Black Holes, Entropy Functionals, and Topological Strings

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Abstract

This thesis is devoted to study of the connection between extremal black holes and topological strings. Important ingredient of this connection is the relation between Hartle-Hawking wave function associated to black holes and topological string partition function. This leads to a natural entropy functional defined on the moduli space of string compactifications. We discuss several examples of such entropy functionals.

We start by proposing a wave function for scalar metric fluctuations on S^3 embedded in a Calabi-Yau. This problem maps to a study of non-critical bosonic string propagating on a circle at the self-dual radius. This can be viewed as a stringy toy model for a quantum cosmology. Then we formulate an entropy functional on the moduli space of Calabi-Yau compactifications. We find that the maximization of the entropy is correlated with the appearance of asymptotic freedom in the effective field theory. The points where the entropy is maximized correspond to points on the moduli which are maximal intersection points of walls of marginal stability for BPS states. We then turn to study of the entropy functional on the moduli space of two dimensional conformal field theories captured by the gauged WZW model whose target space is an abelian variety. This gives rise to the effective action on the moduli space of Riemann surfaces, whose critical points are attractive and correspond to Jacobian varieties admitting complex multiplication. The partition function is a generating function for the number of conformal blocks in rational conformal field theories. Finally, we study non-supersymmetric, extremal 4 dimensional black holes which arise upon compactification of type II superstrings on Calabi-Yau threefolds. We propose a generalization of the OSV conjecture for higher derivative corrections to the non-supersymmetric black hole entropy, in terms of the one parameter refinement of topological string introduced by Nekrasov. We also study the attractor mechanism for non-supersymmetric black holes and show how the inverse problem of fixing charges in terms of the attractor value of Calabi-Yau moduli can be explicitly solved.

Contents

Citations to Previously Published Work

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to my wife Ирина

Chapter 1

Introduction and Summary

One of the major problems in theoretical physics is formulating a quantum theory of gravity, a theory that would unify quantum mechanics and general relativity. Over the years, string theory has proved to be the most reliable candidate for such a theory. In particular, string theory provides a microscopic description of the entropy of certain types of black holes through the counting of D-brane bound states. Black holes are the simplest solutions of the general relativity that reveal quantum properties due to the Hawking radiation, and understanding the quantum physics of the black holes is an important step towards formulating a quantum theory of gravity.

The predictions of the string theory include not only a confirmation of the leading semi-classical black hole entropy formula of Bekenstein and Hawking, which was first confirmed in [150] (see, e.g. [145, 49] for a review and references), but also all the subleading quantum gravitational corrections, which was proposed by Ooguri, Strominger and Vafa in [142] (building on the work of [35, 37, 38, 36, 33]). These higher derivative corrections have been confirmed by explicit microscopic enumeration in a number of examples [159, 5, 46, 4, 47, 153].

As it was shown in [142], the accounting of the black hole entropy is deeply connected with the topological strings, which, roughly speaking, capture the supersymmetric sector of the string theory. In particular, using supergravity results [142] conjectured a simple relation of the form $Z_{\rm BH} = |Z_{\rm top}|^2$ between the (indexed) entropy of a four-dimensional BPS black hole in a Type II string Calabi-Yau compactification, and topological string partition function, evaluated at the attractor point on the moduli space. Viewed as an asymptotic expansion in the limit of large black hole charges, this relation predicts all order perturbative contributions to the black hole entropy due to the F-term corrections in the effective $\mathcal{N} = 2$ supergravity Lagrangian. Over the last few years, this led to significant progress in understanding the spectrum of D-brane BPS states on compact and non-compact Calabi-Yau manifolds, and gave new insights on the topological strings and quantum cosmology [146].

Chapter 2: A Stringy Wave Function for an S^3 Cosmology

The notion of the wave function of the universe, in the mini-superspace description a la Hartle-Hawking [94], has recently been made precise in the context of a certain class of string compactifications [143]. In particular, this work provided an explanation for the appearance of a topological string wave function in the conjecture of [142] relating the entropy of certain extremal $4d$ black holes with the topological string wave function. It is natural to ask whether we can extend this picture in order to obtain a more realistic quantum cosmology within string theory.

In Chapter 2, we take a modest step in this direction by proposing wave function for scalar metric fluctuations on $\mathbf{S}^3$ embedded in a Calabi-Yau. Then we use the relation between topological B model on $T^*\mathbf{S}^3$ and non-critical bosonic string theory on a circle of self-dual radius to compute the wave function for local volume fluctuations on $\mathbf{S}^3$ and some n-point correlation functions. We also discuss some possible toy model cosmologies based on $\mathbf{S}^3$.

The arguments presented in this chapter were obtained in collaboration with Sergei Gukov and Cumrun Vafa [91]

Chapter 3: The Entropic Principle and Asymptotic Freedom

There is little doubt that there exist a large number of consistent superstring vacua. This fact is not new: it has been well known for a while in the context of supersymmetric vacua. More recently, there has been some evidence that the multitude of vacua continues to exist even without supersymmetry (for introduction and references see [106, 65]). Of course, one can stop here and resort to the standard philosophy of physics: choose the theory to be in accord with observation. However, in the context of string theory being a unified theory of all matter, it is natural to explore whether one can say a little more about the selection criteria. In particular, it is desirable to have some sort of a weight function on the space of possible vacua.

This question typically arises in quantum gravity, where one is interested in comparing different possible universes (vacuum states) M and choosing the 'preferred' ones. The original suggestion of Hartle and Hawking [94] is to weigh each vacuum by the probability of creation from nothing (see also recent discussion in [95]). This gives some measure on the landscape of vacua. In [143], this proposal was interpreted in the context of string compactification with fluxes on $\mathrm{AdS}_2 \times \mathbf{S}^2 \times M$, where M is a Calabi-Yau threefold, using the OSV conjecture [142]. The weight, associated to a given M, is the norm of the Hartle-Hawking wave-function, which is related to the entropy $S_{p,q}$ of the dual black hole, obtained by wrapping a $D3$ brane with magnetic and electric charges (p,q) on M. This is called the entropic principle [143]:

$$\langle \Psi_{p,q}(M) | \Psi_{p,q}(M) \rangle \sim \exp(S_{p,q}) \tag{1.1}$$

The complex moduli of M are fixed by the charges (p,q) via the attractor mechanism [76, 155, 35, 124].

In Chapter 3 we further explore this idea and formulate an entropy functional on the moduli space of Calabi-Yau compactifications. We find that the maximization of the entropy is correlated with the appearance of asymptotic freedom in the effective field theory. The points where the entropy is maximized correspond to points on the moduli which are the maximal intersection points of walls of marginal stability for BPS states. We also find an intriguing link between extremizing the entropy functional and the points on the moduli space of Calabi-Yau three-folds which admit a 'quantum deformed' complex multiplication.

The results in this chapter were obtained in collaboration with Sergei Gukov and Cumrun Vafa [90]

Chapter 4: Abelian Varieties, RCFTs, Attractors, and Hitchin Functional in Two Dimensions

The entropic principle implies that one can define corresponding quantum mechanical problem on the moduli space $\mathcal{M}_M$ by summing over all Calabi-Yau manifolds M with the weight (1.1). This path integral can be used, for example, for computing correlation functions of the gravitation fluctuations around the points on the moduli space that correspond to "preferred" string compactifications. This approach can help us shed some light on the fundamental physical problems, such as quantum cosmology and string landscape.

The entropic principle in general can be formulated by saying that the entropy function is the Euler characteristic of the moduli space, associated with the problem. It is expected that the critical points of the entropy function on the moduli space correspond to special manifolds with extra (arithmetic) structures, such as complex multiplication, Lie algebra lattices, etc. There is also a hidden integrality involved coming from the quantization of (at least partially) compact moduli space. As a result, we expect appearance of the nice modular functions and automorphic forms at the critical points of the entropy function.

It was noted in [58, 133, 81] that some new geometric functionals introduced by Hitchin [97, 98] might be useful for the formulation of this problem in the context of topological strings. There are several reasons why the approach based on the Hitchin functional is attractive. Since it is a diffeomorphism-invariant functional depending only on the cohomology classes of some differential forms, it is a proper candidate for the description of topological degrees of freedom. It can also be used for incorporating the generalized geometry moduli. Moreover, at the classical level it reproduces the black hole entropy.

In Chapter 4 we apply the entropy functional idea to the two-dimensional toy model, that has many interesting features which are expected to survive in higher dimensions. We find a two-dimensional sibling of the Hitchin functional, formulate an analog of the entropic principle in $1_{\mathbb{C}} + 1$ dimensions, and describe corresponding quantum theory. It involves a generating function for the number of conformal blocks in rational conformal field theories with an even central charge c on a genus g Riemann surface. We study a special coupling of this theory to two-dimensional gravity. When $c = 2g$, the coupling is non-trivial due to the gravitational instantons, and the action of the theory can be interpreted as a two-dimensional analog of the Hitchin functional for Calabi-Yau manifolds. This gives rise to the effective action on the moduli space of Riemann surfaces, whose critical points are attractive and correspond to Jacobian varieties admitting complex multiplication. The theory that we describe can be viewed as a dimensional reduction of topological M-theory.

The advantage of taking digression to the two dimensions is that in this case (almost) everything becomes solvable. It turns out that this way we find a unified description of all two-dimensional topologies. Moreover, the study of the two-dimensional model leads to a natural generalization of the six-dimensional Hitchin functional, which may be useful for understanding of the topological M-theory at a quantum level. The analysis of this model supports the idea that the quantum partition function of the topological M-theory is given by a generalized index theorem for the moduli space. In particular, this implies that the OSV conjecture [142] should be viewed as a higher dimensional analog of the E. Verlinde's formula for the number of conformal blocks in a two-dimensional conformal field theory.

Chapter 5: Non-supersymmetric Black Holes and Topological Strings

An important feature of extremal black hole solutions in $\mathcal{N} = 2, 4, 8$ supergravity in four space-time dimensions is that some of the scalar fields (lowest components of the vector multiplets) acquire fixed values at the horizon. These values are determined by the magnetic and electric charges (p^I, q_I) of the black hole and does not depend on the asymptotic values of the fields at infinity. The so-called attractor mechanism, which is responsible for such fixed point behavior of the solutions, was first studied in [76, 155, 73, 74] in the context of the BPS black holes in the leading semiclassical approximation. Later, the attractor equations describing these fixed points for BPS black hole solutions were generalized to incorporate the higher derivative corrections to $\mathcal{N} = 2$ supergravity Lagrangian (see [123] for a review).

