chiral* multiplets in our torsion model to get a theory of only *twisted chiral* multiplets, all axially coupled to an otherwise free gauge multiplet. As emphasized by Morrison & Plesser [107] and by Hori & Vafa [83], the resulting theory has a non-perturbative superpotential of the form $W = e^{-Z_I}$, where Z_I are the twisted chirals dual to the original Φ_I. The resulting theories end up looking like complicated generalizations of Liouville theories coupled to a host of scalars.

Going back to our strategy of axially coupling twisted chirals to the gauge multiplets of a chiral GLSM, and vice versa, a little play leads us to the very general form,

$$\mathcal{L} = \mathcal{L}_V(\Phi, \Sigma) + \mathcal{L}_W(Y, S) \ + \int d^2\tilde{\theta}\ \Sigma G(Y) \ + \int d^2\theta\ SF(\Phi) + \text{h.c.},$$

where $\mathcal{L}_{V,W}$ are the Lagrangians for standard chiral (twisted chiral) GLSMs on V (W), F and G are gauge invariant analytic functions of the chiral and twisted chiral fields, repectively, and S is the chiral field-strength of the gauge field in $\mathcal{L}_W$. The resulting geometry has an obvious local product structure, $M \sim V \times W$, but is globally non-trivial – this is a simple extension of the fibration structure discussed above. One annoying feature of all such models is that any model of this form, which has trivial one-loop running of the D-term (*i.e.* all the Ricci-flat manifolds), appears to be, at first blush, non-compact: it is simply impossible to build a non-trivial coupling of this form when V and W are both compact Calabi-Yaus. Something remains missing.

Note that the models described above evaded the "no-go" statement by coupling a $(2,2)$ vector minimally to chiral matter and axially to twisted chirals or vice vera. While these models have a particularly simple presentation, they are by no means the most general $(2,2)$ models one can construct – in particular, there are many more representations than simply chiral and twisted chiral. In fact, as has only recently been proven[99], the most general off-shell $(2,2)$ NLSM can only be written by including semi-chiral multiplets anihilated by a single supercharge. It is reasonable to ask if the same is true of GLSMs.

As it turns out, a large class of generalized geometries [80, 75] only admit gauged linear descriptions using semi-chiral superfields. Suppose we want to couple a $(2,2)$

gauge field to a conserved current; of necessity, that current must be either a chiral or a twisted chiral current. However, the matter fields which appear in the current do not have to be chiral or twisted chiral, only the total current is so constrained. This suggests a simple strategy for constructing a $(2,2)$ GLSM out of semi-chiral fields: begin with a theory of free semi-chiral fields and identify a chiral isometry of this free theory under which the semi-chiral matter fields rotate by a chiral phase. Then, couple the associated current to a canonical $(2,2)$ gauge supermultiplet. The result is a manifestly $(2,2)$ GLSM which, in general, does not reduce to a theory of chirals.

There are many fun $(2,2)$ torsion linear sigma models one can build, with interesting geometric and algebraic properties, but our interests in this note lie with the heterotic string, so we now turn to $(0,2)$ models, leaving a thorough discussion of the $(2,2)$ case (and the intriguing liminal $(1,2)$ case) to another publication.

5.3 Non-Compact $(0,2)$ Models and the Bianchi Identity

Suppose we are handed a well-behaved $(0,2)$ GLSM for a vector bundle $\mathcal{V}_S$ over some happy Kähler manifold S. The FI parameters of the GLSM, t^a, parameterize some of the complexified Kähler moduli of S. As in the $(2,2)$ cases discussed above, introducing non-trivial H into this $(0,2)$ GLSM is a simple matter of promoting some subset of the FI parameters t^a to dynamical fields $Y_{l=1\ldots m}$ in the GLSM. The FI coupling in a $(0,2)$ model is again a superpotential interaction, so the requisite promotion is

$$\frac{i}{4}\int d\theta^+\ t^a\Upsilon_a + \text{h.c.}\ \rightarrow\ \frac{i}{4}\int d\theta^+\ N_l^a Y_l \Upsilon_a + \text{h.c.}\ - i\int d^2\theta\ \overline{Y}_l \partial_- Y_l$$

where $y_l = r_l + i\theta_l \in \mathbb{C}^*$, and with $N_l^a \in \mathbb{Z}$ to ensure single-valuedness of the action. This results in non-trivial NS-NS 3-form flux,

$$B = N_l^a \theta_l H_a \quad \Rightarrow \quad H = N_l^a d\theta_l \wedge H_a,$$

not on S, but on a non-compact fibration $(\mathbb{C}^*)^m \rightarrow \widetilde{X} \rightarrow S$, with H having two legs along S and one along the fibre. (Here, H_a is the a^{th} hyperplane class on S.)

This model has two major limitations. First and foremost is the fact that the Bianchi identity is solved rather trivially: $dH = 0$ by construction, since both $d\theta_l$ and H_a lift trivially to closed forms on the total space of the $(\mathbb{C}^*)^m$-fibration, $\widetilde{X}$. What we are after is an interesting solution to the Bianchi identity. Secondly, the classical moduli space, $\widetilde{X}$, is non-compact. Since the non-compactness is due to the unconstrained real part of the dynamical FI parameters, we might try to simply lift them, leaving the imaginary part dynamical as required for non-trivial H-flux.[4] Unfortunately, this explicitly breaks $(0,2)$ supersymmetry. In the remainder of this section we will focus on correcting the triviality of the Bianchi identity – the thorny problem of compactification we defer to the next section.

To begin, note a curious difference from the $(2,2)$ case above. In a $(0,2)$ gauge theory, the FI parameter does not appear in a twisted chiral superpotential – indeed, there *is* no twisted chiral representation of $(0,2)$ – but in a *chiral* superpotential,

[4]Indeed, this is what happens in the Goldstein-Prokushkin construction [70], whose compact non-Kähler manifolds arise as the unit-circle sub-bundles of two $\mathbb{C}^*$-bundles over a base Calabi-Yau, as described further in appendix C.2.

so the dynamical FI parameters in a $(0,2)$ theory are *chiral*, just like the minimally coupled scalars. This raises an interesting possibility: since supersymmetry no longer forbids the minimal coupling of the gauge fields to the Y_l, we can couple Y_l both axially *and* minimally in a completely supersymmetric fashion:

$$\mathcal{L} = -\frac{1}{2}\int d^2\theta\ (\overline{Y}_l + Y_l + 2M_l^a V_{+a})(i\partial_-[Y_l - \overline{Y}_l] - M_l^a V_{-a}) + \frac{i}{4}\int d\theta^+\ N_l^a Y_l \Upsilon_a + \text{h.c.},$$

where the M_l^a are integers (we will discuss their quantization later). Unfortunately, under a gauge transformation $Y_l \to Y_l - iM_l^b\Lambda_b$, the superpotential transforms as

$$\delta_\Lambda \mathcal{L} = \frac{1}{4}\int d\theta^+\ M_l^b N_l^a \Lambda_b \Upsilon_a + \text{h.c.},$$

which is not a total derivative, so this Lagrangian does not appear to be terribly useful.

However, this gauge variation has the familiar form of the *gauge anomaly* of a $(0,2)$ GLSM. Consider a GLSM for a holomorphic vector bundle $\mathcal{V}_S$ over a Calabi-Yau base, S, built out of chiral superfields Φ_I and fermi superfields Γ_m (see appendix C.1 for conventions). While the classical Lagrangian is manifestly gauge invariant, the measure generically suffers from a set of one-loop exact chiral gauge anomalies[5] of the form

$$\mathcal{D}[\Phi,\Gamma] \xrightarrow{\delta_\Lambda} \mathcal{D}[\Phi,\Gamma]\exp\left(-\frac{iA^{ab}}{8\pi}\int d^2y\left[\int d\theta^+ \Lambda_b \Upsilon_a + \text{h.c.}\right]\right),$$

where A^{ab} is a quadratic form built out of the gauge charges Q_I^a and q_m^a of the right-

[5]Such gauge anomalies are strictly absent in $(2,2)$ models, where left- and right-handed fermions are paired up in $(2,2)$ chiral multiplets to give an overall non-chiral theory; in a $(0,2)$ model, by contrast, left- and right-moving fermions transform in different supersymmetry multiplets and may thus transform differently under the gauge symmetry, leading to the gauge anomaly advertised above.

and left-moving fermions,

$$A^{ab} = \sum_I Q_I^a Q_I^b \; - \sum_m q_m^a q_m^b. \tag{5.3}$$

This can be easily derived by examining the loop diagram with two external gauge bosons. This anomaly, a familiar feature of $(0,2)$ GLSM building, has a natural geometric interpretation. Recall that the right-handed fermions transform as sections of a sheaf $\mathcal{F}_V$ over the ambient toric variety V which restricts over S to the tangent bundle, T_S. Meanwhile, the left-handed fermions transform as sections of a sheaf $\mathcal{V}_V$ which restricts to the gauge bundle, $\mathcal{V}_S$. The gauge anomaly measures

$$\mathcal{A} \propto ch_2(\mathcal{F}_V) - ch_2(\mathcal{V}_V).$$

Since the Bianchi identity is just the restriction of $\mathcal{A}$ to S, the vanishing of the gauge anomaly[6] ensures that the IR NLSM satisfies the heterotic Bianchi identity with $dH = 0$. This connection will be better explored in section 5.4.3.

These two effects – the gauge variance of the classical action and the one-loop gauge anomaly – dovetail beautifully. Consider a GLSM for $\mathcal{V}_S \to S$ with $ch_2(T_S) \neq ch_2(\mathcal{V}_S)$. On its own, this model is anomalous. Now promote some subset of FI parameters to dynamical fields Y_l with axial couplings N_l^a and charges M_l^a. Under a gauge variation, the effective action ($S_{eff} = \frac{1}{4\pi}\int d^2y\, \mathcal{L}_{eff}$) picks up classical terms

[6]Note that the gauge anomaly may fail to vanish even when the classical moduli space of the GLSM has vanishing ch_2 anomaly. For example, consider a $(0,2)$ model for an elliptic curve in $\mathbb{P}^2$ with trivial left-moving bundle. A NLSM on an elliptic curve cannot have a ch_2 anomaly – nonetheless, the GLSM has a gauge anomaly. What is going on? The point is that the gauge anomaly computes the non-vanishing self-intersection number of the hyperplane class in $\mathbb{P}^2$, an intersection which does not restrict to the hypersurface (indeed, there is no four-cohomology on T^2). This is a somewhat familiar fact in $(0,2)$ model building: many geometries for which a NLSM analysis is perfectly consistent do not seem to admit GLSM descriptions due to uncanceled gauge anomalies.

from the axions and one-loop terms from the anomaly,

$$\delta_\Lambda \mathcal{L}_{\mathit{eff}} = \frac{1}{2}\int d\theta^+ \, \left[\frac{1}{2}M_l^b N_l^a - Q_I^a Q_I^b + q_m^a q_m^b\right] \Lambda_b \Upsilon_a + \text{h.c.}\,.$$

Thus, for every solution of the Diophantine equation

$$\frac{1}{2}\sum_l M_l^b N_l^a = \sum_I Q_I^a Q_I^b - \sum_m q_m^a q_m^b \tag{5.4}$$

we have a non-anomalous $(0,2)$ quantum field theory. Since the superpotential of this $(0,2)$ theory is not renormalized beyond one loop in perturbation theory, and since the anomaly is one-loop exact, the path integral remains gauge invariant to all orders in perturbation theory.[7] Note that the ch_2 anomaly in the NLSM is also one-loop exact. We shall refer to a $(0,2)$ GLSM which implements the above cancellation mechanism as a *torsion linear sigma model* (TLSM).

Notice what has happened. First, we have replaced the Kähler geometry S with a non-Kähler $(\mathbb{C}^*)^m$-fibration $\widetilde{X}$ over S such that the curvature 2-forms of the $(\mathbb{C}^*)^m$-fibration are trivial in $H^2(\widetilde{X}, \mathbb{Z})$, the cohomology of the total space. It is important to distinguish $ch_2(T_S) - ch_2(\mathcal{V}_S)$, the anomaly on S, from the very different quantity $ch_2(T_{\widetilde{X}}) - ch_2(\mathcal{V}_{\widetilde{X}})$, the anomaly on the $(\mathbb{C}^*)^m$-fibration $\widetilde{X}$ *over* S. At the end of the day, the physical Bianchi identity lives on $\widetilde{X}$ and says that $dH = ch_2(T_{\widetilde{X}}) - ch_2(\mathcal{V}_{\widetilde{X}})$, so in cohomology on $\widetilde{X}$, $ch_2(T_{\widetilde{X}}) = ch_2(\mathcal{V}_{\widetilde{X}})$. However, since $\widetilde{X}$ is a non-trivial fibration over S, cohomology classes do not trivially lift, or descend (think about the Hopf map). The upshot it that Bianchi identity does *not* imply that $ch_2(T_S) = ch_2(\mathcal{V}_S)$, even in cohomology. However, the 3-form flux $H = N_l^a d\theta_l \wedge H_a$ on the the total space, $\widetilde{X}$, was constructed precisely so as to solve the Bianchi identity when pushed down

[7] We will discuss non-perturbative effects below.

the fibres – this is what led us to introduce the gauge-variant axial coupling in the first place.

This graceful mechanism of anomaly cancellation, a one-loop gauge anomaly canceling the gauge variation of an axionic coupling in the classical Lagrangian, is simply a 2d avatar of the Green-Schwarz anomaly in the target space.

5.4 Compact $(0,2)$ Models and the Torsion Multiplet

Let us summarize the story so far. We begin with a conventional $(0,2)$ GLSM for a Calabi-Yau S equipped with a generic holomorphic bundle $\mathcal{V}_S$. The $ch_2(T_S) \neq ch_2(\mathcal{V}_S)$ anomaly of the associated NLSM is realized in the GLSM as a gauge anomaly. To cancel the gauge anomaly, we promote some of the FI parameters to dynamical axions carrying charges chosen such that the gauge variation of the classical action cancels the one-loop gauge anomaly in a 2d version of the Green-Schwarz mechanism. The IR geometry of the resulting non-anomalous $(0,2)$ GLSM is a non-compact $(\mathbb{C}^*)^m$-fibration $\widetilde{X}$ over S,

$$\begin{array}{cc} (\mathbb{C}^*)^m \rightarrow & \widetilde{X} \\ & \downarrow \\ & S \quad , \end{array}$$

where the curvature two-forms of the $\mathbb{C}^*$-bundles are $M_l^a H_a|_S \in H^2(S,\mathbb{Z})$. Threading this geometry is a non-trivial NS-NS 3-form flux, $H = N_l^a d\theta_l \wedge H_a$, which satisfies the Bianchi identity non-trivially. For simplicity of presentation, we will focus on the special cases $S = K3$ or T^4 with m=2; the generalization to higher dimension and other geometries is straightforward.

Not coincidentally, this is enticingly close to the compact Fu-Yau geometry[8] – all we need to do is restrict to the T^2 sub-bundle of the $(\mathbb{C}^*)^2$ bundle by lifting the real direction along each $\mathbb{C}^*$ fibre. What could be easier?

5.4.1 Decoupling of Radial Fields

In fact, this turns out to be rather non-trivial. The issue is supersymmetry. The target space of any sigma model with a linearly realized $\mathcal{N} = 2$ is a complex manifold, and the specific presentation of the $\mathcal{N} = 2$ corresponds to a specific choice of complex structure. Under the particular $\mathcal{N} = 2$ respected by our GLSM, the real directions along the $\mathbb{C}^*$ fibre, r_l, are paired with the S^1 angles, θ_l, so removing only the radial coordinates would explicitly break our $(0, 2)$ supersymmetry to an all-but-useless $(0, 1)$ subgroup (which we are not allowed to lose since this $(0, 1)$ will be gauged when we couple our matter theory to heterotic worldsheet supergravity). The situation appears to be grim.

To reassure ourselves that there *should* be a $(0, 2)$ on the T^2 sub-bundle, note that

$$(\mathbb{C}^*)^2 = \mathbb{C} \times T^2$$

if the coordinates $y_l = r_l + i\theta_l$ on $(\mathbb{C}^*)^2$ are reorganized into the coordinates $r = r_1 + ir_2$ on $\mathbb{C}$ and $\theta = \theta_1 + i\theta_2$ on T^2. The IR geometry thus must admit an $\mathcal{N} = 2$ corresponding to this choice of complex structure, pairing the two angles into one

[8] A review of the Fu-Yau geometry is given in appendix C.2.

supermultiplet and the two lines into another. Unfortunately, an extensive search for such an $\mathcal{N}=2$ in our UV gauge theory quashes our high expectations.