Recently, the interest toward accounting the entropy of non-supersymmetric extremal black holes spiked again. The attractor behavior of a non-supersymmetric extremal black hole solutions is similar to the BPS black hole case, since it is a consequence of extremality rather than supersymmetry [72]. Therefore, it is natural to look for an extension of the OSV formula (5.2) that can be applied *simultaneously* to both BPS and non-BPS extremal black holes and will describe corrections to their entropy due to higher derivative terms in the Lagrangian as a perturbative series in the inverse charge. Recently, several steps in this direction have been taken from the supergravity side. A general method (the entropy function formalism) for computing the macroscopic entropy of extremal black holes based on $\mathcal{N} = 2$ supergravity action in the presence of higher-derivative interactions was developed in [151, 152].

In Chapter 4 we propose a generalization of the OSV formula [142] that predicts degeneracies of both supersymmetric and non-supersymmetric extremal black holes using topological string partition function. Moreover, we conjectured that corresponding corrected (non-supersymmetric) extremal black hole entropy needs an additional ingredient: Nekrasov's extension of the topological string free energy $F(X^I, \epsilon_1, \epsilon_2)$.

We also study the attractor mechanism for non-supersymmetric black holes and show how the inverse problem of fixing charges in terms of the attractor value of CY moduli can be solved. Explicit solution of the inverse problem in one-modulus Calabi-Yau case is presented.

The results in this chapter were obtained in collaboration with Cumrun Vafa [150].

Chapter 2

A Stringy Wave Function for an S^3 Cosmology

The notion of the wave function of the universe introduced by Hartle and Hawking [94], has reappeared recentely [143] in the context of flux compactifications of type II string theory on a Calabi-Yau three-fold times $\mathbf{S}^2 \times \mathbf{S}^1$, where a wave function on moduli space of Calabi-Yau and the overall size of $\mathbf{S}^2$ was defined. In particular for a given choice of flux in the Calabi-Yau, labeled by magnetic and electric fluxes (P^I, Q_I), we have a wave function, $\psi_{P,Q}(\Phi^I)$, depending on (real) moduli of Calabi-Yau. This wave function is peaked at the attractor values of the moduli of the Calabi-Yau. Also, in general, this wave function depends only on the BPS subspace of the field configurations which thus yields a rather limited information about the full Calabi-Yau wave function. One would like to have a wave function which depends on more local data of the Calabi-Yau geometry, rather than just global moduli. This may seem to be in contradiction with the requirement that the data depends only on

BPS quantities. However this need not be the case, as we will now explain.

For concreteness, let us take type IIB superstring compactified on a Calabi-Yau three-fold and consider the shape of a particular special Lagrangian 3-cycle L inside the Calabi-Yau. For instance, this may be a natural setup for a toy model of our universe obtained by wrapping some D-branes on L. In this setup, the question about the wave function as a function of the shape of L translates into the wave function for our universe. In general, varying the moduli of Calabi-Yau will induce changes in the shape of L. So, at least we have a wave function on a subset of the moduli of L. More precisely, since Calabi-Yau space has a 3-form which coincides with the volume form on special Lagrangian submanifolds, we are effectively asking about a wave function for some subset of local volume fluctuations on L. On the other hand, since the issue is local, we can consider a local model of Calabi-Yau near L, which is given by T^*L. In this context, global aspects of Calabi-Yau will not provide any obstruction in arbitrary local deformations of the shape of L. We could thus write a wave function which is a function of *arbitrary local volume fluctuations of* L. In particular, if we know how to compute topological string wave function on T^*L we will be able to write the full wave function for arbitrary local volume fluctuations (scalar metric perturbations) of L.

A particularly interesting choice of L is $L = \mathbf{S}^3$. Not only is this the most natural choice in the context of quantum cosmology, but luckily it also turns out to be the case already well studied in topological string theory: As is well known, the topological B model on the conifold $T^*\mathbf{S}^3$ gets mapped to non-critical bosonic string propagating on a circle of self-dual radius [82]. Hence, we can use the results on the $c = 1$ non-critical bosonic string theory to write a wave function for scalar metric fluctuations of the $\mathbf{S}^3$. This is the main goal of this chapter. We will show how the known results of the non-critical bosonic strings can be used to yield arbitrary 2-point fluctuations. In particular we find the following result

$$\langle \phi_{\vec{k}} \phi_{-\vec{k}} \rangle \sim g_s^2 |\vec{k}| \tag{2.1}$$

where $\phi_{\vec{k}}$ denotes the Fourier modes of the conformal rescalings of the metric, which differs from the scale invariant spectrum in the standard cosmology:

$$\langle \phi_{\vec{k}} \phi_{-\vec{k}} \rangle \sim |\vec{k}|^{-3} \tag{2.2}$$

One can also in principle compute arbitrary n-point fluctuations. However, in general, for this one would need to know arbitrary momentum and winding correlation functions of the non-critical bosonic string which are not yet available (see, however, the recent work [121]). Nevertheless, from the known results about the correlation functions of the momentum modes of $c = 1$ string theory we can obtain arbitrary n-point fluctuations for scalar fluctuations on a large circle $\mathbf{S}^1 \subset \mathbf{S}^3$.

The organization of this chapter is as follows: In section 2.1, we review the notion of the wave function for topological strings and its relation to the wave function for moduli of a Calabi-Yau in flux compactifications [143]. In section 2.2, we review non-critical bosonic string theory on a circle of self-dual radius and its relation to the topological B model on $T^*\mathbf{S}^3$. In section 2.3, we use these relations to compute the wave function for local volume fluctuations on $\mathbf{S}^3$ and compute some n-point correlation functions. We also discuss some possible toy model cosmologies based on $\mathbf{S}^3$.

2.1 Stringy Hartle-Hawking Wave Function

In this section we briefly review the work of [143]. Consider a flux compactification of type IIB string on a Calabi-Yau space M times $\mathbf{S}^2 \times \mathbf{S}^1$, with a 5-form field strength flux threading through $\mathbf{S}^2$ and a 3-cycle of M. We choose a canonical symplectic basis for the three cycles on M, denoted by A_I, B^J. In this basis, the magnetic/electric flux can be denoted by (P^I, Q_J). The wave function of the "universe" in the mini-superspace will be a function of the moduli of M and the sizes of $\mathbf{S}^2$ and $\mathbf{S}^1$. It turns out that it does not depend on the size[1] of the $\mathbf{S}^1$ and its dependence on the size of $\mathbf{S}^2$ can be recast by writing the wave function in terms of the projective coordinate on the moduli space of M.

The moduli space of a Calabi-Yau is naturally parameterized by the periods of the holomorphic 3-form Ω on the 3-cycles. In particular, if we denote the periods by

$$\int_{A_I} \Omega = X^I \tag{2.3}$$

$$\int_{B^J} \Omega = F_J \tag{2.4}$$

we can use the X^I as projective coordinates on the moduli space of the Calabi-Yau (in particular special geometry implies that F_J is determined in terms of X^I as gradients of the prepotential $\mathcal{F}_0$, i.e. $F_J = \partial_J \mathcal{F}_0(X^I)$). However, as observed in [143], X^I and $\overline{X}^I$ do not commute in the BPS mini-superspace. Therefore, to write the wave function we have to choose a commuting subspace. A natural such choice is to parameterize this subspace by either real or imaginary part of X^I. Let us call these

[1]If we change the boundary conditions on the fermions to be anti-periodic, then the wave function does depend on the radius of $\mathbf{S}^1$ and its norm increases as the value of the supersymmetry breaking parameter increases [60].

variables Φ^I. Then, the wave function is given by

$$\psi_{P^I,Q_J}(\Phi^I) = \psi_{top}(P^I + \frac{i\Phi^I}{\pi}) \exp(Q_J \Phi^J/2) \tag{2.5}$$

where

$$\psi_{top}(X^I) = \exp(\mathcal{F}_{top}(X^I)) \tag{2.6}$$

is the B-model topological string partition function. For compact case there are only finite number of moduli, but for the non-compact case, which we are interested in here, there are infinitely many moduli and I runs over an infinite set. It should be understood that this expression for the wave function is only an asymptotic expansion (see [57] for a discussion of non-perturbative corrections to this). The overall rescaling of the charges is identified with the inverse of topological string coupling constant and we assume it to be large, so that the string expansion is valid. The wave function is peaked at the attractor value where

$$\mathrm{Re}X^I = P^I, \tag{2.7}$$

$$\mathrm{Re}F_J = Q_J. \tag{2.8}$$

We will be interested in a non-compact Calabi-Yau space, where the same formalism continues to hold (one can view it, at least formally, as a limit of a compact Calabi-Yau). In this case, we will have an infinite set of moduli. This is similar to [5, 4], where the incorporation of the infinitely many moduli in the non-compact case was shown to be crucial for reproducing the conjecture of [142]. Specifically, in this paper we will be considering the conifold, $T^*\mathbf{S}^3$. In this case, we can turn on a set of fluxes which result in a round $\mathbf{S}^3$ at the attractor point, and then consider the fluctuations of the metric captured by the topological string wave function. Before we proceed to this analysis, let us review some aspects of the topological strings on the conifold and its relation to non-critical bosonic strings on a circle of self-dual radius.

2.2 Topological Strings on the Conifold and Non-critical c=1 String

In this section we review the B-model topological string on the conifold

$$xy - zw = \mu \tag{2.9}$$

deformed by the terms of the form $\epsilon(x, y, z, w)$. The canonical compact 3-cycle of the conifold is $\mathbf{S}^3$. If we rewrite 2.9 as

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \mu \tag{2.10}$$

using appropriate change of the variables the real slice is exactly this $\mathbf{S}^3$ with the radius equal to $\sqrt{\mathrm{Re}\mu}$.

Let us recall that these deformations are in 1-to-1 correspondence with spin (j, j) representations of the $SO(4) \cong SU(2) \times SU(2)$ symmetry group [167, 174, 113]. Here, the variables x_i transform in the $(\frac{1}{2}, \frac{1}{2})$ representation of the $SU(2) \times SU(2)$. Thus, we can write x_i as $x^{AA'}$ where $A, A' = 1, 2$ are the spinor indices. In these notations, infinitesimal deformations of the hypersurface (2.9) can be represented by monomials of the form

$$\epsilon(x) = t_{A_1 A_2 \ldots A_n; A'_1 A'_2 \ldots A'_n} x^{A_1 A'_1} x^{A_2 A'_2} \ldots x^{A_n A'_n} \tag{2.11}$$

where the deformation parameters $t_{A_1 A_2 \ldots A_n; A'_1 A'_2 \ldots A'_n}$ are completely symmetric in all

A_i and all A'_i:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \mu + \epsilon(x) \tag{2.12}$$

We shall label a generic deformation of the form (2.11) by its quantum numbers:

$$t_{A_1 A_2 \ldots A_n; A'_1 A'_2 \ldots A'_n} \;\longrightarrow\; t_{|j,j;m,m'\rangle}. \tag{2.13}$$

where $j = n/2$ and

$$m = \sum_{i=1}^{n} (A_i - 3/2) \tag{2.14}$$

$$m' = \sum_{i=1}^{n} (A'_i - 3/2) \tag{2.15}$$

As is well known [82, 64, 129], topological B-model partition function of the conifold $Z_{\rm top}$ considered as a function of the deformation parameters (2.13) can be identified with the partition function of the $c = 1$ non-critical bosonic string theory at the self-dual radius

$$Z_{\rm top}(t) = Z_{c=1}(t) \tag{2.16}$$

The partition function (2.16) of the $c = 1$ theory is a generating functional for all correlation functions and has a natural genus expansion in the string coupling constant g_s. From the point of view of the conformal algebra the equation (2.9) describes a relation among four generators of the ground ring [167], and "cosmological constant" μ is interpreted as the conifold deformation parameter. Moreover the $SU(2) \times SU(2)$ used in classifying deformation parameters of the conifold get identified with the $SU(2) \times SU(2)$ symmetry of the conformal theory of $c = 1$ at the self-dual radius. Turning on only the momentum modes leads to deformations which depend on two of the parameters $\epsilon(x, y)$, whereas turning on the winding modes corresponds to deformations of the other two variables $\epsilon(z, w)$. Turning on all modes corresponds to an arbitrary deformation of the conifold $\epsilon(x, y, z, w)$, captured by (2.11).

The most well studied part of the amplitudes of $c = 1$ involves turning on momentum modes only. This corresponds to deformation

$$xy - zw = \mu + \sum_{n>0} (t_n x^n + t_{-n} y^n) + \dots \tag{2.17}$$

Here the dots stand for the terms of higher order in t_n which are only a function of x and y. The deformations t_n, associated with momentum n states, have the $SU(2) \times SU(2)$ quantum numbers

$$t_n \qquad \longleftrightarrow \qquad |\frac{|n|}{2}, \frac{|n|}{2}; \frac{n}{2}, \frac{n}{2}\rangle \tag{2.18}$$

where n runs over all integers.

The partition function (2.16) for this subset of deformations is equal to the τ-function of the Toda hierarchy. In particular, it depends on infinite set of couplings which are sources for the *amputated* tachyon modes[2]

$$\langle \mathcal{T}_{n_1} \dots \mathcal{T}_{n_k} \rangle = \frac{\partial}{\partial t_{n_1}} \dots \frac{\partial}{\partial t_{n_k}} \mathcal{F}_{c=1}(t)|_{t=0} \tag{2.19}$$

where on the left hand side we have connected amplitudes. The conservation of momentum implies that the sum of n's in each non-vanishing amplitude should be equal to zero,

$$\langle \mathcal{T}_{n_1} \mathcal{T}_{n_2} \dots \mathcal{T}_{n_k} \rangle = 0 \quad \text{unless} \quad \sum_{i=1}^{k} n_i = 0 \tag{2.20}$$

[2]Here, $\mathcal{T}_n = \frac{\Gamma(|n|)}{\Gamma(-|n|)} T_n$ where $T_n = \int d^2\sigma e^{(2-|n|)\phi/\sqrt{2}} e^{inX/\sqrt{2}}$ is the standard tachyon vertex. (We follow the conventions in [125, 59], which are slightly different from the conventions used in [87, 112].) For $n \in \mathbb{Z}$, the vertex operator $\mathcal{T}_n$ is a linear combination of a "special state" and the tachyon vertex [126, 125].

The tachyon correlators (2.19) can be computed using the $\mathcal{W}_{1+\infty}$ recursion relations of 2D string theory [59]. For example, for genus 0 amplitudes we have

$$\begin{aligned} &\langle \mathcal{T}_n \mathcal{T}_{-n} \rangle = -\tfrac{\mu^{|n|}}{g_s^2}\tfrac{1}{|n|} \\ &\langle \mathcal{T}_n \mathcal{T}_{n_1} \mathcal{T}_{n_2} \rangle = \tfrac{1}{g_s^2}\mu^{\frac{1}{2}(|n|+|n_2|+|n_3|)-1} \\ &\langle \mathcal{T}_{n_1} \mathcal{T}_{n_2} \mathcal{T}_{n_3} \mathcal{T}_{n_4} \rangle = \tfrac{1}{g_s^2}\mu^{\frac{1}{2}(|n_1|+|n_2|+|n_3|+|n_4|)-2}(1-\max\{|n_i|\}) \\ &\ldots \end{aligned} \tag{2.21}$$

The genus expansion of the free energy has the form

$$\mathcal{F}_{c=1} = \sum_{g=0}^{\infty} \left(\frac{\mu}{g_s}\right)^{2-2g} \mathcal{F}_g(t) \tag{2.22}$$

This is a good expansion in the regime $\mu >> g_s$. Usually, it is convenient to absorb the string coupling constant in the definition of μ, and make a suitable redefinition of the t_n's. However, in our case it is convenient *not* to do this; the advantage is that t_n's appear in the deformed conifold equation (2.17) without any extra factors.

It is instructive to note that, in this set of conventions, all the parameters μ, g_s, and t_n are dimensionful:

$$\begin{aligned} &\mu \sim [\text{length}]^2 \\ &g_s \sim [\text{length}]^2 \\ &t_n \sim [\text{length}]^{2-|n|} \end{aligned} \tag{2.23}$$

In particular, the ratio $(\frac{\mu}{g_s})$ is dimensionless, and t_n has the same dimension as $\mu^{1-\frac{|n|}{2}}$. Since the genus-g term in the free energy (5.192) should be independent of g_s, it follows that $\mathcal{F}_g(t)$ depends on t_n only via the combination $t_n \mu^{\frac{|n|}{2}-1}$. This is consistent with the fact that when all the t_n's are zero $\mathcal{F}_g$ is just a number

$$\mathcal{F}_g(t_n = 0) = \frac{(-1)^{g+1} B_{2g}}{2g(2g-2)} \quad , \quad g > 1 \tag{2.24}$$

In general, $\mathcal{F}_g(t)$ has the following structure [125, 112]

$$\mathcal{F}_g(t) = \sum_m P_g^m(n_i) \prod_{i=1}^m t_{n_i} \mu^{\frac{|n_i|}{2}-1} \tag{2.25}$$

where $P_g^m(n_i)$ is a polynomial in the momenta n_i of fixed degree depending on m and g. For example, $\deg P_g^2(n_i) = 4g-1$ and

$$P_1^2(n) = \frac{1}{24}(|n|-1)(n^2-|n|-1) \tag{2.26}$$

Also, $P_0^m(n_i)$ is a linear polynomial and for $m > 2$ is given by

$$P_0^m(n_i) = (-1)^{m-1}\frac{\mu^{m-2}}{m!}\Big(\psi_{m-2} + \frac{\max\{|n_i|\}}{2}\sum_{r=1}^{m-3}\frac{(m-2)!}{r!(m-2-r)!}\psi_{m-2-r}\psi_r\Big), \tag{2.27}$$

where $\psi_r := \left(\frac{d}{d\mu}\right)^r \log\mu$. Notice that these expressions for $P_0^3(n_i)$ and $P_0^4(n_i)$ agree with (2.21). Thus, the leading genus zero terms have the following form

$$\mathcal{F}_{c=1} = -\frac{1}{g_s^2}\sum_{n>0}\frac{1}{n}\mu^n t_n t_{-n} + \frac{1}{3!g_s^2}\sum_{n_1+n_2+n_3=0}\mu^{\frac{1}{2}(|n_1|+|n_2|+|n_3|)-1} t_{n_1}t_{n_2}t_{n_3} + \dots \tag{2.28}$$

It is easy to check that all the terms in this formula scale as μ^2. It also leads to the tachyon correlation functions (2.19) consistent with the KPZ scaling [114] (see also [87, 112]):

$$\langle \mathcal{T}_{n_1}\mathcal{T}_{n_2}\dots\mathcal{T}_{n_k}\rangle_g \sim \mu^{2(1-g)-k+\frac{1}{2}\sum_{i=1}^k |n_i|} \tag{2.29}$$

2.3 Local Volume Form Fluctuations on $\mathbf{S}^3$

We are interested in making a toy model of quantum cosmology. In this regard we are interested in the quantum metric fluctuations of an $\mathbf{S}^3$ inside the Calabi-Yau. More precisely, we take the nine-dimensional spatial geometry as

$$M^9 = \mathbf{S}^1 \times \mathbf{S}^2 \times T^*\mathbf{S}^3$$

and study the fluctuations of the metric on $\mathbf{S}^3 \subset T^*\mathbf{S}^3$. Usually we view the Calabi-Yau scales as much smaller than the macroscopic scales $\mathbf{S}^1$ and $\mathbf{S}^2$, but nothing in the formalism of Hartle-Hawking wave function prevents us from considering larger Calabi-Yau. In particular we will be assuming that $\mathbf{S}^3$ has a very large macroscopic size, which we wish to identify with our observed universe. One may view our world, in this toy model, as for example coming from branes wrapped over this $\mathbf{S}^3$, as we will discuss later in this section. For the purposes of this section we assume we have certain fluxes turned on, such that the classically preferred geometry for $\mathbf{S}^3$ is a large, round metric, and we study what kind of fluctuations are implied away from this round metric, in the context of Hartle-Hawking wave function in string theory. We ask, for example, if the metric fluctuation spectrum implied by this wave function is scale invariant?

From the point of view of the flux compactification considered in section 2.1 we set all electric fluxes to zero and turn on only one magnetic flux:

$$Q_I = 0, \tag{2.30}$$

$$P^{I\neq 0} = 0, \tag{2.31}$$

$$P^0 = N \tag{2.32}$$

Then the value of μ is fixed by the attractor mechanism:

$$\mathrm{Re}\mu = \frac{1}{2} N g_s \tag{2.33}$$

We will fix $N \gg 1$ leading to $\mathrm{Re}\mu/g_s \gg 1$. Note that in this limit the topological string partition function has a well defined perturbative expansion.

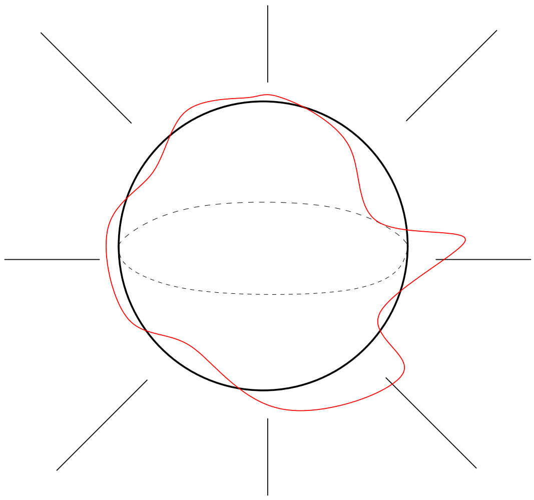


Figure 2.1: Fluctuations of the $\mathbf{S}^3$ inside a Calabi-Yau.