Let's explore this apparent failure more explicitly. The relevant terms in the action are, in components,

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{K3} - k_l^2(\partial r_l)^2 - k_l^2(\partial\theta_l + M_l^a v_a)^2 + 2ik_l^2\bar{\chi}_l\partial_-\chi_l + 2N_l^a\theta_l v_{+-a} \\ &\quad + \left(2k_l^2 M_l^a - N_l^a\right)\left[r_l D_a + \frac{i}{\sqrt{2}}\chi_l\lambda_a\right] + \dots .\end{aligned}$$

Meanwhile, under the linearly realized $(0,2)$ supersymmetry

$$\delta_\epsilon\lambda_a = i\epsilon(D_a + 2iv_{+-a}) \tag{5.5}$$

Now, suppose we attempt reorganize the Y_l superfields into superfields that respect the $\mathbb{C}\times T^2$ complex structure: $R \sim r_1 + ir_2 + \dots$, $\Theta \sim \theta_1 + i\theta_2 + \dots$. The problem is that the variation of λ_a yields terms of the form $\epsilon\chi_l D_a$ and $\epsilon\chi_l v_{+-a}$. The only way to cancel these terms is for the variation of both r_l and θ_l to include terms of the form $\epsilon\chi_l$. This makes it appear impossible to split r_l and θ_l into two separate supermultiplets for generic charges.

The key word here is "generic". Note that our troublesome terms are both proportional to $(2k_l^2M_l^a - N_l^a)$, where $M_l^a, N_l^a \in \mathbb{Z}$ and $k_l \in \mathbb{R}$. If we fix k_l and N_l^a so that $N_l^a = 2k_l^2M_l^a$, these terms disappear from the action! Repeating our analysis, we find that there *is* a $(0,2)$ supersymmetry with exactly the desired properties:

$$R = (r_1 - ir_2) + i\sqrt{2}\theta^+(\chi_1^I + i\chi_2^I) + \dots \qquad \Theta = (\theta_1 + i\theta_2) + \sqrt{2}\theta^+(\chi_1^R - i\chi_2^R) + \dots$$

where R and I superscripts refer to the real and imaginary parts of the fermions, respectively. In fact, the R-multiplet is free and *entirely decouples!* What's more, since k_l, which measures the radius of the T^2 in string units, is fixed in terms of two integers, the volume of the fibre is quantized in terms of the torsion flux, just as it is in Fu-Yau.[9]

Life is now sweet and easy. Based on the above, we define

$$\begin{aligned} &\theta = \theta_1 + i\theta_2 \qquad \chi = \chi_1^R - i\chi_2^R \qquad N^a = 2k^2 M^a = 2k^2(M_1^a + iM_2^a) \\ &r = r_1 - ir_2 \qquad \tilde{\chi} = i\chi_1^I - \chi_2^I \qquad \nabla_\pm\theta = \partial_\pm\theta + M^a v_{\pm a}, \end{aligned} \tag{5.6}$$

which transform under $\mathcal{N} = 2$ supersymmetry as

$$\begin{aligned} &\delta_\epsilon\theta = -\sqrt{2}\epsilon\chi \qquad \delta_\epsilon\chi = 2\sqrt{2}i\bar{\epsilon}\ \nabla_+\theta \\ &\delta_\epsilon r = -\sqrt{2}\epsilon\tilde{\chi} \qquad \delta_\epsilon\tilde{\chi} = 2\sqrt{2}i\bar{\epsilon}\ \partial_+ r. \end{aligned} \tag{5.7}$$

In these coordinates, the action reduces to

$$\begin{aligned} \mathcal{L} \ = \ & \mathcal{L}_{K3} + 2\nabla_+\bar{\theta}\nabla_-\theta + 2\nabla_+\theta\nabla_-\bar{\theta} + 2i\bar{\chi}\partial_-\chi + 2(N^a\bar{\theta} + \overline{N}^a\theta)v_{+-a} \\ & -2|\partial r|^2 + 2i\bar{\tilde{\chi}}\partial_-\tilde{\chi}. \end{aligned} \tag{5.8}$$

We may now drop the radial supermultiplet $R = r + \sqrt{2}\theta^+\tilde{\chi} - 2i\theta^+\bar{\theta}^+\partial_+ r$, as it is entirely decoupled.

It is important to verify that the truncated Lagrangian is invariant under the $(0, 2)$ supersymmetry defined above. However, $\delta_{susy}^2 = \delta_{gauge}$ in WZ gauge, so the gauge variance of the classical action rears its stupefying head and some care is required. Under a supersymmetry transformation, the classical action transforms non-trivially,

$$\delta_\epsilon\mathcal{L} = 2(M^a\bar{M}^b + \bar{M}^a M^b)v_{+b}(i\epsilon\bar{\lambda}_a + i\bar{\epsilon}\lambda_a).$$

---

[9] Since $\int d^2y\, v_{+-a} \in \pi\mathbb{Z}$, θ_l is automatically periodic, $\theta_l \sim \theta_l + 2\pi L_l$, such that $N_l^a L_l \in 2\mathbb{Z}$. Fixing $N_l^a = 2k_l^2 M_l^a$ then implies that $k_l^2 M_l^a L_l = n_a \in \mathbb{Z}$, so M_l^a is quantized in terms of k_l and L_l. Meanwhile, the anomaly cancellation condition implies that that $\frac{n_a^2}{k_l^2 L_l^2}$ should be an integer, since the Q_I and q_m are integers. Since the physical radius of the S^1 is $k_l L_l$, this means that the radius is quantized as claimed. For the rest of this chapter, we will work with $k_l = L_l = 1$ for simplicity.

This is not a disaster because the gauge transformation needed to return us to WZ gauge (which we have been using throughout), $\alpha_a = -4i\theta^+\bar{\epsilon}v_{+a}$, induces a shift in the effective action from the anomalous measure:

$$\delta_{WZ}\mathcal{D}[\Phi,\Gamma] = \mathcal{D}[\Phi,\Gamma]\exp\left\{\frac{A^{ab}}{\pi}\int d^2y\, v_{+a}(\epsilon\bar{\lambda}_b + i\bar{\epsilon}\lambda_b)\right\}.$$

Anomaly cancellation ensures that this cancels the supersymmetry variation of the action.

At this point, we can play various games to simplify the presentation of the theory. For example, we can build a superfield out of the T^2 multiplet,

$$\Theta = \theta + \sqrt{2}\theta^+\chi - 2i\theta^+\bar{\theta}^+\nabla_+\theta.$$

This looks a lot more convenient than it actually is. While it has the usual field content, this is *not* a standard chiral multiplet: the gauging is complex, with both real and imaginary components of θ shifting under gauge transformations. Since no other superfield transforms in the same strange way, gauge multiplet included, it is extremely hard to build gauge covariant or invariant operators out of Θ. In fact, the only gauge-invariant dressed field is $(\partial_-\Theta + \frac{1}{2}M^aV_{-a})$. Meanwhile, the only chiral operator we can build out of Θ is $(\Theta + iM^aV_{+a})$, which we cannot add to the superpotential in a gauge invariant fashion. Indeed, it is impossible to build a supersymmetric and gauge invariant action for this multiplet since the supersymmetry variation of the kinetic terms cancels against the variation of the axial superpotential. To emphasize its peculiar role, we call Θ a *torsion multiplet.*

5.4.2 The IR Geometry

Setting $N_l^a = 2k^2 M_l^a$ has decoupled the R multiplet, leaving us with a non-Kähler T^2 sub-bundle $X \subset \widetilde{X}$ with torsionful $SU(3)$-structure induced from $\widetilde{X}$. In other words, the semi-classical IR geometry of our TLSM is a compact holomorphic T^2 fibration X over a Calabi-Yau S, endowed with a Hermitian metric, a stable holomorphic sheaf $\mathcal{V}_X = \pi^* \mathcal{V}_S$ pulled back from S, and an NS-NS 3-form H satisfying the Bianchi identity on $\mathcal{V}_X \to X$. Moreover, the radii of the T^2 fibres are fixed to discrete values in terms of the integral curvatures of the T^2-bundles, given as integer classes on the base $K3$. Up to uninteresting changes of coordinates, this is the Fu-Yau construction.

It is revealing to derive this IR geometry explicitly from the final TLSM. Let's begin by writing out the component Lagrangian in all its majesty. To simplify our lives, we will call all the chiral multiplets ϕ_I whether their charges are positive, negative, or zero, and leave all obvious sums implicit. This is easy to unpack when we focus on specific models. After integrating out the auxillary fields, the kinetic terms are,

$$\begin{aligned}
\mathcal{L}_{kin} \quad = \quad & -|(\partial + iQ_I^a v_a)\phi_I|^2 + 2i\bar{\psi}_I(\partial_- + iQ_I^a v_{-a})\psi_I \\
& +4(\partial_+\theta_l + M_l^a v_{+a})(\partial_-\theta_l + M_l^a v_{-a}) + 4M_l^a \theta_l v_{+-a} + 2i\bar{\chi}_l \partial_- \chi_l \\
& +2i\bar{\gamma}_m(\partial_+ + iq_m^a v_{+a})\gamma_m + \frac{2}{e_a^2}\left[(v_{+-a})^2 + i\bar{\lambda}_a \partial_+ \lambda_a\right],
\end{aligned}$$

and the scalar potential is

$$U = \sum_m \left(|E_m|^2 + |J^m|^2\right) + \sum_a \frac{e_a^2}{2}\left(\sum_I Q_I^a |\phi_I|^2 - r^a\right)^2 \qquad (5.9)$$

where $\overline{\mathcal{D}}_+\Gamma_m = \sqrt{2}E_m(\Phi)$ and $J^m(\Phi)$ is a $(0,2)$ superpotential satisfying $\sum_m E_m J^m = 0$. For completeness, the Yukawa terms are

$$\mathcal{L}_{Yuk} = -\sqrt{2}iQ_I^a\lambda_a\psi_I\bar{\phi}_I - \bar{\gamma}_m\psi_I\frac{\partial E_m}{\partial\phi_I} - \gamma_m\psi_I\frac{\partial J^m}{\partial\phi_I} + \text{h.c.}\,. \tag{5.10}$$

As in the case of $(2,2)$ GLSMs on Kähler geometries, the Hermitian geometry of the Higgs branch of our TLSM may be computed by integrating out the massive vectors and scalars in the gauge theory to derive a Born-Oppenheimer effective action on the classical moduli space. However, since the classical action of our TLSM is not gauge invariant, the story is slightly more subtle than usual.

Suppose, for example, that we simply integrate out the massive vector as usual – let us work in polar variables where $\phi_I = \rho_I e^{i\varphi_I}$. This replaces the gauge connection v_μ with a non-trivial implicit connection $v_\mu(\rho_I, \varphi_I, \theta_l, \ldots)$ on the classical moduli space. The chiral fermion content then leads to an anomaly in the resulting non-linear sigma model – an anomaly which cancels against the classical variation of the action due to the torsion multiplet. This presentation has the advantage of making the role of the anomalous gauge transformation in the NLSM manifest, but it complicates the computation of the effective metric.

Alternatively, we can take a lesson from Fujikawa and change coordinates in field space to work with uncharged fermions *before* integrating out the massive vector[56, 57]. The Jacobian of this field redefinition introduces a gauge variant operator to the action whose gauge variation cancels against that of the classical torsion terms, leaving the action gauge invariant. We can then integrate out the massive vector and massive scalars to compute the effective metric on moduli space.

Let's take the second approach and change variables to gauge invariant fermions.

For each right-moving fermion ψ_I, there is a natural choice of uncharged dressed fermion $\tilde{\psi}_I = e^{-i\varphi_I}\psi_I$; for the left-movers, there is generically no model-independent choice, so we choose an arbitrary linear combination $\hat{\varphi}_m = l^I_m \varphi_I$ of phases with the correct charges to make the dressed fermion $\tilde{\gamma}_m = e^{-i\hat{\varphi}_m}\gamma_m$ gauge neutral, *i.e.* such that $\delta_\alpha \hat{\varphi}_m = -q^a_m \alpha_a$. The Jacobian for this change of variables shifts the action by a simple term

$$\mathcal{L}_{Jac} = -4\omega^a v_{+-a}, \qquad \omega^a \equiv Q^a_I \varphi_I - q^a_m \hat{\varphi}_m \equiv T^a_I \varphi_I,$$

whose gauge variation is just the familiar anomaly,

$$\delta_\alpha \mathcal{L}_{Jac} = 4\left(Q^a_I Q^b_I - q^a_m q^b_m\right)\alpha_a v_{+-b}.$$

The total axial coupling is thus

$$\mathcal{L}_{axial} = 4\left(M^a_l \theta_l - \omega^a\right) v_{+-a},$$

which is gauge invariant by construction. The typical next step is to fix a gauge. However, since the Faddeev-Popov measure for the simplest gauge choice, $\theta_l = 0$, is trivial, it is just as easy to work in gauge unfixed presentation; the decoupled longitudinal mode will simply cancel the volume of the gauge group in the path integral.

With the action and measure now both independently gauge invariant, we can consistently integrate out the massive vector. Since the action is quadratic in the vector, this is straightforward. Solving the classical EOM for the two components of our massive vector, and splitting them into fermionic and bosonic components, yields

$$\begin{aligned}
v_{-a} &= (\Delta^{-1})^{ab}\left(\frac{1}{2}\bar{\tilde{\gamma}}_m \tilde{\gamma}_m q^b_m - \rho^2_I \partial_- \varphi_I Q^b_I + \partial_- \omega^b - 2M^b_l \partial_- \theta_l\right) = v^F_{-a} + v^B_{-a} \\
v_{+a} &= (\Delta^{-1})^{ab}\left(\frac{1}{2}\bar{\tilde{\psi}}_I \tilde{\psi}_I Q^b_I - \rho^2_I \partial_+ \varphi_I Q^b_I - \partial_+ \omega^b\right) = v^F_{+a} + v^B_{+a}
\end{aligned}$$

where we define

$$\Delta^{ab} \equiv \rho_I^2 Q_I^a Q_I^b + M_l^a M_l^b \equiv \Delta_Q + \Delta_M ,$$

which is naturally symmetric in the gauge indices. It is easy to check that both components of v transform covariantly under gauge transformations.

Thus prepared, we are finally ready to compute the effective metric on the Higgs branch. After a tedious but miserable calculation, the bosonic effective action reduces to

$$\begin{aligned} \mathcal{L}^B_{kin} &= 4\partial_+\rho_I\partial_-\rho_I + 4\partial_+\varphi_I\partial_-\varphi_J\left[\rho_I^2\delta_{IJ} - \rho_I^2\rho_J^2(\Delta^{-1})_{ab}Q_I^aQ_J^b\right] \\ &\quad +4(\Delta^{-1})_{ab}\partial_+\omega^a\partial_-\omega^b + 4\partial_+\theta_l\partial_-\theta_l - 8(\Delta^{-1})_{ab}\left(\rho_I^2\partial_+\varphi_I Q_I^a + \partial_+\omega^a\right)\partial_-\theta_l M_l^b \\ &\quad -8(\Delta^{-1})_{ab}\rho_I^2 Q_I^a \partial_{[+}\omega^b\partial_{-]}\varphi_I , \end{aligned}$$

and the fermionic effective action to

$$\begin{aligned} \mathcal{L}^F_{kin} &= 2i\bar{\tilde{\psi}}_I(\partial_- + iQ_I^a v^B_{-a} + i\partial_-\varphi_I)\tilde{\psi}_I + 2i\bar{\chi}_l\partial_-\chi_l + 2i\bar{\tilde{\gamma}}_m(\partial_+ + iq_m^a v^B_{+a} + i\partial_+\hat{\varphi}_m)\tilde{\gamma}_m \\ &\quad -(\Delta^{-1})_{ab}\bar{\tilde{\psi}}_I\tilde{\psi}_I\bar{\tilde{\gamma}}_m\tilde{\gamma}_m Q_I^a q_m^b , \end{aligned}$$

where $A_{[+}B_{-]} \equiv \frac{1}{2}(A_+B_- - A_-B_+)$, $A_{(+}B_{-)} = \frac{1}{2}(A_+B_- + A_-B_+)$. We will also find it useful to define $\Delta_2^{-1} \equiv \Delta^{-1} - \Delta_Q^{-1} = -\Delta_Q^{-1}\Delta_M\Delta^{-1}$, and to make a habit of suppressing gauge indices, representing them instead by matrix multiplication.

Since one of the features we would like to make manifest is the natural complex structure on the total space X, it is natural to return to complex variables ϕ_I and θ, as well as $M^a \equiv M_1^a + iM_2^a$. It is also natural to split the Lagrangian into terms symmetric and anti-symmetric in the derivatives, corresponding to the pullback to the worldsheet of the metric and B-field, respectively. The symmetric terms we will refer to as ds^2, where we will also use the shorthand

$$dAdB \equiv \partial_{(+}A\partial_{-)}B \qquad dA\wedge dB \equiv \partial_{[+}A\partial_{-]}B,$$

remembering that the "differentials" dA and dB are symmeterized without the $\wedge$.

Using these conventions and the definition of Δ_2^{-1}, we can easily factor out the usual kinetic terms for the ambient variety V:

$$ds_V^2 = 4|d\phi_I|^2 - 4(\bar\phi_I d\phi_I)(\phi_J d\bar\phi_J)Q_I^T\Delta_Q^{-1}Q_J.$$

The metric can then be written as

$$\begin{aligned} ds^2 &= ds_V^2 - 4|\phi_I|^2|\phi_J|^2(d\ln\bar\phi_I d\ln\phi_J)Q_I^T\Delta_2^{-1}Q_J - \left[d\ln\frac{\phi_I}{\bar\phi_I}\right]\left[d\ln\frac{\phi_J}{\bar\phi_J}\right]T_I^T\Delta^{-1}T_J \\ &\quad +4|d\theta|^2 + 2i\left[d\ln\frac{\phi_I}{\bar\phi_I}\right]\left(|\phi_I|^2Q_I^T + T_I^T\right)\Delta^{-1}\left(Md\bar\theta + \bar M d\theta\right) \end{aligned} \tag{5.11}$$

where we have used $dr^a = \sum_I Q_I^a(\phi_I d\bar\phi_I + \bar\phi_I d\phi_I) = 0$ to simplify the expression. Working patchwise on V makes the geometry somewhat more transparent. We can cover V by patches on which s of the homogeneous coordinates, say $\phi_{\sigma=N+1,\ldots,N+s}$, are nonzero and for which Q_σ^a is an invertible $s\times s$ matrix. We can then define gauge invariant coordinates on each patch,

$$z_A \equiv \phi_A\prod_{\sigma=N+1}^{N+s}\phi_\sigma^{-(Q^{-1})_a^\sigma Q_A^a}, \qquad \zeta \equiv \theta + i(Q^{-1})_a^\sigma M^a\ln\phi_\sigma,$$

where $A = 1,\ldots,N$.

All of these coordinates transform holomorphically as we move from one patch of V to another. Furthermore, from the gauge variant coordinates it is clear that there are no fixed points of the T^2 action (complex shifts of θ). Thus, as long as $S\subset V$ is smooth, our construction will yield a principal holomorphic T^2 bundle over S à *la* Goldstein and Prokushkin (see appendix C.2). In these manifestly holomorphic coordinates, the metric can be written in Hermitian form,

$$\begin{aligned} ds_H^2 &= ds_V^2 + 4\left| d\zeta - \frac{iM^T}{2}\left(\partial P - \Delta^{-1}(Q_A|\phi_A|^2 + T_A)d\ln z_A\right)\right|^2 \\ &\quad + \left(\partial P^T\Delta_M + d\ln z_A\, T_A^T\right)\left(2\Delta^{-1} - \Delta^{-1}M\bar{M}^T\Delta^{-1}\right)\left(\Delta_M\bar{\partial}P + T_B\, d\ln\bar{z}_B\right) \end{aligned}$$

where $P_a \equiv \sum_\sigma (Q^{-1})_a^\sigma \ln|\phi_\sigma|^2$ and

$$ds_V^2 = 4|\phi_A|^2|d\ln z_A|^2 - 4\left(|\phi_A|^2 Q_A^T\Delta_Q^{-1}Q_B|\phi_B|^2\right)[d\ln z_A]\,[d\ln\bar{z}_B]$$

is the analog of the Fubini-Study metric for V (and reduces to it in the case of $\mathbb{P}^N$). A similarly tedious but straightforward computation gives the resulting B-field,

$$\begin{aligned} B &= 2i\left[d\ln\frac{z_A}{\bar{z}_A}\right]\wedge(|\phi_A|^2Q_A^T + T_A^T)\Delta^{-1}\left[Md\bar{\zeta} + \bar{M}d\zeta\right] \\ &\quad -2(|\phi_A|^2Q_A^T\Delta^{-1}T_B)\left[d\ln\frac{z_A}{\bar{z}_A}\right]\wedge\left[d\ln\frac{z_B}{\bar{z}_B}\right] \\ &\quad -(\bar{M}^aM^T - M^a\bar{M}^T)\Delta^{-1}(Q_A|\phi_A|^2 + T_A)\left[d\ln\frac{z_A}{\bar{z}_A}\right]\wedge dP_a. \end{aligned}$$

We thus have a manifestly Hermitian metric on a smooth principal holomorphic T^2-bundle over S, with non-vanishing H threading the total space. This is precisely the geometry we were expecting to find.

5.4.3 The Bianchi Identity

As we sketched in section 5.3, the one-loop exact spacetime Bianchi identity is realized in the TLSM by the one-loop exact gauge anomaly. However, the gauge anomaly is independent of the superpotential and thus naturally lives on the ambient toric variety V, while the Bianchi identity lives on the space X, so the connection between the Bianchi identity and the gauge anomaly requires some work to explicate.

Their relationship is most transparent when the Bianchi identity is pushed down to the base, S. In the Fu-Yau case, it has been shown on purely geometric grounds that [55, 14]

$$dH = \pi^* \left(\omega \wedge *_S\overline{\omega}\right) + \ldots, \qquad ch(T_X) = \pi^* \left(ch(T_S)\right) + \ldots\,,$$

where $\omega = \omega_1 + i\omega_2$ is the anti-self-dual[10] (1,1) curvature form of the T^2 bundle, and the omitted terms are all exact forms on S and thus vanish in cohomology on the base. Meanwhile, by construction, $ch(\mathcal{V}_X) = \pi^* \left(ch(\mathcal{V}_S)\right)$, so the Bianchi identity reduces to a simple equation in the cohomology of S:

$$\omega \wedge *_S\overline{\omega} = -\omega_1^2 - \omega_2^2 = 2ch_2(\mathcal{V}_S) - 2ch_2(T_S). \tag{5.12}$$

All the quantities in this equation can now be written in terms of the defining charges of the TLSM. The second Chern characters can be calculated from the short exact sequences $(C.10)$ and $(C.11)$ to be

$$\begin{aligned} ch_2(T_S) &= \frac{1}{2}\sum_{a,b}\left(\sum_i Q_i^a Q_i^b - d^a d^b\right)(H_a \wedge H_b)|_S\,, \\ ch_2(\mathcal{V}_S) &= \frac{1}{2}\sum_{a,b}\left(\sum_m q_m^a q_m^b - m^a m^b\right)(H_a \wedge H_b)|_S\,, \end{aligned}$$

Meanwhile, the curvature ω of the T^2 fibration can be expressed as $\omega = (M_1^a + iM_2^a)H_a$, so the Bianchi identity pushes down to S to give

$$\sum_{a,b}\left(M^{(a}\bar{M}^{b)} - \sum_i Q_i^a Q_i^b + d^a d^b + \sum_m q_m^a q_m^b - m^a m^b\right)(H_a \wedge H_b)|_S = 0. \tag{5.13}$$

This is precisely the condition for the cancelation of the gauge anomaly of the TLSM.

[10] Strictly speaking, there can also be a self-dual $(2,0)$ ω-form, but it is automatically absent in the TLSM construction.

5.4.4 Ruling Out T^4

The case $S = T^4$ provides a revealing test case for our construction. Since T_{T^4} is (utterly) trivial, the Bianchi identity takes a particularly simple form – in fact, it is so simple that there are no non-trivial solutions [14]. This can be seen by integrating (5.1) over the base using the restricted forms of dH and $[ch(\mathcal{V}_X) - ch(T_X)]$ given in the previous section. Since F_S – the curvature of the bundle $\mathcal{V}_S$ – is anti-Hermitian and anti-self-dual, and since $ch_2(T_{T^4}) = 0$, the right-hand side of (5.12) is non-positive for $S = T^4$ while the left-hand side is manifestly non-negative for anti-self-dual ω (and only 0 when ω is exact). We would like to see this directly in the TLSM, or at least in a specific example.

To this end, we build the base $S = T^4$ as the product of two $T^2 \subset \mathbb{P}^2$, but with H-flux lacing both factors. This ensures that any 4-form on the base must be proportional to $H_1 \wedge H_2|_S$, where H_1 and H_2 are the hyperplane classes of the two $\mathbb{P}^2$s (the restrictions of H_i^2 vanish trivially). Since the Hodge star on T^4 acts as

$$*_S H_1 = H_2 \qquad *_S H_2 = H_1,$$

$(H_1 - H_2)$ is the only anti-self-dual 2-form on T^4 constructed from hyperplane classes. Since the Fu-Yau construction requires ω be anti-self-dual, we must have $\omega = M(H_1 - H_2)$. Two further conditions apply: (1) for our embedding of T^4, none of the coordinate fields are charged under both $U(1)$s, and so $d^1 d^2 = Q_i^1 Q_i^2 = 0$; and (2) the condition that $c_1(\mathcal{V}_S) = 0$ translates into $m^a = \sum_m q_m^a$. Plugging this into (5.13), only the $H_1 \wedge H_2$ cross-term does not vanish upon restriction to S and we find

$$\sum_{m \neq n} q_m^1 q_n^2 = -|M|^2. \tag{5.14}$$

But for the gauge bundle to be stable, all charges must satisfy $q^a_m \geq 0$ [46], in which case the equation has no solution unless $M = 0$. We conclude that our TLSM does not allow us to build a non-trivial T^2-bundle over this T^4-base, in agreement with the the supergravity result.

5.4.5 Global Anomalies

Of course, vanishing of the gauge anomaly and satisfaction of the Bianchi identity are not sufficient to ensure that the TLSM flows to a consistent vacuum of the heterotic string. In order to couple to worldsheet supergravity, our theory must flow to a superconformal fixed point which admits a chiral GSO projection. This in turn requires [46, 45] the existence of a non-anomalous right-moving $U(1)$ R-current, J_R, and a non-anomalous left-moving flavor symmetry, J_L, leading to additional constraints on allowed charges beyond quantum gauge invariance. The relevant anomalies are thus the various mixed gauge-global and global-global anomalies; consistency of the gauge theory requires that they cancel.

Let's start with the R-current. R-invariance of the $\Upsilon\Theta$ terms in the superpotential require Θ to be an R-scalar, though it may carry a shift-charge under R-symmetry. This implies that the fermion χ in Θ carries R-charge $+1$. However, since χ is gauge neutral, it does not contribute to the mixed gauge-R anomaly. Since the chiral superfields Φ_i typically appear in quasi-homogeneous polynomials in the superpotential $\Gamma_0 G(\Phi_i)$ (see appendix C.1), it is most natural to assign them R-charges proportional to their gauge charges rQ_i – this also fixes the R-charge of Γ_0 to $-rd-1$. Then one has the fermi supermultiplets Γ_m appearing in the superpotential via $\Phi_0\Gamma_m J^m(\Phi_i)$, restricting charge assignments for Φ_0 and Γ_m to be $p-rm$ and rq_m-p-1, respectively. This additional shift of p is a freedom not available to us in $(2,2)$ models.

The anomaly in the left-moving flavor symmetry can be treated similarly. For example, by setting the flavor charge of each field proportional to its gauge charge, and assigning to Θ an anomalous shift-charge under the flavor $U(1)$, vanishing of the gauge anomaly ensures the non-anomaly of the left-moving flavor symmetry. Note that the contribution of the torsion multiplet to the currents J_L, J_R, and J_{gauge}, is of the form $J_\Theta \sim \partial\theta$, so its contributions to the anomalies actually come from tree-diagrams rather than loops.

Two final anomaly relations are important. First, for J_R and J_L to be purely right- and left-moving, their mixed anomaly must also vanish, giving one integer constraint. Finally, the $J_R J_R$ OPE measures the conformal anomaly, which must be equal to 9, giving one last integer equation on the charges. In the typical model of interest, there are many more fields than equations, making it easy to satisfy these constraints.

5.4.6 Caveat Emptor: Spacetime vs. Worldsheet Constraints

One very important elision in the above is distinguishing which conditions on the charges are required on *a priori* 2d grounds, and which derive from spacetime arguments. For example, in a $(2,2)$ model the running of the D-term is equivalent to the R-anomaly, which in turn is equivalent to the vanishing first Chern class of the IR geometry, $c_1(T_S)$. However, in a $(0,2)$ model these three effects are decoupled.

The running of t is decoupled because we can always add a pair of massive spectators to the theory – a chiral and a fermi superfield – whose contributions to all gauge and global anomalies vanish, but whose gauge charges can be chosen to limit the running of t to a finite shift [46, 45], something not possible in more familiar $(2,2)$ models. Meanwhile, the chiral content of the theory yields enough freedom in assigning R-charges that the R-anomaly is decoupled from $c_1(T_S) = 0$. Similarly, the conditions that $c_1(\mathcal{V}_S)$=0, that ω be anti-self-dual, and that $\mathcal{V}_S$ be stable, are all required to ensure spacetime supersymmetry in the supergravity construction of the Fu-Yau compactification but do not appear as necessary constraints for the consistency of our 2d gauge theories.

A natural guess is that ensuring spacetime supersymmetry of the massless modes of our theory requires the imposition of these constraints on the charges and fields in the TLSM. Checking this requires a more detailed discussion of the exact spectrum of our models than we have presented in this note; for now we will simply impose these conditions, as is often done in $(0,2)$ models, because we can and because doing so matches us precisely onto the Fu-Yau construction. We will return to this question in a sequel.

5.5 The Conformal Limit

So far, we have shown that our compact $(0,2)$ TLSMs exist as non-anomalous, 2d $\mathcal{N} = 2$ quantum field theories which have Fu-Yau-type geometries as their one-loop classical moduli spaces. These are principal holomorphic T^2-bundles over Calabi-Yaus with torsionful G-structures which non-trivially satisfy the Green-Schwarz anomaly constraints. However, since Fu-Yau geometries are necessarily finite radius and generally contain small-volume cycles, the semi-classical geometric analysis is not obviously reliable. What we would like to argue is that the IR conformal fixed points to which these massive TLSMs flow should be taken to *define* the Fu-Yau CFT. For this to make sense, however, we must demonstrate that these TLSMs in fact flow to non-trivial CFTs in the IR.

This will take some work. The first step is to observe that the superpotential in a $(0,2)$ model is one-loop exact, so the vacuum is not destabilized at any order in perturbation theory; the concern is thus worldsheet instantons. It has long been understood that the perturbative moduli spaces of generic $(0,2)$ models are lifted by instanton effects [43, 44]. It has more recently been understood that $(0,2)$ GLSMs on Kähler targets with arbitrary (not necessarily linear) stable vector bundles are not lifted by instanton effects. This has been demonstrated in the class of "half-linear" models via an analysis of the analytic structure of the spacetime superpotential in a paper by Beasley & Witten [11] and, in the more limited case of GLSMs, via a generalized Konishi anomaly argument by Basu & Sethi[10]. Due to the gauge anomaly and gauge variance of the classical Lagrangian, neither of these analyses directly apply to our torsion models; however, the basic structure of the Beasley-Witten argument obtains, which suggests that the vacuum is indeed stable to worldsheet instanton corrections.

The basic ingredients in [11] were that the spacetime superpotential is a holomorphic section of a simple line bundle; that poles can appear only if the instanton moduli space has a non-compact dimension along which worldsheet correlators can diverge; that a simple residue theorem ensures that the sum over all poles is zero; and that the worldsheet theory respect a linearly realized $(0,2)$ with non-anomalous $U(1)$

R-symmetry. In the case of our TLSMs, the crucial step is verifying that the instanton moduli space is in fact compact; the rest appears to follow rather straightforwardly.

The instantons in our TLSM fall into two classes: those involving gauge fields coupled to torsion multiplets and those involving gauge fields coupled only to chiral multiplets. The latter class is identical to those studied in [11, 117] and have compact moduli spaces for the same reasons; these correspond to the homologically non-trivial lifts of holomorphic curves on the base Calabi-Yau. The former is more subtle. Recall that all that matters for the lifting of the massless vacuum are contributions to the chiral superpotential from BPS instantons. Significantly, BPS instantons in the torsion sector must satisfy an unusual BPS equation

$$\delta\psi = \partial_+\theta + M^a v_{+a} = 0.$$

Since v_{+a} is singular for an instanton background, instantons aligned along M^a in G do not have finite action, so we appear to have *no* instantons along the curve associated to M^a. Actually, this makes a great deal of sense. The one-form on $K3$ associated to $M^a v_{+a}$ is α_M (see appendix C.2); since α_M is not a globally-defined form, $\omega_M = d\alpha_M$ – the 2-form curvature of the T^2-bundle – is non-trivial in $H^2(K3,\mathbb{Z})$. However, the connection 1-form on X, $d\theta + \pi^*\alpha_M$, *is* a globally defined 1-form on X, so $d(d\theta + \pi^*\alpha_M)$ is trivial in $H^2(X,\mathbb{Z})$. Thus, there is no 2-cycle in X associated with this gauge field.