Now, let us consider a 3-sphere, $\mathbf{S}^3$, defined by the real values of the x_i, which satisfy (2.10). For non-zero (real) value of the deformation parameter μ and zero values of the t's, the induced metric on the $\mathbf{S}^3$ is the standard round metric, $g^{(0)}_{\mu\nu}$. The fluctuations of the moduli, δt, lead to perturbations of the Calabi-Yau metric on the conifold (2.10), and thus to perturbations of the metric, $g_{\mu\nu}$, induced on the 3-sphere

$$\delta t \longrightarrow \delta g_{\mu\nu} \tag{2.34}$$

In the topological B-model, the theory depends *only* on the complex structure deformations. This in particular means that not all deformations of the metric are observable. However, we recall that in the B-model the fundamental field is the holomorphic 3-form Ω and its variations. Moreover, on a special Lagrangian submanifold the volume form coincides with the restriction of a real form of Ω. In particular, an analog of the scalar fluctuations would be a fluctuation of the "conformal factor" ϕ, where

$$\Omega = e^{\phi}\Omega_0 \tag{2.35}$$

In particular the field ϕ and its fluctuations on a Lagrangian submanifold would be observable in our wave function induced from the B-model topological string. In particular we would be interested in the fluctuations of the field ϕ on the special Lagranigan $\mathbf{S}^3$ inside the conifold. Before discussing how we do this in more detail, let us return to what kind of cosmological models would this question be relevant for.

2.3.1 Toy Models of $\mathbf{S}^3$ Cosmology

So far we have discussed a supersymmetric (morally static) situation, where we ask the typical local shape of an $\mathbf{S}^3$ inside a Calabi-Yau. It is natural to ask if we can make a toy cosmology with this data, where the fluctuations we have studied would be observed as some kind of seed for inhomogeneity of fluctuations of matter.

In order to do this we need to add a few more ingredients to our story: First of all, we need to have the observed universe be identified with what is going on in an $\mathbf{S}^3$. The most obvious way to accomplish this would be in the scenario where we identify our world with some number of D3 branes wrapping $\mathbf{S}^3$. In this situation the inhomogeneities of the metric on $\mathbf{S}^3$ will be inherited by the D3 brane observer. A second ingredient we need to add to our story would be time dependence. This would necessarily mean going away from the supersymmetric context–an assumption which has been critical throughout our discussion. The least intrusive way, would be to have our discussion be applicable in an adiabatic context where we have a small supersymmetry breaking. In particular we imagine a situation where time dependence of the fields which break supersymmetry is sufficiently mild, that we can still trust a mini-superspace approximation in the supersymmetric sector of the theory.

To be concrete we propose one toy model setup where both of these can in principle be achieved. We have started with no D3 branes wrapped around $\mathbf{S}^3$. In the context of attractor mechanism this means that

$$\mathrm{Re}\widehat{\mu} = P$$

$$\mathrm{Re}\frac{i}{2\pi}\widehat{\mu}\log\frac{\widehat{\mu}}{\Lambda} = Q = 0$$

(where $\widehat{\mu} = 2\mu/g_s$) which we realize by taking $\widehat{\mu}$ to be real and equal to P and Λ to be real. The value of Λ is set by the data at infinity of the conifold. Let us write

$$\Lambda = \Lambda_0 \exp(i\varphi)$$

and imagine making Λ time dependent by taking a time dependent $\varphi(t)$. This can be viewed as a "time dependent axion field" induced from data at infinity. This leads to creation of flux corresponding to D3 brane wrapping $\mathbf{S}^3$ as is clear from the attractor mechanism. Indeed each time φ goes through 2π the number of D3 branes wrapping $\mathbf{S}^3$ increases by P units. This in turn can nucleate the corresponding D3 branes.

To bring in dynamics leading to evolution of radius of $\mathbf{S}^3$ we can imagine the following possibilities: Make the magnetic charge P time dependent by bringing in branes from infinity in the same class (or perhaps by the magnetic brane leaving and annihilating other magnetic anti-branes, leading to shrinking $\mathbf{S}^3$). This can in principle be done in an adiabatic way, thus making our story consistent with a slight time dependent μ. Another possibility which would be less under control would be to inject some energy on the D3 branes. It is likely that this leads to some interesting evolution for $\mathbf{S}^3$, though this needs to be studied. In particular, this can be accomplished by making the $\varphi(t)$ undergo partial unwinding motion. In this way we would create some number of anti-D3 branes which would annihilate some of the D3 branes. It is interesting to study what kind of cosmology this would lead to. Keeping this toy model motivation in mind we now return to the study of volume fluctuations in the supersymmetric model.

2.3.2 Setup for Computation of Volume Fluctuations

In principle if we know the full amplitudes of the $c = 1$ theory at the self-dual radius we can compute all correlation functions of ϕ. However the full amplitudes for $c = 1$ are not currently known (see [121] for recent work in this direction). We thus focus on the amplitudes which are known, which include the momentum mode correlations, as discussed in section 2.2. For two point correlation functions, as we will note below, the general amplitudes can be read off from this subspace of deformations, due to $SU(2) \times SU(2)$ symmetry of $\mathbf{S}^3$.

The momentum induced deformation of the 3-sphere in the conifold geometry (2.17) is obtained by specializing to a real three-dimensional submanifold, described by the equation

$$p + x_3^2 + x_4^2 = \mu + \epsilon(p, \theta) \tag{2.36}$$

where x_3, x_4 are real and without loss of generality, μ is assumed to be real and

$$x = p^{1/2} e^{i\theta} \tag{2.37}$$

$$y = p^{1/2} e^{-i\theta} \tag{2.38}$$

In these variables, the restriction of the holomorphic 3-form Ω to the hypersurface (2.36), which is the volume form on it, is given by

$$\Omega = \frac{dx_3 dx_4 d\theta}{1 - \partial_p \epsilon(p, \theta)}, \tag{2.39}$$

In particular, in the linear approximation

$$\epsilon(p, \theta) \approx \mathrm{Re} \sum_{n \neq 0} p^{|n|/2} e^{in\theta} t_n \tag{2.40}$$

The fluctuations of the "conformal factor" are given by

$$\phi = \log \frac{\Omega}{\Omega_0} = -\log(1 - \partial_p \epsilon) = \partial_p \epsilon + \tfrac{1}{2}(\partial_p \epsilon)^2 + \ldots \tag{2.41}$$

Now let us look at the absolute value squared of the Hartle-Hawking wave function (2.6). Remember that relation between the $c = 1$ theory at the self-dual radius and the B-model topological string on the conifold (2.16) implies $\mathcal{F}_{top}(t) = \mathcal{F}_{c=1}(t)$. Therefore,

$$|\Psi|^2 = \exp\Big(- \tfrac{1}{g_s^2} \textstyle\sum_{n>0} \tfrac{2}{n} \mu^n \mathrm{Re}(t_n t_{-n}) + \tfrac{1}{3g_s^2} \textstyle\sum_{n_1+n_2+n_3=0} \mu^{\frac{1}{2}(n_1+n_2+n_3)-1} \mathrm{Re}(t_{n_1} t_{n_2} t_{n_3}) + \ldots \Big) \tag{2.42}$$

We are going to use this wave function density to evaluate correlation functions of the form:

$$\langle \phi_1 \phi_2 \ldots \phi_n \rangle = \frac{\int \mathcal{D}t (\phi_1 \phi_2 \ldots \phi_n) |\Psi(t)|^2}{\int \mathcal{D}t |\Psi(t)|^2} \tag{2.43}$$

where $\phi_k = \phi(p = \mu, \theta_k)$ is the conformal factor at a point on the "large circle" of the $\mathbf{S}^3$, defined by (2.36) with $x_3 = x_4 = 0$. As we already noted, a computation of more general correlations functions (where ϕ_k are in general position on the $\mathbf{S}^3$) would require the information about the correlation functions of both momentum and winding modes of the $c = 1$ model. The reason for this is that $p \neq \mu$ implies $x_3^2 + x_4^2 \neq 0$ and, therefore, leads to generic deformations $\epsilon(p, \theta, x_3, x_4)$ in (2.36). Since from now on we will always consider only the correlation functions of the conformal factor on the large circle, $p = \mu$, we shall often write $\phi(\mu, \theta) = \phi(\theta)$. We will now turn to the two point function for which the momentum correlation functions are sufficient to yield the general correlation function due to $SU(2) \times SU(2)$ symmetry.

2.3.3 2-point Function at Tree Level

We start with evaluating a two point correlation function:

$$\langle \phi_1 \phi_2 \rangle$$

Because of the $SO(4)$ symmetry of the $\mathbf{S}^3$, one can always assume that ϕ_1 and ϕ_2 are evaluated at two points on the large circle $p = \mu$. To the leading order in g_s/μ, one can keep only the linear terms in (2.41). The contribution of non-linear terms in (2.41) is suppressed by g_s and will be discussed later. Thus, in the linear approximation to (2.41) and substituting (2.40),

$$\langle \phi(\theta) \phi(0) \rangle = \frac{1}{4\mu^2} \sum_{n,m} |nm| \mu^{\frac{|n|+|m|}{2}} e^{in\theta} \langle t_n t_m \rangle \tag{2.44}$$

where we used the fact that t_n and t_{-n} are complex conjugate after reduction to the 3-sphere.

Similarly, restricting (2.42) to the 3-sphere, we get

$$|\Psi_{\mathbf{S}^3}(t)|^2 = \exp\left(-\frac{2}{g_s^2} \sum_{n>0} \frac{\mu^{|n|}}{|n|} t_n t_{-n} + \frac{1}{3 g_s^2 \mu} \sum_{n_1+n_2=-n_3} \mu^{\frac{|n_1|+|n_2|+|n_3|}{2}} t_{n_1} t_{n_2} t_{n_3} + \ldots \right) \tag{2.45}$$

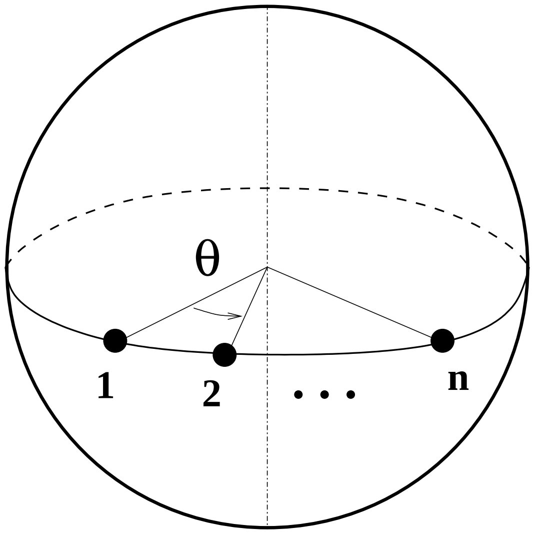


Figure 2.2: n points on the large circle of the $\mathbf{S}^3$.

In particular, it gives

$$\langle t_n t_m \rangle = \frac{\int \mathcal{D}t |\Psi_{\mathbf{S}^3}(t)|^2 t_n t_m}{\int \mathcal{D}t |\Psi_{\mathbf{S}^3}(t)|^2} = \frac{|n|}{2} g_s^2 \mu^{-|n|} \delta_{n+m,0} + \mathcal{O}((\mu/g_s)^{-|n|-2}) \tag{2.46}$$

Evaluating the path integral we treat non-quadratic terms in (2.45) as perturbations. In general, this will give a highly non-trivial theory with all types of interactions. However, using scaling properties (2.23), one can show that contribution from the k-tuple interaction vertex is proportional to $(g_s/\mu)^{k-2}$ (see discussion below). Therefore, all loop corrections to the leading term are suppressed in the limit of large $\mathbf{S}^3$ radius and small string coupling, g_s. As a result, using (2.46) we find

$$\langle \phi(\theta)\phi(0) \rangle = \frac{g_s^2}{8\mu^2} \sum_n |n|^3 e^{in\theta} + \dots \tag{2.47}$$

or, equivalently,

$$\langle \phi_n \phi_{-n} \rangle = \frac{g_s^2}{8\mu^2} |n|^3 \tag{2.48}$$

where

$$\phi_n := \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{i\theta n} \phi(\theta) \tag{2.49}$$

This has to be compared with the usual Fourier transform, given by a 3-dimensional integral

$$\langle \phi_{\vec{k}} \phi_{-\vec{k}} \rangle \sim \int d^3x e^{i\vec{k}\cdot\vec{x}} \langle \phi(\vec{x})\phi(0) \rangle \tag{2.50}$$

$$\sim g_s^2 |\vec{k}| \tag{2.51}$$

Note that a scale invariant power spectrum would correspond to $|k|^{-3}$ fluctuation correlation. Thus the fluctuation spectrum we have on $\mathbf{S}^3$ is *not* scale invariant.

After performing the summation over n in (2.48), we get the 2-point function

$$\langle \phi(\theta)\phi(0) \rangle = \frac{g_s^2}{32\mu^2} \frac{\cos\theta + 2}{\sin^4\theta/2} \tag{2.52}$$

in the coordinate representation. Although this expression appears to have a singularity at $\theta = 0$, as we explain in the next subsection, our approximation cannot be trusted at large momenta or, equivalently, small $\theta < \sqrt{g_s/\mu}$.

2.3.4 General Structure of g_s Corrections

There are three sources for the g_s corrections to the 2-point function: *i*) one due to higher genus terms in the free energy expansion (5.192), *ii*) corrections due to loops made from the k-point vertices (with $k > 2$) in the "effective action" $\mathcal{F}(t)$ as well as due to non-linear terms in the expansion (2.41) of ϕ in terms of ϵ, and *iii*) corrections due to non-linear relation between ϵ and the deformation parameters t_n induced by the deformations of the geometry [59, 6]:

$$xy = \mu - x_3^2 - x_4^2 + \sum_{n>0}(t_n x^n + t_{-n} y^n) - \frac{1}{2\mu} \sum_{\substack{m>0 \\ n>0}} t_n t_{-m} m x^n y^m + \dots \tag{2.53}$$

It is easy to check that all kinds of corrections are suppressed by powers of $(g_s/\mu)^2$. In the case *i*) this is manifest from the form of (5.192). In the case *ii*), *iii*), this can be seen in the language of the Feynman diagrams for the fluctuating fields t_n, that follow from the effective action (2.45):

$$\text{propagator}: \qquad \tfrac{1}{2} g_s^2 |n| \mu^{-|n|} + \dots \tag{2.54}$$

$$k \geq 3 \text{ vertex}: \qquad \frac{2}{g_s^2 k!} \mu^{\frac{|n_1|+\dots+|n_k|}{2}+2-k} P_0^k(n_i) + \dots \tag{2.55}$$

Here $P_0^k(n_i)$ is a linear polynomial (2.27) in momenta n_i, and the dots stand for higher-genus terms. In particular, a genus-g contribution comes with an extra factor of $(g_s/\mu)^{2g}$.

In general, we find that the genus-g contribution (contribution from g loops) to the 2-point function looks like

$$\langle \phi_n \phi_{-n} \rangle_g \sim \Big(\frac{g_s}{\mu}\Big)^{2g+2} |n|^{4g+3} \tag{2.56}$$

where ϕ_n is defined in (2.49). We can read off the higher genus corrections to the propagator $\langle t_n t_{-n} \rangle$ from the quadratic terms in (2.25):

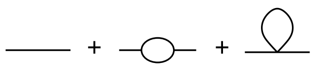
$$\langle t_n t_{-n} \rangle|_{tree} = \frac{\frac{1}{2} g_s^2 |n| \mu^{-|n|}}{1 + \sum_{g \geq 1} (g_s/\mu)^{2g} |n| P_g^2(n)} \tag{2.57}$$

Notice that $\deg P_g^2(n) = 4g-1$ and therefore for large momenta we have an asymptotic expansion of the form:

$$\langle t_n t_{-n}\rangle|_{tree} = \frac{\frac{1}{2}g_s^2|n|\mu^{-|n|}}{1+\sum\limits_{g\geq 1} p_g\left(\frac{n^2 g_s}{\mu}\right)^{2g}+\dots} \tag{2.58}$$

where constants p_g are determined by the polynomial $P_g^2(n)$ and dots stand for the terms suppressed by powers of n. Now it is clear that the good expansion parameter is $\frac{n^2 g_s}{\mu}$ rather than $\frac{g_s}{\mu}$ which means that our approximation is valid only for momenta n small compared to μ/g_s. In other words, we should fix some high-energy cut-off parameter $\Lambda^2 < \mu/g_s$ and consider only deformations with momentum number $n < \Lambda$.

Now let us incorporate corrections due to loops in Feynman diagrams generated by (2.54). For example, if we take into account genus one corrections and one-loop corrections we get the following expression for the propagator:



$$+\dots = \frac{\frac{1}{2}g_s^2|n|\mu^{-|n|}+\dots}{1-\frac{|n|}{24}\left(\frac{g_s}{\mu}\right)^2(|n|-1)(n^2-|n|-1)+\dots} \tag{2.59}$$

Notice that due to (2.56) g-loops corrections dependence on momenta is similar to the genus g corrections. Thus, the general structure of the 2-point function is given by:

$$\langle\phi_n\phi_{-n}\rangle_g = \frac{g_s^2}{8\mu^2}|n|^3\frac{1+\sum\limits_g b_g(\frac{n}{\Lambda})\left(\frac{\Lambda^2 g_s}{\mu}\right)^{2g}+\dots}{1+\sum\limits_g p_g\left(\frac{n^2 g_s}{\mu}\right)^{2g}+\dots} \tag{2.60}$$

where the polynomials b_g depend on the ratio n/Λ, which should be small in order for the perturbation theory to be valid.

2.3.5 n-point Function for the Perturbations on the Large Circle of $\mathbf{S}^3$

Here we briefly discuss the structure of n point function. Unlike the 2-point function where we could compute the general case, for n point functions with the present technology, we can only compute correlations restricted to taking the fluctuations at points on a large circle. Using the Feynman rules (2.54), we find that the contribution of a tree Feynman diagram to a k-point function scales as (to avoid cluttering, we omit polynomials in n_i which do not affect the g_s behavior):

$$\langle t_{n_1} \dots t_{n_k} \rangle_0 \sim g_s^{2k-2} \mu^{-\frac{|n_1|+\dots+|n_k|}{2}+2-k} \tag{2.61}$$

Now, let us consider a g-loop contribution to the k-point function. As we discussed earlier, such contributions come from the vertices with a total of $k+2g$ legs, k of which are connected by propagators to the external legs of the k-point function, and $2g$ of which are pairwise connected by internal propagators. Notice that, for the internal momenta n_j, the factors $\mu^{\frac{|n_j|}{2}}$ cancel out and we get

$$\langle t_{n_1} \dots t_{n_k} \rangle_g \sim g_s^{2k+2g-2} \mu^{-\frac{|n_1|+\dots+|n_k|}{2}+2-k-2g} \tag{2.62}$$

Comparing this expression with (2.61), we see that a g-loop contribution to the k-point correlation function is suppressed by the same factor $(g_s/\mu)^{2g}$ as the contribution from a genus-g term in the free energy (5.192). For the k-point function of the fields ϕ_n this implies

$$\langle \phi_{n_1} \dots \phi_{n_k} \rangle_g \sim \mu^{\frac{|n_1|+\dots+|n_k|}{2}-k} \langle t_{n_1} \dots t_{n_k} \rangle_g \sim \Big(\frac{g_s}{\mu}\Big)^{2k+2g-2}, \tag{2.63}$$

where we used (2.62). Notice, this structure is consistent with our results (2.48) for the 2-point function. For an example of higher point function we now turn to a discussion of the leading correction to the 3-point function.

2.3.6 3-point Function

Now, let us look more carefully at the structure of the 3-point function. Unlike the 2-point function where we studied the general case, since the topological string amplitudes are not known for arbitrary deformations of the conifold, we restrict our attention to the ones corresponding to momentum modes. This means that we consider 3-point functions where all three points lie on the large circle of the $\mathbf{S}^3$. To the leading order in $(g_s/\mu)^2$, from (2.41) we find

$$\langle\phi(\theta_1)\phi(\theta_2)\phi(\theta_3)\rangle = \frac{1}{8\mu^3}\sum_{n,m,l}|nml|\mu^{\frac{|n|+|m|+|l|}{2}}e^{in\theta_1+im\theta_2+il\theta_3}\langle t_n t_m t_l\rangle \tag{2.64}$$

In the momentum representation, this looks like

$$\langle\phi_n\phi_m\phi_l\rangle = \frac{1}{8\mu^3}|nml|\mu^{\frac{|n|+|m|+|l|}{2}}\langle t_n t_m t_l\rangle \tag{2.65}$$

According to (2.45),

$$\langle t_n t_m t_l\rangle = \frac{g_s^4}{4}\delta_{n+m+l,0}|nml|\mu^{-\frac{|n|+|m|+|l|}{2}-1} \tag{2.66}$$

which gives

$$\langle\phi_n\phi_m\phi_l\rangle = \frac{1}{32}\Big(\frac{g_s}{\mu}\Big)^4|nml|^2\delta_{n+m+l,0} \tag{2.67}$$

This is to be compared with the three point function of fluctuations in the inflationary cosmology [120].