Thus the BPS instantons of the TLSM are a refinement of the instantons of the base Calabi-Yau, and the moduli space is consequently compact. Elevating these heuristic arguments to a rigorous proof of the stability of the vacuum to instanton corrections does not appear impossible. We leave a more thorough discussion of instantons in torsion sigma models, and a formal proof of the stability of the vacuum, to future work.

5.6 Conclusions and Speculations

In this note, we have constructed gauged linear sigma models for non-Kähler compactifications of the heterotic string with non-trivial background NS-NS 3-form H satisfying the modified Bianchi identity, and we argued for the exact stability of their vacua to all orders and non-perturbatively in α'. This construction provides a microscopic definition of the Fu-Yau CFT and, via duality, for a related class of KST-like flux-vacua [89] involving non-trivial NS-NS and RR fluxes which stabilize various moduli in a fiducial Calabi-Yau orientifold compactification.

While motivated by the remarkable Fu-Yau construction, this construction is considerably more general, suggesting applications much richer than we have been able to cover explicitly. For example, while we have focused on $K3$ bases for simplicity, it is completely straightforward to construct more general compactifications over higher-dimensional Calabi-Yau bases, leading to 7 and 8 dimensional non-Kähler compactifications corresponding to torsionful G_2 and $Spin(7)$ structure manifolds. It is also natural to try to apply the technology of the torsion multiplet to non-CY bases – say, dP_8 – by suitably adjusting the fibration structure. Perhaps the easiest cases to be studied are the type II examples in section 5.2; there is a rich story to be told there, including non-perturbative existence and a thorough study of the instanton structure of the theory. We will return to all of these points in upcoming publications.

One area where our construction should be of particular use is in the study of the moduli spaces – and hence low-energy phenomenology – of non-Kähler compactifications [17, 29]. The necessary tools for analyzing the spectra of $(0,2)$ GLSMs have long been know [46] and can presumably be applied with minor modifications. Relatedly, TLSMs should also provide a computationally effective tool to study the topological ring which was recently proved to exist for generic $(0,2)$-models [1, 2], as well as the action of mirror symmetry on these stringy geometries. In fact, the action of T-duality and mirror symmetry on these geometries is remarkably subtle – for example, it is easy to check that the T^2 fibre on the Fu-Yau geometry is, in fact, self-dual, corresponding to a pair of $SU(2)$ WZW models at level one. What is the relation between the self-dual circles and the NS-NS flux? Are these WZW models playing the anomaly-cancelling role of the WZW models in the $(0,2)$ Gepner model constructions of Berglund et al [19]? Clearly, a great deal remains to be learned form these torsion linear sigma models, and from the CFTs to which they flow.

Chapter 6

Nearing the Horizon of a Heterotic String

6.1 Introduction and summary

The worldsheet of N stretched, coincident heterotic strings is described by a $(c_L, c_R) = (24N, 12N)$ 2d CFT. General considerations as well as recent investigations, both described below, raise the intriguing possibility that this CFT has an $AdS_3 \times M$ holographic spacetime dual. If so, the first-quantized Hilbert space of the N stretched heterotic strings would be identified with the second-quantized Hilbert space of interacting closed heterotic strings on $AdS_3 \times M$. In this chapter, as reported in [98], we continue these investigations and in particular find some surprising results about the structure of the supersymmetry group.

Why should we expect such a holographic dual? At strong coupling, the heterotic string becomes a $D1$-brane of the type I theory. General low-energy scaling arguments coupled with open-closed duality then suggest the existence of a holographic dual. Because the low-energy limit of the worldvolume theory is conformally invariant, the dual should contain an AdS_3 factor. In addition, N stretched heterotic strings have an exponentially growing spectrum of left-moving BPS excitations. Although the growth is not enough to make a black string with a horizon large compared to the string scale, it is still the case that in the classical limit,[1] the second law of thermodynamics forbids energy from leaking off of the N strings, just as it does for a large black hole. One expects this behavior to be explained in the macroscopic spacetime picture by the appearance of a stringy horizon and associated near-horizon scaling solution. However, the real situation is likely more subtle than these general comments indicate. As we shall see below, simple group theory implies that the situation is highly dimension-dependent. In particular, concrete indicators of a holographic dual in the special case of compactification to $D = 5$, on which we largely concentrate, will be reviewed below.

6.1.1 The Leading-Order Solution

The string frame classical geometry sourced by the N stretched heterotic strings in the leading α' approximation for a compactification to $D \geq 5$ dimensions was found some time ago [34] using the supergravity equations:

$$ds^2 = \frac{dx^+ dx^-}{1 + N(\frac{r_h}{r})^{D-4}} - d\vec{x} \cdot d\vec{x} - ds^2_{10-D}, \tag{6.1}$$

[1] With the spacetime momentum density along the string, and all spacetime fields fixed while $\hbar \to 0$.

where $r_h^{D-4} = \frac{g_{10}^2}{8\pi^5 V_{10-D}}$, $\vec{x}$ is a transverse $D-2$ vector and $r^2 = \vec{x}\cdot\vec{x}$. The string coupling behaves as

$$e^{2\Phi} = \frac{e^{2\Phi_0}}{1+N(\frac{r_h}{r})^{D-4}}, \tag{6.2}$$

and there is also a Kalb-Ramond field

$$H = dx^+ dx^- de^{2(\Phi-\Phi_0)}. \tag{6.3}$$

This spacetime is singular at the core of the string $r = 0$. Interestingly the string coupling goes to zero while the curvature diverges. This suggests the possibility that the singularity might be resolved within classical string theory by α' corrections.

6.1.2 Small 4d Black Holes and Small 5d Black Strings

Recently there have been compelling indicators [31, 33, 32, 116, 35, 27, 26, 28] that such a stringy resolution in fact occurs for the case $D = 5$. The story began with an S^1 compactification from 5 to 4 dimensions, in which the stretched strings are wrapped around the S^1 and become a pointlike object in $D = 4$. This "small black hole" has BPS excitations with momentum-winding (N, k) and a degeneracy that grows at large charges as $e^{4\pi\sqrt{Nk}}$. As above, this growth is not rapid enough to make a large black hole visible in supergravity, for which it is easily seen that the entropy must scale as the square of the charges. Nevertheless, the charges of the small black hole were plugged into the entropy formula derived in an α' expansion for large black holes and found to reproduce – to all orders! – the known BPS degeneracies [31, 33, 32].

This impressive agreement was surprising because the macroscopic derivation of the entropy as a function of the charges employs the known spacetime black hole attractor geometry [120] as an intermediate step. For small black holes, no such solutions were known. Subsequently, it was found that when stringy R^2 corrections to the supergravity equations are included, solutions with string-scale horizon do exist, and furthermore the horizon area scales in the right way with the charges [116]. Of course such solutions can only be regarded as suggestive, due to the ambiguities arising from field redefinitions and uncontrolled effects of R^4 and higher corrections. Nevertheless, the remarkable coherence of the small black hole picture suggests that we take them seriously. Ultimately the existence or not of these solutions should be addressed using worldsheet CFT methods, which can control all the α' corrections.

6.1.3 Small black strings

The near-horizon geometry of the small black holes contains an AdS_2 factor with an electric field associated to the Kaluza-Klein $U(1)$. Hence we have an S^1 fibered over the AdS_2. The total space of such a bundle is a quotient of AdS_3. Taking the cover of this quotient, we obtain the AdS_3 factor of the near horizon geometry of N stretched heterotic strings in 5 dimensions. Indeed the full 5d solutions were seen directly in T^5 compactification to 5 dimensions in a recent elegant paper CDKL (Castro, Davis, Kraus and Larsen) [27]. CDKL begin with the R^2-corrected $D = 5$ $\mathcal{N} = 2$ supersymmetry transformation laws and find half BPS solutions with a near-horizon $AdS_3 \times S^2$ factor and charges corresponding to N heterotic strings. In the heterotic frame, these solutions have a string coupling proportional to $\frac{1}{\sqrt{N}}$.

6.1.4 The near-horizon nonlinear superconformal group

The symmetries of the near-horizon region of this solution are of special interest. One expects the number of supersymmetries in the near horizon region to double from 8 to 16. But there are only four Lie superalgebras of classical type with 16 supercharges and an $SL(2,\mathbb{R})$ factor: $Osp(4^*|4)$, $SU(1,1|4)$, $F(4)$, and $Osp(8|2)$, with bosonic R-symmetry factors $SU(2)\times Sp(4)$, $SU(4)\times U(1)$, $SO(7)$, and $SO(8)$, respectively.[2] This is puzzling for two reasons. Firstly, R-symmetries usually arise geometrically as spacetime isometries – *e.g.* the $SO(6)$ R-symmetry corresponding to S^5 rotations of the near horizon D3 geometry. But in the current context that gives at most the $SU(2)$ rotations of the S^2 and so cannot account for the R-symmetry of any of the above supergroups. Secondly, as shown by Brown and Henneaux [22], when there is an AdS_3 factor the superisometry group must have an affine extension containing a Virasoro algebra. However there are no linear "$\mathcal{N}=8$" superconformal algebras containing any of the above superalgebras as global subalgebras.

We show herein by direct computation that the global superisometry group is in fact $Osp(4^*|4)$. As usual, the $SU(2)$ factor of the R-symmetry arises from the geometric rotational isometries of the S^2 R-symmetry. From the 5d point of view, the $Sp(4)\sim SO(5)$ arises unusually from the global $Sp(4)$ R-symmetry of 5d $\mathcal{N}=4$ supergravity. From the 10d point of view, these are the $SO(5)$ rotations of the spin frame for the T^5. This $SO(5)$ acts only on fermions and is not to be confused with spacetime rotations.

While this explains how the large global R-symmetry arises, it does not explain the

[2]See for example [51]. One may also consider the product group $D(2,1;\alpha)\times D(2,1;\alpha)$.

puzzle with the affine extension. While there are no linear superconformal algebras with more than 4 supercurrents (which means 8 global supercharges in the NS sector), there are a few *nonlinear* algebras with 8 supercurrents. These were classified some time ago by [94, 20, 21], and one of these algebras, let us denote it $Osp\hat{(}4^*|4)$, indeed contains $Osp(4^*|4)$ in the $k \to \infty$ limit. The nonlinearity in the commutation relation is confined to the commutator of the supercurrents, which takes the schematic form

$$\{G^I_r, G^J_s\} \sim 2\delta^{IJ} L_{r+s} + (r-s)(R^{IJ})_{r+s} + \sum_p (R^I{}_K)_{r+s-p}(R^{KJ})_p + \cdots \,, \tag{6.4}$$

where the current R^{IJ} generates the bosonic R-symmetry group. The nonlinear superconformal algebras are a special type of W algebra with only one spin two current and no higher currents. Though known for some time [125], these algebras have not seen many applications in string theory or elsewhere. We note that it is not fully understood when these algebras have unitary representations.

Fortuitously, consistent boundary conditions on AdS_3 with more than eight global supersymmetries and their associated asymptotic symmetry algebras were studied in [79]. The short list contains $Osp\hat{(}4^*|4)$. We conclude that the near-horizon symmetry algebra of the R^2-corrected supergravity solutions corresponding to N stretched heterotic strings is $Osp\hat{(}4^*|4)$.

6.1.5 $D \neq 5$

It is interesting to see how or if a picture could emerge in dimensions other than 5 consistent with the the known supergroups. Near-horizon symmetry enhancement suggests that there should always be 16 near-horizon supersymmetries.[3] In $D = 10$, it is natural to speculate that there is a stretched-string solution with an $AdS_3 \times S^7$ near-horizon region with the $Osp(8|2)$ superisometry group and a geometrically realized $SO(8)$ R-symmetry. In $D = 9$, $F(4)$ could arise with a geometrical $SO(7)$. It could also arise in $D = 3$, but with a nongeometrical $SO(7)$ from spin frame rotations of the T^7. In $D = 8$, the horizon is an S^5, so we could have $SU(1,1|4)$ with the $SU(4) \sim SO(6)$ geometrically realized and the $U(1)$ nongeometrical. $SU(1,1|4)$ is also a candidate for $D = 4$ with a nongeometrical $SO(6)$ and the $U(1)$ realized geometrically as rotations of the S^1 horizon. In $D = 7$, the horizon is an S^4 so one could again have $Osp(4^*|4)$, but with a geometrical $Sp(4) \sim SO(5)$ and a nongeometrical $SU(2)$. In $D = 6$, which is the self-dual dimension for strings, the near horizon geometry would be $AdS_3 \times S^3 \times T^4$. This has both a geometrical and a non-geometrical $SO(4)$, both of which have $SU(2)$ subgroups. This could correspond to two copies of $D(2,1;\alpha)$ with left and right actions, each containing an $SU(2) \times SU(2)$ R-symmetry. So for all $3 \leq D \leq 10$, there are candidate near-horizon supergroups with 16 supercharges.[4] Whether or not the solutions actually exist remains to be seen.

[3] 14 is another possibility, corresponding to the supergroups $G(3)$ with R-symmetry group G_2 or $Osp(7|2)$ with $SO(7)$.

[4] Candidates for near-horizon supergroups of type II strings can be obtained by taking left and right copies of the above, except for the case $D = 6$.

6.1.6 A worldsheet CFT?

As discussed above, the success of the small black hole/string story suggested that N heterotic strings in $D = 5$ have an $AdS_3 \times S^2$ near horizon region. The value of the string coupling goes to zero as $N \to \infty$ so that string loop corrections can be ignored. Such solutions were then found in the classical stringy R^2-corrected supergravity, but R^2-corrected supergravity is unreliable because α' corrections are uncontrolled. Such corrections, however, are controllable using worldsheet CFT methods, so either the authors of [116, 35, 27] and we were misled by the solutions of R^2-corrected supergravity, or an exact worldsheet CFT which describes the near-horizon geometry must exist.

The sought after worldsheet CFT cannot involve a RR background, as there are none in heterotic string theory. Furthermore, the large spacetime symmetry group places strong constraints on the worldsheet CFT [66, 97, 69, 68]. So if this CFT and the associated GSO projection exist, it should be possible to find them. Related and in some cases partial proposals have already appeared in [30, 67, 96] as well as [36, 87] which appeared as the present work was under submission. The closely related problem of finding the CFT which describes the S^2 horizon of a heterotic monopole was solved in [64]. We will review this construction and its application to the current problem in the last section.

Supposing the CFT does not exist for some or all cases, and we have simply been misled by the R^2 solutions, what are the possibilities? One is that there simply is no near horizon solution and that both supergravity and the exact classical string theory are singular at the core of the string. A second possibility, advocated in [67] (but at odds with the picture in CDKL), is that there is a smooth near horizon solution, but that some of the supersymmetries act trivially. Phenomena of this type are known in string theory. For example if we look at the magnetically charged black hole solutions of [64], for magnetic charge ± 1 the horizon part of the "throat" theory is trivial and $SO(3)$ spacetime rotations act trivially. Should this turn out to be the case we will need to understand in what sense the near-horizon spacetime is the holographic dual of the heterotic string CFT.

6.2 Near-Horizon Analysis

In this section we explicitly demonstrate, by finding the unbroken supersymmetries and computing their commutators, that the superisometry group of the near-horizon region of a fundamental heterotic string in R^2-corrected supergravity is $Osp(4^*|4)$. We employ the asymptotically flat solution of the BPS conditions found in CDKL. The CDKL analysis was in turn made possible by the recent supersymmetric completion of the relevant R^2 term in five dimensions [78], which descends from terms related to anomaly cancellation in the M-theory lift.

6.2.1 Supergravity in 5d

Five dimensional supergravity with $8n$ real supercharges, conventionally referred to as $\mathcal{N} = 2n$ supergravity, has an $Sp(2n)$ R-symmetry group with the supersymmetry parameter $\epsilon^i, \quad i = 1, \ldots, 2n$, transforming in the **2n**. CDKL work in an offshell $\mathcal{N} = 2$ formalism (which greatly simplifies the computation), but their solution can be embedded in an $\mathcal{N} = 4$ theory as follows. The $\mathcal{N} = 4$ gravitino variation has relevant terms of the form

$$\delta\psi^i_\mu \sim \nabla_\mu \epsilon^i + (F^{ij}_{\rho\sigma} + G_{\rho\sigma}\Omega^{ij})(\gamma_\mu{}^{\rho\sigma} - 4\delta_\mu{}^\rho \gamma^\sigma)\epsilon_j + \cdots , \tag{6.5}$$

where F^{ij} and G are 2-form field strengths in the **5** and **1** of $Sp(4)$ respectively. However, one can see upon dimensional reduction from $D = 10$ that the F^{ij} come from components of the metric g and anti-symmetric 2-form B which are mixed between $AdS_3 \times S^2$ and T^5, and these vanish in the present context. Under these circumstances, the $Sp(4)$ R symmetry is unbroken by the background and the $\mathcal{N} = 4$ variation looks exactly like that of $\mathcal{N} = 2$ but with $i = 1, \ldots, 4$, instead of $i = 1, 2$.