Chapter 3

The Entropic Principle and Asymptotic Freedom

Consider, following [143], a flux compactification on $\mathrm{AdS}_2 \times \mathbf{S}^2 \times M$, where M is a Calabi-Yau threefold (and where for simplicity we ignore a $\mathbb{Z}$-identification). The norm of the Hartle-Hawking wave function associated with this background can be interpreted holographically as the black hole entropy. In particular, the flux data on the $\mathrm{AdS}_2 \times \mathbf{S}^2$ geometry is mapped to the charge of the dual black hole, and the norm of the wave function satisfies

$$\langle \psi | \psi \rangle = \exp(S) \tag{3.1}$$

where S denotes the entropy of the corresponding black hole. For a fixed flux data, the wave function ψ can be viewed as a function over the moduli space of the Calabi-Yau, together with the choice of the normalization for the holomorphic 3-form, where the overall rescaling of the holomorphic 3-form corresponds to the overall rescaling of the charge of the black hole. Clearly the entropy of the black hole increases as we rescale the overall charge. However, to obtain a wave function on the Calabi-Yau moduli space, one would like to get rid of this extra rescaling. The main purpose of this chapter is to suggest one mechanism of how this may be done: We simply fix one of the magnetic charges, and its electric dual chemical potential. In this way, as we shall argue, the wave function becomes a function on the geometric moduli space of the Calabi-Yau and one can see which Calabi-Yau manifolds are "preferred".

It turns out that this problem can be formulated for both compact and non-compact Calabi-Yau manifolds. It is a bit more motivated in the non-compact case, since in this case there is a canonical choice of the fixed charge (D0 brane in the type IIA context). We find that the condition for a maximum/minimum corresponds to the points of intersection of walls of marginal stability. Moreover we find a set of solutions in various examples. One type of solutions we find corresponds to points on moduli space which admit a complex multiplication type structure. We also find examples where extrema correspond to the appearance of extra massless particles. We find that in these cases the norm of the wave function ψ is *maximized* in correlation with the sign of beta function: Asymptotically free theories yield maximum norm for the wave function.

The organization of this chapter is as follows: we start in section 3.1 with the formulation of the problem. Then, in section 3.2, we find the conditions for maxima/minima of the wave function. We also explain why this favors asymptotically free theories in local examples where there are massless fields. In section 3.3 we give examples of our results and in section 3.4 we end with conclusions and some open questions.

3.1 General Formulation of the Problem

Consider compactifications of type IIB superstrings on $\mathrm{AdS}_2 \times \mathbf{S}^2 \times M$ where M is a Calabi-Yau threefold, which may or may not be compact. In addition we consider fluxes of the 4-form gauge field along $\mathrm{AdS}_2 \times \mathbf{S}^2$ and 3-cycles of M. Let $F_{p,q} = p^I \alpha_I + q_J \beta^J$ denote the flux through M, where α_I and β^J form a canonical symplectic basis for integral 3-form cohomology $H^3(M, \mathbb{Z})$. According to [143], the results of [142] can be interpreted in terms of a Hartle-Hawking type wave function $\psi_{p,q}$ for this geometry on the minisuperspace, with the property that

$$\langle \psi_{p,q} | \psi_{p,q} \rangle = \exp S(p,q) \tag{3.2}$$

Here $S(p,q)$ denotes the entropy of the dual black hole obtained by wrapping a D3 brane with magnetic and electric charges p and q.

In the limit of large fluxes,

$$(p,q) \to \lambda(p,q) \tag{3.3}$$

where $\lambda >> 1$, the entropy $S(p,q)$ has a classical approximation, given by the Bekenstein-Hawking formula. In this limit, the curvature of $\mathrm{AdS}_2 \times \mathbf{S}^2$ becomes small, and the entropy of the black hole is equal to $\frac{1}{4}A(\mathbf{S}^2)$. In particular, the attractor mechanism [76] will freeze the complex structure moduli of the Calabi-Yau space M, so that there exists a holomorphic 3-form Ω on M with the property

$$\mathrm{Re}(\Omega) = F_{p,q} \tag{3.4}$$

Moreover, in this limit the entropy is given by

$$S(p,q) = \frac{A(\mathbf{S}^2)}{4} = -i\frac{\pi}{4}\int_M \Omega \wedge \overline{\Omega}$$

where Ω fixed by (3.4). Furthermore, in this limit, $\mathrm{Im}\Omega$ plays the role of the chemical potential.

Suppose we wish to ask the following question: In type IIB compactification on $\mathbb{R}^4 \times M$, which Calabi-Yau M is "preferred"? One way to tackle this question is to embed it in the geometry $\mathrm{AdS}_2 \times \mathbf{S}^2 \times M$ where the complex structure of M is determined by the fluxes, through the attractor mechanism. Then, the question becomes: For which values of the complex structure moduli the norm of the wave function is maximized or, in other words, for which attractor Calabi-Yau the entropy of the corresponding black hole is maximized? In this formulation of the question, we can view $\mathbb{R}^4$ as a special limit of $\mathrm{AdS}_2 \times \mathbf{S}^2$ where the charge of the black hole is rescaled by an infinite amount $\lambda \to \infty$. Thus $\mathrm{AdS}_2 \times \mathbf{S}^2$ can be viewed as a regulator geometry for $\mathbb{R}^4$.

However, this way of asking the question leads to the following pathology: The entropy of the black hole for large λ scales as λ^2. Therefore, in order to get a reasonable function on the moduli space of M we need to fix the normalization of Ω. One way to do this is to fix the value of $\int \Omega \wedge \overline{\Omega}$ so that it is the same at all points in moduli; however this is precisely the entropy that we wish to maximize. If we fixed the normalization of Ω in this way, we would obtain, tautologically, a flat distribution on the moduli space of M. In this sense there would be no particular preference of one point on the moduli of M over any other. Instead we consider the following mathematically natural alternative: We choose a 3-cycle $A_0 \subset M$ and fix the normalization of Ω by requiring it to have a fixed period along A_0,

$$\int_{A_0} \Omega = \text{fixed} \tag{3.5}$$

Since the overall scale of Ω does not affect the extremum point on the moduli space of M, with no loss of generality we can fix the above period to be 1. Then, we can consider maximizing the entropy, which now depends only on the geometric moduli of M. Thus our problem becomes

$$\text{Maximize} \left| \int \Omega \wedge \overline{\Omega} \right| \qquad \text{subject to} \qquad \int_{A_0} \Omega = 1. \tag{3.6}$$

Physically, what this means is that we fix one of the charges of the black hole, say the magnetic charge $p^0 = 1$, and the corresponding electric chemical potential $\phi^0 = 0$ (determined by the imaginary part of Ω).

Strictly speaking, the above problem is well defined on the moduli space of Calabi-Yau manifolds together with a choice of a 3-cycle. This, in general, is a covering of the moduli space. Nevertheless, any particular maximization of the entropy functional on this covering space will descend to a particular choice of the complex structure moduli of M (by forgetting on which sheet the function is maximized). Of course, it would be interesting to find out whether or not this covering of the moduli space is a finite covering or not. We will discuss some aspects of this in section 3.4. We should point out, however, that there is a *canonical* choice of the cycle A_0 in the mirror type IIA problem, where one is studying even-dimensional D-branes wrapped over even-dimensional cycles of a non-compact Calabi-Yau manifold. In this case, one can choose A_0 to be a point on M and consider a fixed number of $D6$ branes with zero chemical potential for D0 branes. We also note that, in the type IIA setup, fixing one of the periods can be interpreted as fixing the topological string coupling constant

$$X^0 = \frac{4\pi i}{g_s} \tag{3.7}$$

In what follows, we consider both compact and non-compact examples.

3.2 General Conditions for Maxima/Minima

In this section, we derive equations for the critical points of the constrained variational problem described in the previous section, and discuss maxima of the entropy functional

$$S = -i\frac{\pi}{4}\int_M \Omega \wedge \overline{\Omega} \tag{3.8}$$

It turns out that the critical points of (3.8) are described by equations of the form

$$\mathrm{Im}a^D - \tau\,\mathrm{Im}a = 0,$$

where (a, a^D) denote "reduced" periods of the Calabi-Yau and $\tau = \frac{da^D}{da}$ is the coupling constant matrix. The points on the moduli space where the entropy is maximized are those where $\mathrm{Im}\tau > 0$ and all but one of the Calabi-Yau periods have equal phase. As we explain below, these are also the points where the maximal number of the walls of marginal stability for BPS states meet together. Notice, the restriction $\mathrm{Im}\tau > 0$ implies that the effective field theory for the extra massless particles appearing at the maximum point can be decoupled from the gravity. This is a necessary condition for the asymptotically free effective theories.

3.2.1 Critical Points

In order to extremize the functional (3.8) we need to introduce coordinates on the moduli space of a Calabi-Yau manifold with one fixed 3-cycle. Given the symplectic basis of 3-cycles $\{A_I, B^J\}_{I,J=0,\ldots,h^{2,1}}$, such that $\#(A_I, B^J) = \delta_I{}^J$, the periods of the holomorphic 3-form are

$$X^I = \int_{A_I} \Omega \tag{3.9}$$

$$F_I = \int_{B^I} \Omega \tag{3.10}$$

In particular, one can use X^I as the projective coordinates on the Calabi-Yau moduli space, and express the B-periods as derivatives of the prepotential:

$$F_I(X) = \frac{\partial \mathcal{F}_0(X)}{\partial X^I} \tag{3.11}$$

We choose the fixed 3-cycle to be A_0. Then the normalization of Ω is fixed by the condition

$$X^0 = \int_{A_0} \Omega \tag{3.12}$$

As we discussed earlier, we can always set $X^0 = 1$. However, in what follows it will be useful to keep the dependence on X^0 which, in the type IIA context, determines the topological string coupling constant, *cf.* (3.7).