In $4 + 1$ dimensions there is no ordinary Majorana condition, but one can impose a symplectic-Majorana condition via

$$\bar{\xi}^i = \xi^\dagger_i \gamma^{\hat{0}} = \xi^{iT} C, \tag{6.6}$$

where C is the charge conjugation matrix and tangent-space indices are hatted. We will also use the symplectic matrix Ω_{ij} to raise and lower indices by

$$\xi^i = \Omega^{ij}\xi_j \qquad \xi_i = \xi^j \Omega_{ji}. \tag{6.7}$$

We choose a basis in which

$$\Omega_{12} = \Omega_{34} = -\Omega_{21} = -\Omega_{43} = 1. \tag{6.8}$$

The string solution in supergravity has $ISO(1,1)\times SO(3)$ isometry. It is convenient to choose lightcone coordinates along the string $x^\pm = x^0 \pm x^1$, and spherical coordinates r, θ, ϕ for the transverse directions. In conformity with CDKL, we take the tangent space metric to have signature $(+----)$.

We work in a representation of the Clifford algebra with $\gamma^{\hat{0}}$ Hermitian, and the other $\gamma^{\hat{\mu}}$ anti-Hermitian. Consistent with this, we can choose $\gamma^{\hat{1}}$ to be real while the others are pure imaginary. The charge conjugation matrix $C = \gamma^{\hat{0}\hat{1}}$ satisfies

$$C\gamma^{\mu}C^{-1} = \gamma^{\mu T}. \tag{6.9}$$

As this is a non-chiral theory, we choose $\gamma^{\hat{0}\hat{1}\hat{r}\hat{\theta}\hat{\phi}} = 1$ where the indices are tangent space indices, raised and lowered by $-\eta_{\hat{0}\hat{0}} = \eta_{\hat{1}\hat{1}} = \eta_{\hat{r}\hat{r}} = \eta_{\hat{\theta}\hat{\theta}} = \eta_{\hat{\phi}\hat{\phi}} = -1$. Note that $\gamma^{\mu_1 \cdots \mu_p}$ is always the antisymmetric combination divided by $p!$.

6.2.2 Killing spinors

The CDKL solution has an $AdS_3 \times S^2$ near horizon region with metric

$$ds^2 = \frac{r}{l}dx^+dx^- - \frac{l^2}{r^2}dr^2 - l^2 d\Omega_2. \tag{6.10}$$

Choosing the vielbein

$$e^{\hat{+}}{}_{+} = e^{\hat{-}}{}_{-} = \sqrt{\frac{r}{l}}, \qquad e^{\hat{r}}{}_{r} = \frac{l}{r}, \qquad e^{\hat{\theta}}{}_{\theta} = l, \qquad e^{\hat{\phi}}{}_{\phi} = l\sin\theta, \tag{6.11}$$

the only non-zero components of the spin connection are

$$\omega_{\phi}{}^{\hat{\theta}\hat{\phi}} = \cos\theta, \qquad \omega_{+}{}^{\hat{r}\hat{+}} = \omega_{-}{}^{\hat{r}\hat{-}} = \frac{1}{2}\sqrt{\frac{r}{l^3}}. \tag{6.12}$$

The Weyl multiplet of $5d$ $\mathcal{N} = 2$ conformal supergravity contains an auxiliary 2-form $v_{\mu\nu}$ (related to G in (6.5)) which is $v_{\hat{\theta}\hat{\phi}} = \frac{3}{4l}$ in this background. In terms of v the precise version of the gravitino variation (6.5) is

$$\delta_{\epsilon}\psi^i_{\mu} = (\nabla_{\mu} + \frac{1}{2}v_{\nu\rho}\gamma^{\nu\rho}{}_{\mu} - \frac{1}{3}v_{\nu\rho}\gamma_{\mu}\gamma^{\nu\rho})\epsilon^i. \tag{6.13}$$

As mentioned above, because our background preserves R-symmery the R-symmetry index just goes along for the ride. There is also a second fermion χ as well as gauginos whose variations determine the scalar auxilliary field D and field strengths, but turn out to not further constrain the Killing spinor and so shall not concern us here.

Let's first consider the $\delta\psi^i_r$ variation. There are two solutions with r-dependence $(r/l)^{\pm 1/4}$ and satisfyng the projection $\gamma^{\hat{r}\hat{\theta}\hat{\phi}}\epsilon^i_\pm = \pm\epsilon^i_\pm$. Further, the two solutions are related by $\epsilon^i_- = (\sqrt{l/r})\gamma^{\hat{+}\hat{r}}\epsilon^i_+$. Denote by ϵ^i the spinor satisfying $\gamma^{\hat{r}\hat{\theta}\hat{\phi}}\epsilon^i = \epsilon^i$. From the $\delta\psi^i_\pm$ variations we find that ϵ^i is independent of $x^\pm$, and that there is another solution of the form[5]

$$\lambda^i = -\frac{x^+}{l}\epsilon^i + \sqrt{\frac{l}{r}}\gamma^{\hat{+}\hat{r}}\epsilon^i. \tag{6.14}$$

Solving the angular variations for ϵ^i gives

$$\epsilon^i = (\frac{r}{l})^{1/4} e^{\frac{\theta}{2}\gamma^{\hat{\phi}}} e^{-\frac{\phi}{2}\gamma^{\hat{\theta}\hat{\phi}}}\epsilon^i_0, \tag{6.15}$$

where ϵ_0 is a constant spinor which satisfies $\gamma^{\hat{r}\hat{\theta}\hat{\phi}}\epsilon^i_0 = \epsilon^i_0$. In addition, the λ^i given in terms of ϵ^i in (6.14) remain solutions as well since these angular variations commute with $\gamma^{\hat{+}\hat{r}}$. So all in all we have 16 near horizon supersymmetries.

6.2.3 Killing Vectors

In order to determine the complete supergroup, we need to understand the action of the (right-handed) $SL(2,\mathbb{R})$ bosonic symmetries on the Killing spinors. The $SL(2,\mathbb{R})$ Killing vectors are

$$L_{-1} = l\partial_+, \qquad L_0 = -x^+\partial_+ + r\partial_r, \qquad L_1 = \frac{(x^+)^2}{l}\partial_+ - \frac{2x^+r}{l}\partial_r + \frac{4l^2}{r}\partial_-. \tag{6.16}$$

[5]The λ^i are the enhanced supersymmetries of the near-horizon region. This equation expresses them in terms of the Lie derivative with respect to an $SL(2,\mathbb{R})$ Killing vector acting on ϵ^i.

Using these we find that[6]

$$L_0\epsilon^i = \frac{1}{2}\epsilon^i, \qquad L_0\lambda^i = -\frac{1}{2}\lambda^i, \qquad L_1\epsilon^i = \lambda^i, \qquad L_{-1}\lambda^i = -\epsilon^i, \tag{6.17}$$

which identifies ϵ^i and λ^i respectively with the $-\frac{1}{2}$ and $+\frac{1}{2}$ modes of G obeying $[L_m, G_r] = (\frac{m}{2} - r)G_{m+r}$.

Similarly the $SU(2)$ action is generated by

$$J_0^3 = -i\partial_\phi, \qquad J_0^\pm = e^{\pm i\phi}(-i\partial_\theta \pm \cot\theta\partial_\phi). \tag{6.18}$$

Since $\gamma^{\hat{r}\hat{\theta}\hat{\phi}}$ and $\gamma^{\hat{\theta}\hat{\phi}}$ commute we can define

$$\gamma^{\hat{\theta}\hat{\phi}}\epsilon_0^i = \mp i\epsilon_0^i, \tag{6.19}$$

and it is easy to check that these satisfy

$$J_0^3\epsilon^i = \pm\frac{1}{2}\epsilon^i. \tag{6.20}$$

Suppose we start with a constant spinor obeying $\gamma^{\hat{\theta}\hat{\phi}}\epsilon_0 = -i\epsilon_0$ as well as $\epsilon_0 = -i\gamma^{\hat{0}\hat{\theta}}\epsilon_0^*$, and normalized to $\epsilon_0^\dagger\epsilon_0 = \frac{1}{4}$. Then we can define

$$\begin{aligned}
\xi_-^1 &= (\frac{r}{l})^{1/4}e^{\frac{\theta}{2}\gamma^{\hat{\phi}}}e^{-\frac{i}{2}\phi}(\gamma^{\hat{\theta}}\epsilon_0), \\
\xi_+^1 &= (\frac{r}{l})^{1/4}e^{\frac{\theta}{2}\gamma^{\hat{\phi}}}e^{\frac{i}{2}\phi}\epsilon_0, \\
\xi_-^2 &= (\tfrac{r}{l})^{1/4}e^{\frac{\theta}{2}\gamma^{\hat{\phi}}}e^{-\frac{i}{2}\phi}(-\gamma^{\hat{\theta}}\epsilon_0^*), \\
\xi_+^2 &= (\tfrac{r}{l})^{1/4}e^{\frac{\theta}{2}\gamma^{\hat{\phi}}}e^{\frac{i}{2}\phi}(-\gamma^{\hat{0}\hat{\theta}}\epsilon_0^*),
\end{aligned} \tag{6.21}$$

where ξ^a is a **2** of $SU(2)$, $J_0^\pm\xi_\pm^a = 0$ and $J_0^\pm\xi_\mp^a = \xi_\pm^a$. We can organize these into

[6] With the action defined via the Lie derivative $\mathcal{L}_K\epsilon = K^\mu\nabla_\mu\epsilon + \frac{1}{4}\partial_\mu K_\nu\gamma^{\mu\nu}\epsilon$.

symplectic-Majorana Killing spinors

$$\epsilon^{(1)} = \begin{pmatrix} \xi_+^1 \\ -\xi_-^2 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^{(2)} = \begin{pmatrix} \xi_+^2 \\ -\xi_-^1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^{(3)} = \begin{pmatrix} \xi_-^1 \\ \xi_+^2 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^{(4)} = \begin{pmatrix} \xi_+^1 \\ -\xi_-^2 \\ 0 \\ 0 \end{pmatrix}$$
$$\epsilon^{(5)} = \begin{pmatrix} 0 \\ 0 \\ \xi_+^1 \\ -\xi_-^2 \end{pmatrix}, \quad \epsilon^{(6)} = \begin{pmatrix} 0 \\ 0 \\ \xi_+^2 \\ -\xi_-^1 \end{pmatrix}, \quad \epsilon^{(7)} = \begin{pmatrix} 0 \\ 0 \\ \xi_-^1 \\ \xi_+^2 \end{pmatrix}, \quad \epsilon^{(8)} = \begin{pmatrix} 0 \\ 0 \\ -\xi_+^2 \\ -\xi_+^1 \end{pmatrix} \qquad (6.22)$$

where each $\epsilon^{(I)}$ transforms as a **4** of $Sp(4)$ by left-multiplication (see appendix D). We will identify these with $G^I_{-\frac{1}{2}}$, $I = 1, \ldots, 8$. Following the same procedure we can define

$$\eta^a_\pm = -\frac{x^+}{l}\xi^a_\pm + \sqrt{\frac{l}{r}}\gamma^{\hat{+}\hat{r}}\xi^a_\pm \qquad (6.23)$$

and group them into symplectic-Majorana **4**'s which will be identified with $G^I_{\frac{1}{2}}$.

6.2.4 Supercharge commutators

Commutators of supercharges can be expressed as fermion bilinears involving the corresponding Killing spinors. In particular, [58] determines

$$\begin{aligned} \{G^I_r, G^J_s\} \;\sim\; & \Omega_{ij}\left((\bar{\epsilon}^{(I)}_r)^i\gamma^\mu(\epsilon^{(J)}_s)^j + (\bar{\epsilon}^{(J)}_s)^i\gamma^\mu(\epsilon^{(I)}_r)^j\right)\partial_\mu \\ & + \left((\bar{\epsilon}^{(I)}_r)_i\gamma^{\hat{\theta}\hat{\phi}}(\epsilon^{(J)}_s)^j + (\bar{\epsilon}^{(J)}_s)_i\gamma^{\hat{\theta}\hat{\phi}}(\epsilon^{(I)}_r)^j\right), \end{aligned} \qquad (6.24)$$

where $\epsilon^{(I)}_{-\frac{1}{2}} = \epsilon^{(I)}$ and $\epsilon^{(I)}_{\frac{1}{2}} = \lambda^{(I)}$. The first line of (6.24) involves the spacetime Killing vectors of $SL(2,\mathbb{R}) \times SU(2)$ and the second involves the the generators of $Sp(4)$.

Using our previous normalizations and the notation in appendix D, we find

$$\{G^I_{\pm\frac{1}{2}}, G^J_{\pm\frac{1}{2}}\} = -2\delta^{IJ} L_{\pm 1} \tag{6.25}$$

for $I, J = 1, \ldots, 8$. Also,

$$\{G^{I_1}_{\frac{1}{2}}, G^{J_1}_{-\frac{1}{2}}\} = \begin{pmatrix} -2L_0 & -2iJ_0^3 + iA_3 & 2iJ_0^2 + iA_1 & 2iJ_0^1 + iA_2 \\ 2iJ_0^3 - iA_3 & -2L_0 & 2iJ_0^1 - iA_2 & -2iJ_0^2 + iA_1 \\ -2iJ_0^2 - iA_1 & -2iJ_0^1 + iA_2 & -2L_0 & -2iJ_0^3 - iA_3 \\ -2iJ_0^1 - iA_2 & 2iJ_0^2 - iA_1 & 2iJ_0^3 + iA_3 & -2L_0 \end{pmatrix}, \tag{6.26}$$

where $I_1, J_1 = 1, \ldots, 4$. If $I_2, J_2 = 5, \ldots, 8$, the same table arises with A_α replaced by C_α. If $I_1 = 1, \ldots, 4$, and $J_2 = 5, \ldots, 8$,

$$\{G^{I_1}_{\frac{1}{2}}, G^{J_2}_{-\frac{1}{2}}\} = \begin{pmatrix} iB_4 & iB_3 & iB_1 & iB2 \\ -iB_3 & iB_4 & -iB_2 & iB_1 \\ -iB_1 & iB_2 & iB_4 & -iB_3 \\ -iB_2 & -iB_1 & iB_3 & iB_4 \end{pmatrix}. \tag{6.27}$$

These are just the commutation relations of $Osp(4^*|4)$, written below in a more compact form (assuming we rotate $G \to iG$):

$$\begin{aligned} \{G^I_r, G^J_s\} &= 2L_{r+s}\delta^{IJ} + (r-s)(t_\alpha)^{IJ} J_0^\alpha + (r-s)(\rho_A)^{IJ} R_0^A \\ \left[L_m, G^I_s\right] &= (\frac{m}{2} - s) G^I_{m+s}, \\ \left[R_0^A, G^I_r\right] &= (\rho^A)^{IJ} G^J_r \\ \left[J_0^\alpha, G^I_r\right] &= (t^\alpha)^{IJ} G^J_r, \end{aligned} \tag{6.28}$$

where t^α and ρ^A are the representation matrices for $SU(2)$ and $Sp(4)$ respectively, and R^A are the generators of $Sp(4)$. In the first two lines of (6.28), it should be understood we have only computed the global part of the superalgebra.

6.3 Towards an Exact Worldsheet CFT

In this section, we review and point out that the old results of [64] may be relevant to the problem of finding an exact worldsheet dual. We note that with the obvious adaptation of the GSO projection used in [64] one does not realize the needed 16 supercharges, so something more is needed to get a fully viable candidate for the worldsheet CFT.

6.3.1 4d heterotic black monopoles

Heterotic string theory in four dimensions contains macroscopic black hole solutions [61] with magnetic charges lying in a $U(1)$ subgroup of $E_8 \times E_8$. Since the charges are associated with the left-moving sector of the worldsheet, such solutions are generically non-supersymmetric. The near horizon region is the product of 2D Minkowski space with a linear dilaton and an S^2 threaded with magnetic flux.