It is natural to use the following coordinates on the moduli space of Calabi-Yau manifolds with a fixed 3-cycle:

$$a^i = \frac{X^i}{X^0}, \qquad i = 1, \dots, h^{2,1} \tag{3.13}$$

Also, we introduce the "dual" variables

$$a_i^D = \frac{F_i}{X^0} \tag{3.14}$$

and a "rigid" prepotential

$$F(a) = (X^0)^{-2} \mathcal{F}_0(X(a)) \tag{3.15}$$

Then, using the fact that $\mathcal{F}_0$ is a homogeneous holomorphic function of degree two, we find

$$F_0 = X^0 (2F - a^i a_i^D) \tag{3.16}$$

Therefore, the functional (3.8) can be written as[1]

$$S = i\frac{\pi}{4}|X^0|^2\{2(F-\overline{F}) - (a^i - \overline{a}^i)(a_i^D + \overline{a}_i^D)\} \tag{3.17}$$

This is, of course, the standard expression for the Kähler potential $S = \frac{\pi}{4}\, e^{-K}$, written in terms of the special coordinates, see *e.g.* [29].

Extremizing the action (3.17) with respect to a^i and $\overline{a}^i$, we obtain the following system of equations:

$$\mathrm{Im} a_i^D - \tau_{ij}\mathrm{Im} a^j = 0 \tag{3.18}$$

where the coupling constant matrix τ_{ij} is given by

$$\tau_{ij} = \frac{\partial^2 F}{\partial a^i \partial a^j} \equiv \frac{\partial^2 \mathcal{F}_0}{\partial X^i \partial X^j} \tag{3.19}$$

Solutions to the equations (3.18) define critical points on the Calabi-Yau moduli space. Our goal will be to study these points and to understand their physical and/or geometric meaning.

Before we proceed, let us make a few general comments about the form of the equations (3.18). First, note that (3.18) is a system of non-linear complex equations. Even though these equations are not differential, finding their solutions is a challenging and interesting problem. To see this, let us write (3.18) in the following form

$$\mathrm{Im}\, \partial_i F - (\partial_i \partial_j F)\mathrm{Im} a^j = 0 \tag{3.20}$$

where we expressed a_i^D and τ_{ij} in terms of the single function $F(a^i)$. For a given Calabi-Yau space M and a choice of the 3-cycle, the function $F(a^i)$ is fixed; it

[1] Here we used the Riemann bilinear identity $\int_M \alpha \wedge \beta = \sum_I \left(\int_{A_I} \alpha \int_{B^I} \beta - \int_{A_I} \beta \int_{B^I} \alpha \right)$.

is generically a non-trivial transcendental function. Therefore, (3.18) (equivalently (3.20)) represents a system of $n = h^{2,1}$ complex equations for n complex variables a^i, $i = 1, \dots, n$. Therefore one expects that solutions to these equations are isolated points in the moduli space.

Let us note that Calabi-Yau manifolds which correspond to these points admit special structures, analogous to the complex multiplication. The notion of complex multiplication for higher dimensional varieties goes back to the work of Mumford [130]; in the context of Calabi-Yau three-folds it was studied by Borcea [26]. In the physics literature, it appears in the study of black hole attractors [124] and rational conformal field theories [92]. Let us recall the attractor equations [76]

$$2\mathrm{Im} a^i = p^i \tag{3.21}$$

$$2\mathrm{Im} a_i^D = q_i \tag{3.22}$$

where $(p, q) \in \mathbb{Z}^n$ denote magnetic and electric fluxes. One says that a Calabi-Yau manifold admits complex multiplication if the Jacobian $T = \mathbb{C}^n/(\mathbb{Z}^n + \tau\mathbb{Z}^n)$ associated with the coupling constant matrix τ_{ij} admits complex multiplication. This occurs if τ_{ij} satisfies the following second order matrix equation:

$$\tau C \tau + A\tau - \tau D - B = 0 \tag{3.23}$$

where A, B, C, D are some integer matrices. It is straightforward to check that this is indeed the case for a suitable choice of the integer matrices[2]. Therefore, any Calabi-Yau with moduli fixed by the attractor mechanism (3.21) and satisfying equation (3.18) admits complex multiplication. Notice, that Jacobian T for this Calabi-Yau is singular, since (3.18) implies that there are fixed points under the $\mathbb{Z}^n + \tau\mathbb{Z}^n$ action.

[2]For example, $A = n\vec{q} \otimes \vec{q}, \qquad B = 0, \qquad C = m\vec{p} \otimes \vec{p} - n\vec{p} \otimes \vec{q}, \qquad D = m\vec{p} \otimes \vec{q}$

Solutions to (3.18) fall into three families, which can be characterized by the imaginary part of the coupling constant matrix τ_{ij}. To see this, notice that the imaginary part of extremum equations (3.18) is given by

$$\mathrm{Im}\tau_{ij} \cdot \mathrm{Im}a^j = 0 \tag{3.24}$$

Therefore, if $\mathrm{Im}\tau_{ij}$ is non-degenerate (that is if $\det||\mathrm{Im}\tau_{ij}|| \neq 0$), the only possible solution is $\mathrm{Im}a^i = 0$. Moreover, assuming that τ_{ij} remains finite[3], it also follows that $\mathrm{Im}a_i^D = 0$. We shall refer to this family of solutions as solutions of type I:

$$\mathrm{Im}a^i = 0, \qquad \mathrm{Im}a_i^D = 0, \qquad i = 1, \ldots, n \tag{3.25}$$

The expression for the entropy functional calculated at the critical point of type I turns out to be very simple:

$$S_* = i\frac{\pi}{2}|X^0|^2(F - \overline{F}) \tag{3.26}$$

If we go to the conventional topological string notations

$$F_{top} = i(2\pi)^3 F, \quad Z = \exp\frac{1}{g_s^2}F_{top} \tag{3.27}$$

and use (3.7), we see that the probability function for the Calabi-Yau at the critical point is given by the square of the topological wave function $\Psi_{top} = Z$, in accordance with [143, 142]:

$$e^{S_*} = |\Psi_{top}|^2 \tag{3.28}$$

Solutions of type II correspond to $\det||\mathrm{Im}\tau_{ij}|| = 0$. They can be expressed in terms of *real* eigenvectors v^i of the coupling constant matrix, $\mathrm{Im}\tau_{ij} \cdot v^j = 0$,

$$\mathrm{Im}a^i = v^i, \qquad \mathrm{Im}a_i^D = \tau_{ij}v^i, \qquad i, j = 1, \ldots, n \tag{3.29}$$

[3] or if $\tau \sim a^{-\alpha}$, where $\alpha < 1$

Finally, solutions of type III correspond to divergent coupling constant matrix:

$$\mathrm{Im}a^i = 0, \qquad \mathrm{Im}a_i^D = \lim_{\mathrm{Im}a^i \to 0} \mathrm{Re}\tau_{ij} \cdot \mathrm{Im}a^i \neq 0, \qquad i = 1, \ldots, n \tag{3.30}$$

All types of the solutions represent some special points on the Calabi-Yau moduli space.

3.2.2 Conditions for Maximum/Minimum

It is natural to ask which of these critical points are maxima and which are minima. The former correspond to theories which are preferred, while the latter correspond to the theories which are least likely, according to the entropic principle. Therefore, our main goal is to search for the maximum points.

In order to answer this question, we need to look at the second variation of the action at the critical point:

$$\delta^2 S = -\frac{\pi}{2}|X^0|^2(\mathrm{Im}\tau_{ij})\delta a^i \delta\overline{a}^j + \frac{\pi}{4}|X^0|^2 \mathrm{Im}a^i (c_{ijk}\delta a^j \delta a^k + \overline{c}_{ijk}\delta\overline{a}^j \delta\overline{a}^k) \tag{3.31}$$

where c_{ijk} are defined as

$$c_{ijk} = \frac{\partial^3 F}{\partial a^i \partial a^j \partial a^k} \tag{3.32}$$

The bilinear form $\mathrm{Im}a^i c_{ijk}\delta a^j \delta a^k$ does not have a definite signature. This means that if it is non-zero, the critical point is neither minimum nor maximum. Therefore, a necessary condition for the critical point to be a local maximum is

$$c_{ijk}\mathrm{Im}a^k = 0 \tag{3.33}$$

Let us concentrate on the first term in (3.31), assuming that this condition is satisfied. Remember that reduced coupling constant matrix τ_{ij} is part of a full matrix τ_{IJ}.

Imaginary part of this matrix has signature[4] of type $(h^{2,1}, 1)$, as follows from the expression

$$\mathrm{Im}\tau_{IJ} = \frac{i}{2}\int \partial_I\Omega \wedge \overline{\partial}_J\overline{\Omega} \tag{3.34}$$

and decomposition $\partial_I\Omega \in H^{2,1}(M)\oplus H^{3,0}(M)$. Therefore, the signature of the reduced matrix τ_{ij} is either $(h^{2,1}, 0)$ or $(h^{2,1}-1, 1)$. In the first case the form $(\mathrm{Im}\tau_{ij})\delta a^i\delta\overline{a}^j$ is positive definite and therefore, the entropy functional has

$$maximum, \quad \text{if} \quad \mathrm{Im}\tau_{ij} > 0 \tag{3.35}$$

In the second case generically we have a saddle point, and a minimum if $h^{2,1} = 1$. Thus we conclude that, in general, Calabi-Yau models with $\mathrm{Im}\tau_{ij} > 0$ are preferred.

The extremum conditions (3.18) are very restrictive, but it is hard to find a solution in general case. However, if we want to satisfy constraint (3.33), which is necessary for maximization of the entropy, the problem simplifies. Assuming that c_{ijk} is not of special degenerate form, a general solution to this constraint is $\mathrm{Im}a^i = 0$, and therefore we should look at the type I solution

$$\mathrm{Im}a^i = \mathrm{Im}a_i^D = 0 \tag{3.36}$$

Physically, this is a particularly natural choice and we can explain it in yet another way: The extremum of the probability density given by the entropy functional, should naturally pick attractor fixed points. In our problem the electric chemical potential is set to zero and the magnetic charge is fixed in one particular direction. It is not

[4]The simplest illustration is a toy model with the cubic prepotential, $\mathcal{F}_0 = -\frac{1}{3}\frac{(X^1)^3}{X^0}$. It is easy to check that the signature of $\mathrm{Im}\tau_{IJ}$ in this example is (1,1). Notice, that there is a difference between the physical coupling constant matrix, including the graviphoton coupling, and τ_{IJ}. The physical coupling constant matrix is always positive definite (see [43]).

surprising that this means that the rest of the charges are set to zero at the extremum. In particular the equation (3.36) can naturally be interpreted as the attractor point with this set of charges.

Another way to derive (3.36) is to notice that imaginary part of the coupling constant matrix enters the second variation of the action (3.31) and therefore it should be non-degenerate at the local minimum or maximum point:

$$\det||\mathrm{Im}\tau_{ij}|| \neq 0 \tag{3.37}$$

Combining this with the imaginary part of extremum equations (3.24), we get $\mathrm{Im}a^i = 0$.