For every classical solution there should be a corresponding worldsheet CFT. In this case the CFT is rather subtle but was eventually found in [64]. While S^3 factors such as those arising in the near horizon for the NS5-brane are easily recognized as $SU(2)$ WZW models (which have $SU(2)_L$ and $SU(2)_R$ current algebras corresponding to the S^3 isometry group) it is harder to see where an S^2 horizon comes from (which has only one $SU(2)$ isometry). It turns out that it is given by an asymmetric orbifold of level $k = 2|Q^2 - 1|$ WZW model of the form

$$\frac{SU(2)_{2|Q^2-1|} \times SU(2)_{2|Q^2-1|}}{\mathbb{Z}_{2Q+2}}, \tag{6.29}$$

where Q is the monopole charge. (6.29) can be viewed as a two sphere with a left and a right fiber $U(1)_L$ and $U(1)_R$. The $U(1)_L$ fiber comes from the $U(1)$ subgroup of $E_8 \times E_8$ and the Chern class of the fibration is determined by the monopole charge. On the right, one has two fermions which are superpartners of the the two coordinates of the S^2 horizon and live in the tangent bundle. These can be bosonized to a $U(1)_R$ boson which also has a nontrivial fibration. The total space of the S^2 horizon together with its bosonized right-moving superpartners and the left-moving current $U(1)_L$ boson was shown in [64] to be given by (6.29), with a specified action for the $\mathbb{Z}_{2Q+2}$ quotient.

6.3.2 5d monopole-heterotic strings

We wish to consider two modifications of the construction of [64]. First, by trading a compact dimension for a trivial flat dimension, we can uplift the CFT to one describing a monopole string in five dimensions. Second, we replace the 3d $M^2 \times$ (*linear dilaton*) factor with a $(0,1)$ $SL(2,\mathbb{R})_{k+4}$ WZW model[7] (representing an AdS_3 factor) with the same central charges $(c_L, c_R) = (3 + \frac{6}{k+2}, \frac{9}{2} + \frac{6}{k+2})$ and constant dilaton.

The presence of H flux on the AdS_3 factor indicates that the monopole string also carries fundamental string charge. The number N of heterotic strings behaves as

$$N \sim \int_{S^2 \times M_5} e^{-2\Phi} * H \sim \frac{k}{g_5^2}. \tag{6.30}$$

We wish to have weakly coupled string theory so N must be large.

[7]The supersymmetric right side contains bosonic level $k+4$ $SL(2,\mathbb{R})$ current j^A and a supersymmetric level $k+2$ $SL(2,\mathbb{R})$ current J^A.

6.3.3 Q=0: the heterotic string near-horizon

An intriguing feature of the construction of [64] is that it is nonsingular for the case $Q = 0$ which corresponds to $k = 2$. This case was referred to as the "neutral remnant" in [64]. $k = 2$ can be described by 3 left and 3 right free fermions, and the $\mathbb{Z}_2$ quotient in (6.29) acts purely on the left as a 2π rotation. One then expects that with the modifications of the previous subsection, the case $k = 2$ corresponds to the near-horizon geometry of N strings. However, to define the theory we must specify the GSO projection (with the $SL(2, \mathbb{R})$ factor there seems to be more than one way to do this), and the obvious adaptation of the one given in [64] does not give the needed spacetime supersymetries. Possibly a different value of k[8] or modified GSO will give the desired theory.

[8]The construction of [66] naively indicates that the value $k = 0$ (which gives bosonic $SL(2, R)$ currents at level 4), gives the desired spacetime central charge, but this construction requires modifications for a nonlinear superconformal algebra.

Appendix A

Collective Field Description of Matrix Cosmologies

In this Appendix, we will analyze an example which does not fall into the restricted category of solutions analyzed in section 2.4. We return to the general case from section 2.3, and, using only the property (2.14), we write the cubic part of the action as

$$S_{(3)} = \frac{\sqrt{\pi}}{2}\int d\sigma d\tau \frac{1}{6\varphi_0|\partial_x\tau^+\partial_x\tau^-|}\Big\{ \left((\partial_x\tau^-)^3 - (\partial_x\tau^+)^3\right)\left((\partial_\sigma\eta)^3 + 3\partial_\sigma\eta(\partial_\tau\eta)^2\right) - \left((\partial_x\tau^+)^3 + (\partial_x\tau^-)^3\right)\left(3(\partial_\sigma\eta)^2\partial_\tau\eta + (\partial_\tau\eta)^3\right)\Big\} . \qquad (A.1)$$

For the couplings in this action to be time independent, as in equation (2.40), $\partial_x\tau^\pm/\varphi_0$ must be a function of σ only. We will analyze this condition in a specific example.

Consider the Fermi surface given by

$$x^2 - p^2 = 1 + (x-p)^3 e^{3t} . \qquad (A.2)$$

Parametrically, this surface is given by

$$\begin{aligned} x &= \cosh\omega + \frac{1}{2}e^{3t-2\omega} \\ p &= \sinh\omega + \frac{1}{2}e^{3t-2\omega} \; . \end{aligned} \tag{A.3}$$

Since the parametric form is similar to the one given in [7], we use the procedure given there to define the Alexandrov coordinates

$$\tau^+ = t - \omega \; , \quad \tau^- = t - \tilde{\omega} \; , \tag{A.4}$$

where $\tilde{\omega}$ is defined by $x(\tilde{\omega}, t) = x(\omega, t)$ as well as $p(\omega, t) = p_+$ and $p(\tilde{\omega}, t) = p_-$. It is possible to solve for x, t and $p_\pm$ as functions of $\tau^\pm$:

$$\begin{aligned} x(\tau^\pm) &= -\frac{e^{2\tau^+ + 2\tau^-} - e^{\tau^+} - e^{\tau^-}}{2\sqrt{e^{\tau^+ + \tau^-} - e^{2\tau^+ + 3\tau^-} - e^{3\tau^+ + 2\tau^-}}} \\ \exp(t(\tau^\pm)) &= -\frac{\sqrt{e^{\tau^+ + \tau^-} - e^{2\tau^+ + 3\tau^-} - e^{3\tau^+ + 2\tau^-}}}{e^{2\tau^+ + \tau^-} + e^{\tau^+ + 2\tau^-} - 1} \\ p_+(\tau^\pm) &= \frac{e^{2\tau^+ + 2\tau^-} + 2e^{3\tau^+ + \tau^-} + e^{\tau^-} - e^{\tau^+}}{2\sqrt{e^{\tau^+ + \tau^-} - e^{2\tau^+ + 3\tau^-} - e^{3\tau^+ + 2\tau^-}}} \\ p_-(\tau^\pm) &= \frac{e^{2\tau^+ + 2\tau^-} + 2e^{3\tau^- + \tau^+} + e^{\tau^+} - e^{\tau^-}}{2\sqrt{e^{\tau^+ + \tau^-} - e^{2\tau^+ + 3\tau^-} - e^{3\tau^+ + 2\tau^-}}} \; . \end{aligned} \tag{A.5}$$

The coordinates given here have the property that the edge of the Fermi sea ($p_+ = p_-$) is at $2\sigma = \tau^+ - \tau^- = 0$. It is now possible to compute $\partial_x \tau^\pm / \varphi_0$. Not surprisingly, this is not a function of σ only. The question is whether, by a suitable conformal change of coordinates to $\bar{\tau}^\pm$, this condition could be satisfied. The change of coordinates would have to map $\sigma = 0$ to itself to maintain a static Fermi sea edge in the new coordinates. Thus, the change of coordinates must be of the form $\tau^\pm = f(\bar{\tau}^\pm)$, with $f(\cdot)$ an arbitrary function. Define $Q_\pm \equiv \partial_x \tau^\pm / \varphi_0$. The necessary condition is then

$$0 = \partial_{\bar{\tau}} Q_\pm = f'(\bar{\tau}^+) \partial_{\tau^+} Q_\pm + f'(\bar{\tau}^-) \partial_{\tau^-} Q_\pm \tag{A.6}$$

implying that $\partial_{\tau^-} Q_\pm / \partial_{\tau^+} Q_\pm$ is of the form

$$W_\pm(\tau^+, \tau^-) \equiv \frac{\partial_{\tau^-} Q_\pm}{\partial_{\tau^+} Q_\pm} = -\frac{f'(\bar{\tau}^+)}{f'(\bar{\tau}^-)} = -\frac{F(\tau^+)}{F(\tau^-)} \ . \tag{A.7}$$

Therefore,

$$W_\pm(\tau^+, \tau^-) W_\pm(\tau^-, \tau^+) = 1 \ . \tag{A.8}$$

By explicit computation, it can be checked that this condition is not satisfied. Therefore, there does not exist a coordinate transformation after which $S_{(3)}$ has no τ dependence.

Appendix B

Towards the Massless Spectrum of Non-Kähler Heterotic

Compactifications

B.1 Einstein/String Frame Actions and EOM's

We start with the ten-dimensional action in Einstein-frame, as is found in Chapter 13 (p325) of [71], with the substitution $\phi_{GSW} = \exp[\phi/2 + 2\ln(\kappa/g)]$:

$$\begin{aligned} L^{(E)}_{Het} &= -\tfrac{1}{2}\sqrt{-G}\Big[\tfrac{1}{\kappa^2}R + \tfrac{1}{2\kappa^2}D_M\phi D^M\phi + \tfrac{1}{2\kappa^2}e^{-\phi/2}\mathrm{Tr}(F^2) + \tfrac{3}{4\kappa^2}e^{-\phi}H^2 \\ &+ \bar{\psi}_M\Gamma^{MNP}D_N\psi_P + \bar{\lambda}\Gamma^M D_M\lambda + \mathrm{Tr}(\bar{\chi}\Gamma^M D_M\chi) \\ &+ \frac{1}{\sqrt{2}}\bar{\psi}_M\Gamma^N\Gamma^M\lambda D_N\phi - \tfrac{1}{8}e^{-\phi/2}\mathrm{Tr}(\bar{\chi}\Gamma^{MNP}\chi)H_{MNP} \\ &+ \frac{1}{2}e^{-\phi/4}\mathrm{Tr}(\bar{\chi}\Gamma^M\Gamma^{NP}(\psi_M + \frac{\sqrt{2}}{12}\Gamma_M\lambda)F_{NP}) \\ &- \tfrac{1}{8}e^{-\phi/2}\big(\bar{\psi}_M\Gamma^{MNPQR}\psi_R + 6\bar{\psi}^N\Gamma^P\psi^Q - \sqrt{2}\bar{\psi}_M\Gamma^{NPQ}\Gamma^M\lambda\big)H_{NPQ} \\ &+ (\mathrm{Fermions})^4\Big], \end{aligned} \tag{B.1}$$

where D_M is the Levi-Civita connection plus the gauge connection. We know that if we rescale the metric $G_{MN} \to e^{p\phi}G_{MN}$, then $\sqrt{-G} \to e^{pd\phi/2}\sqrt{-G}$ and

$$R \to e^{-p\phi}\left(R - p(d-1)D^2\phi - p^2\frac{(d-1)(d-2)}{4}D_M\phi D^M\phi\right) \tag{B.2}$$

for GSW conventions.

Since we want an overall factor of $e^{-2\phi}$ in front in string frame, we must choose

$p = -\frac{1}{2}$. Under this rescaling, we have

$$\begin{aligned} G^{(E)}_{MN} &= e^{-\phi/2} G_{MN}, & R^{(E)} &= e^{\phi/2}(R - \tfrac{9}{2} D_M\phi D^M\phi), \\ \lambda^{(E)} &= e^{\phi/8}\lambda, & \chi^{(E)} &= e^{\phi/8}\chi, \\ \psi^{(E)}_M &= e^{-\phi/8}\psi_M, & \Gamma^{(E)}_M &= e^{-\phi/4}\Gamma_M, \\ \epsilon^{(E)} &= e^{-\phi/8}\epsilon, \end{aligned} \tag{B.3}$$

where ϵ is the spinor appearing in the supersymmetry variations. The Levi-Civita connection has additional terms depending on derivatives of ϕ:

$$\begin{aligned} D^{(E)}_M \lambda &= D_M\lambda + \tfrac{1}{8}\Gamma^N{}_M \lambda D_N\phi, \\ D^{(E)}_M V_N &= D_M V_N + \tfrac{1}{4}\big(V_M D_N\phi + V_N D_M\phi - G_{MN} V^P D_P\phi\big), \\ \Gamma^{(E)M}{}_{NP} &= \Gamma^M_{NP} - \tfrac{1}{4}\big(\delta^M_N D_P\phi + \delta^M_P D_N\phi - G_{NP} D^M\phi\big), \end{aligned} \tag{B.4}$$

where V_M is a spacetime 1-form.

After some simplification, the string-frame action is

$$\begin{aligned} L^{(S)}_{Het} &= -\tfrac{1}{2} e^{-2\phi}\sqrt{-G}\Big[\tfrac{1}{\kappa^2}R - \tfrac{4}{\kappa^2} D_M\phi D^M\phi + \tfrac{1}{2\kappa^2}\mathrm{Tr}(F^2) + \tfrac{3}{4\kappa^2}H^2 \\ &+ \bar\psi_M \Gamma^{MNP} D_N \psi_P + \bar\psi_M \Gamma^{MP}\Gamma^N \psi_P D_N\phi + \bar\lambda \Gamma^M D_M \lambda \\ &- \bar\lambda\Gamma^M \lambda D_M\phi + \mathrm{Tr}\big(\bar\chi\Gamma^M D_M\chi - \bar\chi\Gamma^M\chi D_M\phi\big) \\ &+ \frac{1}{2}\mathrm{Tr}\Big(\bar\chi\Gamma^M\Gamma^{NP}(\psi_M + \frac{\sqrt{2}}{12}\Gamma_M\lambda)F_{NP}\Big) + \tfrac{1}{\sqrt{2}}\bar\psi_M\Gamma^N\Gamma^M\lambda D_N\phi \\ &- \frac{1}{8}\mathrm{Tr}\big(\bar\chi\Gamma^{MNP}\chi\big)H_{MNP} - \frac{1}{8}\Big(\bar\psi_M\Gamma^{MNPQR}\psi_R + 6\bar\psi^N\Gamma^P\psi^Q \\ &- \sqrt{2}\bar\psi_M\Gamma^{NPQ}\Gamma^M\lambda\Big)H_{NPQ} + (\text{Fermions})^4. \end{aligned} \tag{B.5}$$

B.2 Useful Relations

B.2.1 SUSY Implications

A few things to note. First, working in string-frame implies that

$$D_\mu \eta = 0. \tag{B.6}$$

Second, for $D_M\Omega$ we have

$$\begin{aligned} D_\mu \Omega_{abc} &= 0, \\ D_d \Omega_{abc} &= -3(D_d\phi)\Omega_{abc}, \\ D_{\bar{a}} \Omega_{abc} &= -(D_{\bar{a}}\phi)\Omega_{abc}, \end{aligned} \tag{B.7}$$

all of which follow from the fermionic supersymmetry variations, as do

$$\begin{aligned} \frac{3}{2} H_{\bar{a}d}{}^{d} &= D_{\bar{a}}\phi, \\ \frac{3}{2} H_{ad}{}^{d} &= -D_a\phi, \\ 4\gamma^m D_m \phi \eta &= H_{mnp}\gamma^{mnp}\eta. \end{aligned} \tag{B.8}$$

See [118] for details.

B.2.2 A Note about Non-Kähler Manifolds

One of the drawbacks to working with non-Kähler geometries is that, contrary to one's naive expectation,

$$\nabla_m(\Omega_{abc}\gamma^{abc}) \neq \gamma^{abc}\nabla_m\Omega_{abc}. \tag{B.9}$$

This arises from the fact that we are only summing over holomorphic indices and not all real indices; thus, the complex structure is implicitly used and we recall that, unlike in the Kähler case, the complex structure is not covariantly constant with respect to the Levi-Civita connection. To use the product rule, we must write everything in real indices.

Define

$$P_{\pm m}{}^{n} \equiv \frac{1}{2}(\delta_m{}^n \mp iJ_m{}^n) \tag{B.10}$$

so that $P_{+a}{}^{b} = \delta_a{}^b$, $P_{-\bar{a}}{}^{\bar{b}} = \delta_{\bar{a}}{}^{\bar{b}}$, and all other components are zero. Since $\nabla^{(H)}_m J = 0$, we find

$$\nabla_m P_{\pm n}{}^{p} = \mp\frac{3i}{4}\left(H^p{}_{ms}J_n{}^s - H^s{}_{mn}J_s{}^p\right). \tag{B.11}$$

Thus,

$$\begin{aligned}\nabla_m(C_{abc}\gamma^{abc}) &= \nabla_m\left(C_{pst}\gamma^{nqr}P_{+n}{}^{p}P_{+q}{}^{s}P_{+r}{}^{t}\right)\\ &= \gamma^{abc}\nabla_m C_{abc} + 3C_{pbc}\gamma^{nbc}\nabla_m P_{+n}{}^{p}\\ &= \gamma^{abc}\nabla_m C_{abc} + \frac{9}{2}C_{abc}\gamma^{\bar{a}bc}H^a{}_{m\bar{a}}.\end{aligned} \tag{B.12}$$

Similarly,

$$\nabla_m(C_{ab}\gamma^{ab}) = \gamma^{ab}\nabla_m C_{ab} + 3C_{ab}\gamma^{\bar{a}b}H^a{}_{m\bar{a}} \tag{B.13}$$

and

$$\nabla_m(C_a\gamma^a) = \gamma^a\nabla_m C_a + \frac{3}{2}C_a\gamma^{\bar{a}}H^a{}_{m\bar{a}}. \tag{B.14}$$

B.3 Derivation of (4.30)

When we consider a gauge bundle with structure group G and embed this into a larger group H ($E_8 \times E_8$ or $SO(32)$), the adjoint of H decomposes into a sum containing the adjoint of G. Since we focus on variations of the gaugino orthogonal to the adjoint of G, the linearized gaugino equation of motion becomes

$$0 = 2\Gamma^M D_M\chi - 2\Gamma^M \chi D_M\phi - \frac{1}{4}\Gamma^{MNP}\chi H_{MNP}, \tag{B.15}$$

where we have suppressed gauge indices.

The most general Ansatz for the variation of the gaugino is

$$\delta\chi = \epsilon_- \otimes \left(C\eta + C_{ab}\gamma^{ab}\eta\right) - \epsilon_+ \otimes \left(\bar{C}\eta^* + \bar{C}_{\bar{a}\bar{b}}\gamma^{\bar{a}\bar{b}}\eta^*\right), \tag{B.16}$$

where $C, C_{ab} \in \Omega^*(K;V)$ and $\epsilon_\pm$ are covariantly constant spinors on $\mathcal{M}_4$ with chiralities ± 1 respectively. To simplify the gaugino equation of motion, we note that

$$\nabla_a\eta = \tfrac{3}{4}H_{ab\bar{a}}\gamma^{b\bar{a}}\eta + \tfrac{3}{8}H_{a\bar{a}\bar{b}}\gamma^{\bar{a}\bar{b}}\eta = -\tfrac{3}{4}H_{ab}{}^{b}\eta = \tfrac{1}{2}(\partial_a\phi)\eta \tag{B.17}$$

$$\nabla_{\bar{a}}\eta = \tfrac{3}{4}H_{\bar{a}\bar{b}a}\gamma^{\bar{b}a}\eta + \tfrac{3}{8}H_{\bar{a}ab}\gamma^{ab}\eta = -\tfrac{1}{2}(\partial_{\bar{a}}\phi)\eta + \tfrac{3}{8}H_{\bar{a}ab}\gamma^{ab}\eta, \tag{B.18}$$

which follow from $\nabla^{(H)}_m\eta = 0$, $\gamma^{\bar{a}}\eta = 0$, and (B.8).

We will focus on the terms involving ϵ_- as the others are obtained by complex conjugation and multiplication by the ten-dimensional charge conjugation operator. Using the relations above, (B.13), and the fact that the product of more than three gamma matrices with all holomorphic or anti-holomorphic indices is zero (since we work on a complex 3-fold), we have

$$
\begin{aligned}
\Gamma^M D_M \delta\chi\Big|_{\epsilon_-} &= (\gamma^\mu D_\mu \epsilon_-) \otimes (C\eta + C_{ab}\gamma^{ab}\eta) - \epsilon_- \otimes \gamma^m D_m (C\eta + C_{ab}\gamma^{ab}\eta) \\
&= -\epsilon_- \otimes \big\{ \gamma^a \left[\left(D_a (C + C_{bc}\gamma^{bc}) \right) \eta + (C + C_{bc}\gamma^{bc}) D_a \eta \right] \\
&\qquad + \gamma^{\bar a} \left[\left(D_{\bar a} (C + C_{bc}\gamma^{bc}) \right) \eta + (C + C_{bc}\gamma^{bc}) D_{\bar a} \eta \right] \big\} \\
&= -\epsilon_- \otimes \Big\{ \Big[\left((D_a C)\gamma^a + (D_a C_{bc})\gamma^{abc} + 3 C_{bc} H^b{}_a{}^c \gamma^a \right) \eta \\
&\qquad + \left(\tfrac{1}{2} (\partial_a \phi) C \gamma^a + \tfrac{1}{2} (\partial_a \phi) C_{bc} \gamma^{abc} \right) \eta \Big] + \Big[\left((D_{\bar a} C_{bc}) \gamma^{\bar a} \gamma^{bc} \right) \eta \\
&\qquad + \left(\tfrac{3}{8} C H_{\bar a ab} \gamma^{\bar a} \gamma^{ab} - \tfrac{1}{2} (\partial_{\bar a} \phi) C_{bc} \gamma^{\bar a} \gamma^{bc} \right) \eta \Big] \Big\} \\
&= -\epsilon_- \otimes \big\{ \left[D_a C + \tfrac{3}{2} (\partial_a \phi) C - 3 C_{bc} H_a{}^{bc} + 4 D^b C_{ba} - 2 (\partial^b \phi) C_{ba} \right] \gamma^a \eta \\
&\qquad + \left[D_a C_{bc} + \tfrac{1}{2} (\partial_a \phi) C_{bc} \right] \gamma^{abc} \eta \big\} .
\end{aligned}
\tag{B.19}
$$

Similarly,

$$
\begin{aligned}
-\Gamma^M \delta\chi \partial_M \phi \Big|_{\epsilon_-} &= \epsilon_- \otimes \left\{ (\gamma^a \partial_a \phi + \gamma^{\bar a} \partial_{\bar a} \phi)(C\eta + C_{bc}\gamma^{bc}\eta) \right\} \\
&= \epsilon_- \otimes \big\{ (\partial_a \phi) C \gamma^a \eta + (\partial_a \phi) C_{bc} \gamma^{abc} \eta + 4 (\partial^b \phi) C_{ba} \gamma^a \eta \big\}
\end{aligned}
\tag{B.20}
$$

and

$$
\begin{aligned}
-\tfrac{1}{8} \Gamma^{MNP} H_{MNP}\, \delta\chi \Big|_{\epsilon_-} &= \tfrac{1}{8} \epsilon_- \otimes \gamma^{mnp} H_{mnp} (C\eta + C_{ab}\gamma^{ab}\eta) \\
&= \tfrac{3}{8} \epsilon_- \otimes \Big\{ C H_{ab\bar a} \gamma^{ab\bar a} \eta + C_{ab} H_{\bar a cd} \gamma^{\bar a cd} \gamma^{ab} \eta + C_{ab} H_{c\bar a \bar b} \gamma^{c \bar a \bar b} \gamma^{ab} \eta \Big\} \\
&= \tfrac{3}{8} \epsilon_- \otimes \Big\{ \tfrac{4}{3} (\partial_a \phi) C \gamma^a \eta - \tfrac{4}{3} (\partial_a \phi) C_{bc} \gamma^{abc} \eta - \tfrac{4}{3} (\partial_{\bar a} \phi) C_{bc} \gamma^{\bar a} \gamma^{bc} \eta \\
&\qquad + C_{ab} H_{c \bar a \bar b} \gamma^c \gamma^{\bar a \bar b} \gamma^{ab} \eta \Big\} \\
&= \tfrac{1}{2} \epsilon_- \otimes \Big\{ (\partial_a \phi) C \gamma^a \eta - (\partial_a \phi) C_{bc} \gamma^{abc} \eta - 4 (\partial^b \phi) C_{ba} \gamma^a \eta \\
&\qquad - 6 C_{bc} H_a{}^{bc} \gamma^a \eta \Big\} .
\end{aligned}
\tag{B.21}
$$

Combining these, the gaugino equation of motion reduces to

$$0 = -2\left(D_a C + 4D^b C_{ba} - 4(\partial^b \phi) C_{ba}\right)\gamma^a \eta - 2\left(D_a C_{bc}\right)\gamma^{abc}\eta \tag{B.22}$$

as claimed.

Appendix C

Linear Models for Flux Vacua

C.1 Review of $(0,2)$ and $(2,2)$ GLSMs

The following is a lightning review of the salient features of $(0,2)$ gauged linear sigma models; for more complete discussions see [124, 45]. Our conventions and notation follow [81], with all factors of α' suppressed throughout the paper. We take the $(0,2)$ superspace coordinates to be $(y^+, y^-, \theta^+, \bar{\theta}^+)$, where $y^\pm = (y^0 \pm y^1)$. We begin with the gauge multiplet.

The right-moving gauge covariant superderivatives $\mathcal{D}_+, \overline{\mathcal{D}}_+$, satisfy the algebra

$$\mathcal{D}_+^2 = \overline{\mathcal{D}}_+^2 = 0, \qquad -\tfrac{i}{4}\{\,\mathcal{D}_+, \overline{\mathcal{D}}_+\}\ = \nabla_+ = \partial_+ + iQv_+, \tag{C.1}$$

where Q is the charge of the field on which they act. These imply that in a suitable basis we can identify

$$\mathcal{D}_+ = \frac{\partial}{\partial\theta^+} - 2i\bar{\theta}^+\nabla_+, \qquad \overline{\mathcal{D}}_+ = -\frac{\partial}{\partial\bar{\theta}^+} + 2i\theta^+\nabla_+, \qquad \mathcal{D}_- = \partial_- + \tfrac{i}{2}QV_-,$$

where $V_\pm$ are real vector superfields which transform under a gauge transformation with (uncharged) chiral gauge parameter $\overline{\mathcal{D}}_+\Lambda = 0$ as $\delta_\Lambda V_- = \partial_-(\Lambda + \overline{\Lambda})$ and $\delta_\Lambda V_+ = \frac{i}{2}(\Lambda - \overline{\Lambda})$; $\nabla_\pm$ are the usual gauge covariant derivatives. This allows us to fix to Wess-Zumino gauge in which

$$V_+ = \theta^+\bar{\theta}^+ 2v_+ \qquad V_- = 2v_- - 2i\theta^+\bar{\lambda}_- - 2i\bar{\theta}^+\lambda_- + 2\theta^+\bar{\theta}^+ D.$$

Note that V_- contains a complex left-moving gaugino. Finally, the natural field strength is a fermionic chiral superfield,

$$\Upsilon = 2[\overline{\mathcal{D}}_+, \mathcal{D}_-] = \overline{\mathcal{D}}_+(2\partial_- V_+ + iV_-) = -2\{\lambda_- - i\theta^+(D + 2iv_{+-}) - 2i\theta^+\bar{\theta}^+\partial_+\lambda_-\}, \quad \text{(C.2)}$$

for which the natural action is

$$S_\Upsilon = -\frac{1}{8e^2}\int \mathrm{d}^2y\, d\theta^+ d\bar{\theta}^+\ \overline{\Upsilon}\Upsilon = \frac{1}{e^2}\int \mathrm{d}^2y\ \left\{2v_{+-}^2 + 2i\bar{\lambda}_-\partial_+\lambda_- + \frac{1}{2}D^2\right\}, \quad \text{(C.3)}$$

where $d^2y = dy^0 dy^1$ and we use conventions where $\int d\theta^+\theta^+ = \int \bar{\theta}^+ d\bar{\theta}^+ = 1$.

Matter multiplets are similarly straightforward. A bosonic superfield satisfying $\overline{\mathcal{D}}_+\Phi = 0$ is called a *chiral* supermultiplet and contains a complex scalar and a right-moving complex fermion $\Phi = \phi + \sqrt{2}\theta^+\psi_+ - 2i\theta^+\bar{\theta}^+\nabla_+\phi$, and under gauge transformations $\Phi \to e^{-iQ(\Lambda+\overline{\Lambda})/2}\Phi$. The gauge invariant Lagrangian is given by

$$\begin{aligned} S_\Phi &= -i\int \mathrm{d}^2y\, \mathrm{d}^2\theta\ \overline{\Phi}\mathcal{D}_-\Phi, & \text{(C.4)} \\ &= \int \mathrm{d}^2y \Big\{ -|\nabla_\alpha\phi|^2 + 2i\bar{\psi}_+\nabla_-\psi_+ - iQ\sqrt{2}\bar{\phi}\lambda_-\psi_+ + iQ\sqrt{2}\phi\bar{\psi}_+\bar{\lambda}_- + QD|\phi|^2\Big\}, \end{aligned}$$

where the metric is given by $\eta^{+-} = -2$.

Left-moving fermions transform in their own supermultiplet, the *fermi* supermultiplet, which satisfies the chiral constraint

$$\overline{\mathcal{D}}_+\Gamma = \sqrt{2}E \quad \text{(C.5)}$$

and has component expansion $\Gamma = \gamma_- - \sqrt{2}\theta^+ G - 2i\theta^+\bar{\theta}^+\nabla_+\gamma_- - \sqrt{2}\bar{\theta}^+ E$, where $\overline{\mathcal{D}}_+ E = 0$ is a bosonic chiral superfield with the same gauge charge as Γ. The action for Γ is given by

$$\begin{aligned} S_\Gamma &= -\frac{1}{2}\int \mathrm{d}^2y\, d^2\theta\, \overline{\Gamma}\Gamma \\ &= \int \mathrm{d}^2y \left\{ 2i\bar{\gamma}_-\nabla_+\gamma_- + |G|^2 - |E|^2 - \left(\bar{\gamma}_- \frac{\partial E}{\partial\phi_i}\psi_{+i} + \bar{\psi}_{+i}\frac{\partial \overline{E}}{\partial\bar{\phi}_i}\gamma_- \right) \right\}. \end{aligned} \tag{C.6}$$

In general, we can add superpotential terms to our Lagrangian. Since these are integrals over a single supercoordinate, the superpotential can be written as a sum of fermi superfields Γ_m times holomorphic functions J^m of the chiral superfields,

$$\begin{aligned} S_{\mathcal{W}} &= \frac{1}{\sqrt{2}}\int \mathrm{d}^2y\, d\theta^+\ \Gamma_m J^m|_{\bar{\theta}^+=0} + \text{h.c.}, \\ &= -\int \mathrm{d}^2y \left\{ G_m J^m(\phi_i) + \gamma_{-m}\psi_{+i}\frac{\partial J^m}{\partial \phi_i} \right\} + \text{h.c.}. \end{aligned} \tag{C.7}$$

Since Γ_m is not an honest chiral superfield but satisfies (C.5), we need to impose the condition

$$E \cdot J = 0 \tag{C.8}$$

to ensure that the superpotential is chiral. Finally, since Υ is a chiral fermion, we can also add an FI term of the form

$$S_{\mathrm{FI}} = \frac{it}{4}\int \mathrm{d}^2y\, d\theta^+\ \Upsilon|_{\bar{\theta}^+=0} + \text{h.c.} = \int d^2y\,(-rD + 2\theta v_{+-}) \tag{C.9}$$

where $t = r + i\theta$ is the complexified FI parameter.

C.1.1 Our Canonical Example: $\mathcal{V} \to K3$

Our canonical example begins with a vector bundle $\mathcal{V} \to S$ over a $K3$ hypersurface S in a resolved weighted projective space $W\mathbb{P}^3$. The associated GLSM includes the gauge group $G = U(1)^s$ with s gauge field-strengths Υ_a, $(3+s)$ chiral scalars $\Phi_{i=1,\ldots 3+s}$ with charges Q_i^a, a set of c neutral scalars $\Sigma_{A=1\ldots c}$, a single chiral scalar Φ_0 with charges $-d^a$, r fermi multiplets $\Gamma_{m=1,\ldots r}$ with charges q_m^a satisfying the constraints $\overline{\mathcal{D}}_+\Gamma_m = \sqrt{2}\Sigma_A E_m^A(\Phi)$, a single chiral fermion Γ_0 with charges $-m^a$, and spectators as needed to ensure vanishing of the one-loop tadpole for D^a [46, 45], all interacting according to the canonical Lagrangian density

$$\begin{aligned}\mathcal{L} &= -\int d^2\theta \, \left[\frac{1}{8e_a^2}\overline{\Upsilon}_a\Upsilon_a + \frac{1}{2}\overline{\Gamma}_m\Gamma_m + i\overline{\Sigma}_A\partial_-\Sigma_A + i\overline{\Phi}_i(\partial_- + \tfrac{i}{2}Q_i^a V_{-a})\Phi_i\right] \\ &\quad + \frac{1}{\sqrt{2}}\int d\theta^+ \, [\Gamma_0 G(\Phi_i) + \Gamma_m \Phi_0 J^m(\Phi_i) + \frac{\sqrt{2}}{4} it^a\Upsilon_a] + \text{h.c.}.\end{aligned}$$

Integrating out the auxiliary fields results in a scalar potential

$$\begin{aligned}U = &\sum_a \frac{e_a^2}{2}\left(\textstyle\sum_i Q_i^a|\phi_i|^2 - m^a|\phi_0|^2 - r^a\right)^2 + |G(\phi)|^2 \\ &+ \sum_m \left(|\phi_0|^2|J^m(\phi)|^2 + \left|\textstyle\sum_A \sigma_A E_m^A(\phi)\right|^2\right).\end{aligned}$$

Non-singularity of the relevant geometric phase requires $G(\Phi_i)$ and $J^m(\Phi_i)$ to be transverse,

$$\begin{aligned}G = \frac{\partial G}{\partial\phi_1} = \frac{\partial G}{\partial\phi_2} = \cdots = 0 \qquad &\Longleftrightarrow \qquad \forall i : \phi_i = 0, \\ G = J^1 = J^2 = \cdots = 0 \qquad &\Longleftrightarrow \qquad \forall i : \phi_i = 0.\end{aligned}$$

In the relevant geometric phase of the Kähler cone, the Yukawa interactions

$$\begin{aligned}\mathcal{L}_{Yuk} &= -\Bigg[\gamma_{-m}\left(\psi_{+0}J^m + \phi_0\psi_{+i}\frac{\partial J^m}{\partial\phi_i}\right) + \sum_i \sqrt{2}iQ_i^a\bar{\phi}_i\lambda_{-a}\psi_{+i} \\ &\quad + \bar{\gamma}_{-m}\left(\eta_{+A}E_m^A + \psi_{+i}\sigma_A\frac{\partial E_m^A}{\partial\phi_i}\right) + \gamma_{-0}\psi_{+i}\frac{\partial G}{\partial\phi_i}\Bigg] + \text{h.c.}\end{aligned}$$

give masses to various linear combinations of the right- and left-moving fermions. The massless right-moving fermions couple to a bundle which fits into two exact sequences.

For instance, for a single $U(1)$ we have

$$0 \to \mathcal{O}_{W\mathbb{P}} \xrightarrow{Q_i\phi_i} \oplus_i \mathcal{O}_{W\mathbb{P}}(Q_i) \to T_{W\mathbb{P}} \to 0 \tag{C.10}$$
$$0 \to T_S \to T_{W\mathbb{P}}|_S \xrightarrow{\partial_{\phi_i}G} \mathcal{O}_S(d) \to 0,$$

so the massless right-moving fermions couple to T_S. Similarly, the bundle $\mathcal{V}_S$ to which the massless left-moving fermions couple fits into a pair of short exact sequences,

$$0 \to \oplus_A \mathcal{O}_{W\mathbb{P}} \xrightarrow{E^A_m} \oplus_m \mathcal{O}(q_m) \to \mathcal{V}_{W\mathbb{P}} \to 0 \tag{C.11}$$
$$0 \to \mathcal{V}_S \to \mathcal{V}_{W\mathbb{P}}|_S \xrightarrow{J^m} \mathcal{O}(m) \to 0.$$

C.1.2 GLSMs with $(2,2)$ Supersymmetry

A special class of $(0,2)$ theories have enhanced $(2,2)$ supersymmetry. To describe these theories, we enlarge our superspace by adding two fermionic coordinates, $(y^+, y^-, \theta^+, \bar{\theta}^+, \theta^-, \bar{\theta}^-)$, and introduce supercovariant derivatives

$$\mathrm{D}_\pm = \frac{\partial}{\partial\theta^\pm} - 2\mathrm{i}\bar{\theta}^\pm\partial_\pm \qquad \overline{\mathrm{D}}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} + 2\mathrm{i}\theta^\pm\partial_\pm. \tag{C.12}$$

Unlike the $(0,2)$ case, there are two kinds of $(2,2)$ chiral multiplets, *chiral* multiplets satisfying

$$\overline{\mathrm{D}}_+\Phi_{2,2} = \overline{\mathrm{D}}_-\Phi_{2,2} = 0, \tag{C.13}$$

and *twisted chiral* multiplets satisfying

$$\overline{\mathrm{D}}_+ Y_{2,2} = \mathrm{D}_-\mathrm{Y}_{2,2} = 0. \tag{C.14}$$

Both have the field content of one $(0,2)$ chiral and one $(0,2)$ fermi multiplet,

$$\Phi_{2,2} = \Phi + \sqrt{2}\theta^-\Gamma_- - 2i\theta^-\bar{\theta}^-\partial_-\Phi \qquad Y_{2,2} = Y + \sqrt{2}\bar{\theta}^- F + 2i\theta^-\bar{\theta}^-\partial_- Y.$$

The $(2,2)$ vector superfield, $V_{2,2}$, whose field strength is a twisted chiral multiplet $\Sigma = \frac{1}{\sqrt{2}}\overline{D}_+ D_- V_{2,2}$, is built out of an uncharged $(0,2)$ chiral multiplet Σ_0 and a $(0,2)$ vector multiplet $V_\pm$ as

$$V_{2,2} = V_+ + \theta^-\bar{\theta}^- V_- + \sqrt{2}\bar{\theta}^+\theta^-\Sigma_0 + \sqrt{2}\bar{\theta}^-\theta^+\overline{\Sigma}_0, \tag{C.15}$$

where $\Sigma_0 = \sigma - i\sqrt{2}\theta^+\bar{\lambda}_+ - 2i\theta^+\bar{\theta}^+\partial_+\sigma$ for agreement with [124], and $\delta_g V_{2,2} = \frac{i}{2}\left(\Lambda_{2,2} - \overline{\Lambda}_{2,2}\right)$. The standard FI-term is

$$\mathcal{L}_{FI} = \frac{-t}{2\sqrt{2}}\int d\theta^+ d\bar{\theta}^- \Sigma + \text{h.c.} = -rD + 2\theta v_{+-},$$

where $t = r + i\theta$.

Lastly, we note that a $(2,2)$ chiral multiplet with $U(1)$ charge Q reduces to a charged $(0,2)$ chiral multiplet Φ and a charged fermi multiplet Γ satisfying

$$\overline{\mathcal{D}}_+\Gamma = \sqrt{2}E$$

in $(0,2)$ notation, and where E is given by

$$E = \sqrt{2}Q\Sigma_0\Phi. \tag{C.16}$$

We will omit the subscripts "2,2" in the main text, as it should always be clear from the context to which supersymmetry we refer.

C.2 The Fu-Yau Geometry

C.2.1 Supersymmetry Constraints

Consider compactification of the heterotic string on a 6-dimensional manifold, X. Preserving $\mathcal{N}$=1 supersymmetry in 4d requires that X admit a nowhere vanishing spinor, η. This immediately implies that X admits an almost complex structure. The existence of a nowhere-vanishing spinor on an almost-complex 3-fold implies that the frame bundle admits a connection of $SU(3)$ holonomy – *i.e.* that X is a special-holonomy manifold with $SU(3)$-structure. However, the connection of special holonomy need not be the Levi-Civita connection, and in general the nowhere-vanishing spinor is *not* annihilated by the metric connection, ∇_g, but by a (unique) torsionful connection,

$$(\nabla_g + H)\eta = 0.$$

H is called the intrinsic torsion of the $SU(3)$-structure. In the special case $H = 0$, when the nowhere-vanishing spinor is covariantly constant according to the metric connection, X admits a metric of $SU(3)$ holonomy and is thus Calabi-Yau.

$\mathcal{N} = 1$ supersymmetry in 4d further requires the vanishing of the supersymmetry variations of the gravitino, dilatino, and gaugino. Together with the Jacobi identity for the resulting superalgebra, these constraints imply that X admits an *integrable* complex structure, a nowhere-vanishing Hermitian metric corresponding to a globally-defined Hermitian (1,1)-form, $J_{a\bar{b}} = \eta^\dagger \Gamma_{a\bar{b}} \eta$, a nowhere-vanishing holomorphic (3,0)-form, $\Omega_{abc} = \eta^\dagger \Gamma_{abc} \eta$, and comes equipped with a Hermitian-Yang-Mills gauge field, $F^{(2,0)} = F^{(0,2)} = F_{mn} J^{mn} = 0$. They also imply that

$$H = i(\overline{\partial} - \partial)J,$$

so H is the obstruction to X being Kähler. Instead, X is *conformally balanced*,

$$d(e^{-2\phi} J \wedge J) = 0,$$

where ϕ is the Einstein-frame dilaton. While more complicated than the simple $H = 0$

Calabi-Yau case, the general X would so far appear to be on a similar footing.

The Green-Schwarz anomaly completely changes the story. Including the one-loop gravitational correction, the vanishing of the anomaly implies

$$dH = \alpha' \left(\mathrm{tr} R \wedge R - \mathrm{Tr} F \wedge F \right),$$

where R is the curvature of the Hermitian connection on X. This changes the story in several dramatic ways. First, since the left and right hand sides of this equation scale inhomogenously in the global conformal mode of the metric, any solution to this equation has fixed volume modulus. Crucially, this means any solution to this equation does *not* have a large radius limit, so supergravity perturbation theory has a finite, fixed expansion parameter and must be taken with a sizeable grain of salt. Secondly, this equation is spectacularly nonlinear, so even proving the existence of solutions is a profoundly difficult problem.

Happily, in at least one special case there exists an existence proof by Fu and Yau for solutions to the full set of conditions outlined above, including the anomaly equation, analogous to Yau's proof of the existence of a Ricci-flat Kähler metric on manifolds of $SU(3)$-holonomy. Unlike the Yau proof of the Calabi conjecture, however, the Fu-Yau proof begins with a very specific Ansatz for the metric, torsion, and holomorphic 3-form [55].

C.2.2 GP Manifolds and the FY Compactification

The underlying manifold satisfying all of the supersymmetry constraints unrelated to the gauge bundle was first constructed by Goldstein and Prokushkin [70]. Their solution involved constructing the complex 3-fold as a T^2 bundle over a T^4 or $K3$

base. Fu and Yau [55] used this underlying manifold and constructed a gauge bundle satisfying the remaining supersymmetry constraints as well as the modified Bianchi identity, which was a monumental accomplishment since it is a complicated differential equation. We start by explaining the GP manifold.

Let S be a complex Hermitian 2-fold and choose[1]

$$\frac{\omega_P}{2\pi}, \frac{\omega_Q}{2\pi} \in H^2(S;\mathbb{Z}) \cap \Lambda^{1,1}T^*_S. \tag{C.17}$$

where ω_P and ω_Q are anti self-dual. Being elements of integer cohomology, there are two $\mathbb{C}^*$-bundles over S, call them P and Q, whose curvature 2-forms are ω_P and ω_Q, respectively. We can then restrict to unit-circle bundles S^1_P and S^1_Q of P and Q respectively, and take the product of the two circles over each point in S to form a T^2 bundle over S which we will refer to as X ($T^2 \to X \xrightarrow{\pi} S$).

Given this setup, Goldstein and Prokushkin showed that if S admits a non-vanishing, holomorphic $(2,0)$-form, then X admits a non-vanishing, holomorphic $(3,0)$-form. Furthermore, they showed that if ω_P or ω_Q are nontrivial in cohomology on S, then X admits *no* Kähler metric. They constructed the non-vanishing holomorphic $(3,0)$-form and a Hermitian metric on X from data on S.

The curvature 2-form ω_P determines a non-unique connection ∇ on S^1_P (and similarly for ω_Q on S^1_Q). A connection determines a split of T_X into a vertical and horizontal subbundle – the horizontal subbundle is composed of the elements of T_X that are annihilated by the connection 1-form, the vertical subbundle is then, roughly speaking, the elements of T_X tangent to the fibres. Over an open subset $U \subset S$, we

[1] Actually, Goldstein and Prokushkin only required that $\omega_P + i\omega_Q$ have no $(0,2)$-component, but Fu and Yau used the restriction that we have stated.

have a local trivialization of X and we can use unit-norm sections, $\xi \in \Gamma(U; S^1_P)$ and $\zeta \in \Gamma(U; S^1_Q)$, to define local coordinates for $z \in U \times T^2$ by

$$z = (p, e^{i\theta_P}\xi(p), e^{i\theta_Q}\zeta(p)), \tag{C.18}$$

where $p = \pi(z) \in U$. The sections ξ and ζ also define connection 1-forms via

$$\nabla\xi = i\alpha_P \otimes \xi \qquad \text{and} \qquad \nabla\zeta = i\alpha_Q \otimes \zeta, \tag{C.19}$$

where $\omega_P = d\alpha_P$ and $\omega_Q = d\alpha_Q$ on U, and the α_i are necessarily real to preserve the unit-norms of ξ and ζ.

The complex structure is given on the fibres by $\partial_{\theta_P} \to \partial_{\theta_Q}$ and $\partial_{\theta_Q} \to -\partial_{\theta_P}$ while on the horizontal distribution it is induced by projection onto S.[2] Given a Hermitian 2-form ω_S on S, the 2-form

$$\omega_u = \pi^*\left(e^u \omega_M\right) + \left(d\theta_P + \pi^*\alpha_P\right) \wedge \left(d\theta_Q + \pi^*\alpha_Q\right), \tag{C.20}$$

where u is some smooth function on S, is a Hermitian 2-form on X with respect to this complex structure. The connection 1-form

$$\vartheta = (d\theta_P + \pi^*\alpha_P) + i(d\theta_Q + \pi^*\alpha_Q) \tag{C.21}$$

annihilates elements of the horizontal distribution of T_X while reducing to $d\theta_P + id\theta_Q$ along the fibres. These data define the complex Hermitian 3-fold (X, ω_u), which we

[2] Actually, this just gives an almost complex structure, but Goldstein and Prokushkin proved that it is integrable [70]

call the GP manifold [70]. Explicitly,

$$\begin{aligned} ds_X^2 &= \pi^*\left(e^u ds_S^2\right) + (d\theta_P + \pi^*\alpha_P)^2 + (d\theta_Q + \pi^*\alpha_Q)^2 \\ J_X &= \pi^*\left(e^u J_S\right) + \frac{1}{2}\vartheta \wedge \bar{\vartheta} \\ \Omega_X &= \pi^*\left(\Omega_S\right) \wedge \vartheta \\ H &= \sum_{i=P,Q} (d\theta_i + \pi^*\alpha_i) \wedge \pi^*\omega_i, \end{aligned}$$

where Ω_S is the nowhere-vanishing, holomorphic $(2,0)$-form on S ($K3$ or T^4). It is straightforward check that all the supersymmetry constraints are satisfied by this Ansatz, however for a valid heterotic compactifications a gauge bundle still needed to be constructed to satisfy the Bianchi identity.

Fu and Yau undertook the more difficult problem of proving the existence of gauge bundles over the GP manifold with Hermitian-Yang-Mills connections satisfying the Bianchi identity (5.1). They took the Hermitian form (C.20) and converted the Bianchi identity into a differential equation for the function u. Under the assumption

$$\left(\int_{K3} e^{-4u}\frac{\omega_{K3}^2}{2}\right)^{1/4} \ll 1 = \int_{K3}\frac{\omega_{K3}^2}{2}, \tag{C.22}$$

they showed that there exists a solution u to the Bianchi identity for *any* compatible choice of gauge bundle $\mathcal{V}_X$ and curvatures ω_P and ω_Q such that the gauge bundle $\mathcal{V}_X$ over X is the pullback of a stable, degree 0 bundle $\mathcal{V}_{K3}$ over $K3$, $\mathcal{V}_X = \pi^*\mathcal{V}_{K3}$ [55]; this is what we call the Fu-Yau geometry.

Note that by a "compatible" choice of gauge bundle and ω_i's we mean the following: choose the gauge bundle $\mathcal{V}_X$ and the curvature forms to satisfy the integrated Bianchi identity

$$\chi(S) - \mathrm{Tr}F^2 = \int_S \sum_i \omega_i^2. \tag{C.23}$$

In particular, note that the right-hand side and $\mathrm{Tr}F^2$ are manifestly non-negative, since $*_S F = -F$ and F is anti-Hermitian. Hence, the only possible solution for a T^4 base is to take the gauge bundle *and* the T^2 bundle to be trivial, leaving us with a Calabi-Yau solution $T^2 \times T^4$ [14, 55]. This is in agreement with arguments from string duality ruling out the Iwasawa manifold as a solution to the heterotic supersymmetry constraints [63].

Appendix D

Nearing the Horizon of a Heterotic String: $Sp(4)$

An element $g \in Sp(4)$ satisfies $g^\dagger g = 1$ and $g^T \Omega g = \Omega$. If we parameterize g as e^{iM}, $M \in \mathbf{sp}(4)$, then $M = M^\dagger$ and $M^T\Omega + \Omega M = 0$. This determines

$$\mathbf{sp}(4) = \mathrm{span}_{\mathbb{R}}\{A_1, A_2, A_3, B_1, B_2, B_3, B_4, C_1, C_2, C_3\}. \qquad \text{(D.1)}$$

Writing these in 2×2 blocks,

$$A_\alpha = \begin{pmatrix} \sigma^\alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad C_\alpha = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^\alpha \end{pmatrix},$$

$$B_\alpha = \frac{1}{2}\begin{pmatrix} 0 & i^{\delta_{\alpha,2}}\sigma^\alpha \\ (i^{\delta_{\alpha,2}}\sigma^\alpha)^\dagger & 0 \end{pmatrix}, \quad B_4 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \text{(D.2)}$$

where σ^α are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad \text{(D.3)}$$

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