One of the interesting examples of the maximum entropy solutions is the one with the logarithmic behavior of the coupling constant matrix (explicit examples of this are discussed in the next section):

$$\tau = \tau_0 + i\beta \log \frac{a^2}{\Lambda^2} \tag{3.38}$$

near the critical point $a = 0$. From the point of view of the corresponding effective field theory in four dimensions this expression describes an RG flow of the couplings, and β is the one-loop beta function of the effective field theory near the point $a = 0$. Combining this with the maximum condition $\mathrm{Im}\tau_{ij} > 0$ discussed earlier, we conclude that for the theories with $\beta < 0$, that is, *for asymptotically free theories the probability density is maximized.*

3.2.3 Marginal Stability Curves and the Entropy

As we will explain below, the conditions (3.36) imply that *the points where the entropy is maximized correspond to points on the moduli space which are maximal intersection points of the marginal stability walls for BPS states.* The solutions to (3.36) can be characterized in purely geometric terms. Let us consider the type IIB setup and look at the periods (3.9) - (3.10) of the Calabi-Yau. Since we used the gauge $X^0 = 1$, the condition (3.25) means that exactly $2h^{2,1} + 1$ periods, namely $(X^0, X^i; F_i)$ are real. However, the last period $F_0 = \frac{2}{X^0}\mathcal{F}_0 - \frac{X^i}{X^0}F_i$ does not have to be real, since the phase of the prepotential is not fixed by the phase of its derivatives. In fact, if all of the periods were real, the holomorphic volume of the Calabi-Yau $\int \Omega \wedge \overline{\Omega} = (X^I \overline{F}_I - \overline{X}^I F_I)$ would be zero. Considering such singular Calabi-Yau would hardly make sense, as it implies that Ω is pointwise zero on the Calabi-Yau. Fortunately, this is not the case since F_0 is not real. Thus the geometry of periods is as given in Figure 3.1.

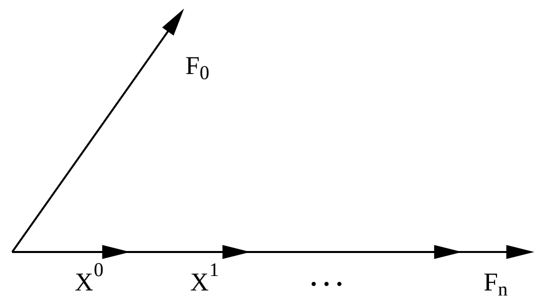


Figure 3.1: Calabi-Yau periods at the maximum entropy point on the moduli space.

Note that a point where all periods but one are aligned is a point where a maximal number of walls of marginal stability for BPS states meet together. In fact, this is the strongest condition we can have; generically, it is impossible to have all periods aligned as the number of constraints would be one higher than the number of parameters. So, the condition for a maximum for the probability is the same as maximal marginality for BPS bound states.

One can actually relax the condition $X^0 = 1$, and introduce an arbitrary phase $X^0 = e^{i\phi}$ instead. This will accordingly rotate all other periods, resulting in an equivalent Calabi-Yau manifold. Therefore, we can formulate an alternative maximum criterion: *The entropy functional is maximized for the points on the moduli space where all but one of the Calabi-Yau periods are aligned on the complex plane, and* $\mathrm{Im}\tau_{ij} > 0$.

Suppose now that we can find such a point, where exactly $(2h^{2,1}+1)$ of the periods are aligned. We should stress that for a given prepotential these aligned periods can actually be some linear combinations of the canonical A and B periods (3.9) -(3.10). Is there a freedom to choose, which of them we should use to fix the normalization of Ω in the maximization problem (3.6), or there is a *canonical* choice of the cycle A_0? The answer to the last question is positive: the cycle A_0 is dual to the 3-cycle which is not aligned with the rest of the 3-cycles. In other words, A_0 corresponds to the null vector in the space of $(2h^{2,1} + 1)$ aligned cycles with respect to the intersection pairing. Thus, given the point on the moduli space where all but one of the Calabi-Yau periods are aligned, the cycle A_0 is determined uniquely. In the next section, we will illustrate this with a simple example of the quintic three-fold.

Similarly, in the type IIA setup, we can consider bound states of D0, D2, and D4 branes with charges n_0, $\vec{n}_2$, and $\vec{n}_4$, respectively. The BPS mass of such states is given by the standard formula

$$M_{BPS} = |Z| = |n_0 + \vec{n}_2 \cdot \vec{a} + \vec{n}_4 \cdot \vec{a}^D| \tag{3.39}$$

Since in our case $\mathrm{Im} a_i = \mathrm{Im} a_i^D = 0$, it follows from the BPS formula (3.39) that the critical points of type I are precisely the points in the moduli space, where all bound states of D0, D2, and D4 branes become marginal,

$$M_{BPS}(n_0, \vec{n}_2, \vec{n}_4) = M_{BPS}(n_0, 0, 0) + M_{BPS}(0, \vec{n}_2, 0) + M_{BPS}(0, 0, \vec{n}_4) \tag{3.40}$$

3.3 Examples

In this section we will discuss two types of examples, corresponding to non-compact and compact Calabi-Yau cases. There is a crucial difference between these two cases in the type IIA setup. Namely, on a compact Calabi-Yau manifold M, the cycles undergo monodromies as one goes around singularities in the moduli space, while on a non-compact Calabi-Yau there always exists at least one cycle (0-cycle in the type IIA frame) which does not undergo monodromy. Therefore, this is a canonical cycle to fix the period. However, as we will see, the general approach based on (3.6) works in both cases.

As we discussed in the previous section, the problem of finding the maximum points on the moduli space is equivalent to the problem of finding the points where all but one of the Calabi-Yau periods are aligned. Unfortunately, at the moment it is unknown how to find all such points for a given class of Calabi-Yau manifolds.

Our approach below is to look at the well known special points on the moduli space (singularities, large complex structure, etc.) as potential candidates. Therefore, our list of examples is far from complete and serves just as an illustration of the general idea. We find two types of solutions: Solutions which correspond to points on Calabi-Yau moduli which admit a structure similar to complex multiplication. The other type of solutions corresponds to points on the moduli space where we have massless particles. In the non-compact case the only set of examples where we actually find a *maximum*, as opposed to minimum or other extremum points, is when we have extra massless fields which lead to an asymptotically free gauge theory.

3.3.1 The Local $\mathbf{CP}^1$

Let us start with the two simplest local models for a non-compact Calabi-Yau manifolds, the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{CP}^1$ (conifold singularity), and the total space of $\mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbf{CP}^1$. (The exact stringy wave function for the conifold model with an infinite set of non-normalizable deformations was studied in [91].) It is instructive to look first at the infinite product representation of the topological string partition function on the A-model side [85]. For the conifold we have[5]:

$$Z_{(-1,-1)} = \frac{\prod_{n=1}^{\infty} (1 - q^n Q)^n}{\prod_{n=1}^{\infty} (1 - q^n)^n} \tag{3.41}$$

[5] Here we use topological string conventions: $t = 2\pi(J - iB)$. Sometimes an alternative convention $t = B + iJ$ is used in the literature, since then the mirror map is given by $t = \frac{X^1}{X^0}$. In notations (3.13) then we have $a = B + iJ$.

where $q = e^{-g_s}$ and $Q = e^{-t}$, while for $\mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbf{CP}^1$ case:

$$Z_{(0,-2)} = \frac{\prod_{n=1}^{\infty} (1-q^n)^n}{\prod_{n=1}^{\infty} (1-q^n Q)^n} \tag{3.42}$$

In the semiclassical limit $q \to 1$, which is the limit we are most interested in, the above expressions depend on Q only. Moreover, the first expression decreases while the second increases rapidly as $Q \to 1$. And as we will see in a moment, the entropy functional has a minimum for the conifold, and a maximum for $\mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbf{CP}^1$ at $t = 0$. Indeed, since $Z = \exp\left(\frac{1}{g_s^2} F_{top}\right)$, the entropy functional (3.17) in this representation is given by

$$e^S = |Z|^2 e^{-\frac{1}{2}(t+\bar{t})\left(\frac{\partial}{\partial t}\log Z + \frac{\partial}{\partial \bar{t}}\log \overline{Z}\right)} \tag{3.43}$$

As expected, the second variation of the functional at the extremum is equal to

$$\frac{\delta^2 S}{\delta t \delta \bar{t}} = -\frac{2\pi}{g_s^2}\mathrm{Im}\tau \tag{3.44}$$

where

$$\tau = i\frac{g_s^2}{2\pi}\frac{\partial^2}{\partial t^2}\log Z \tag{3.45}$$

At the conifold point we have $t = t^D = 0$. Therefore (3.25) is satisfied and the conifold point is an extremum. Moreover, near $t = 0$ we have

$$\tau = \frac{i}{2\pi}\log t + \ldots \tag{3.46}$$

Hence, $\mathrm{Im}\tau < 0$, which means that the conifold point is a minimum. On the contrary, since the infinite product expression (3.42) for the total space of $\mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbf{CP}^1$ is given by the inverse of (3.41), in this case we have $\mathrm{Im}\tau > 0$ at $t = 0$. Therefore, this is a maximum point.

One can arrive to the same conclusion by looking at the genus-zero prepotential, which is equivalent to the approach based on the infinite-product formula (3.41). For example, for the conifold, we have

$$F^0_{top} = \frac{1}{12}t^3 - \frac{\pi^2}{6}t + \zeta(3) - \sum_{n=0}^{\infty} \frac{e^{-nt}}{n^3} \tag{3.47}$$

and therefore

$$\tau = \frac{i}{2\pi} \log(1 - e^{-t}) + \dots \tag{3.48}$$

in agreement with (3.46). For the total space of $\mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbf{CP}^1$ we get the same expression with an extra minus sign.

The difference between these two examples is very instructive: At the conifold point, where the entropy in minimized, we have extra massless hypermultiplet, while at $t = 0$ locus of $\mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbf{CP}^1$, where the entropy in maximized, an extra massless vector multiplet appears. The morale of the story is that Calabi-Yau manifold providing the vector multiplet is preferred.

Let us conclude with a few general remarks. Suppose we use topological strings to obtain information about the effective supersymmetric four-dimensional gauge theories with matter fields and interactions which, in principle, can have applications to phenomenological models of particle physics. This can be done, for example, in the context of the geometrical engineering program in string theory. Here, topological strings provide a nice laboratory, since many exact results for the topological string partition function are available.

In the example we just studied, the distinction between these two cases directly corresponds to whether or not the effective field theory is asymptotically free. In particular, the appearance of a vector multiplet, which corresponds to an asymptotically free theory, is preferred over the non-asymptotically free theory, where we have an extra massless hypermultiplet. From the effective field theory viewpoint, this is translated into the statement that *asymptotically free effective theories are preferred*. One can also study other special points in moduli space, such as points which correspond to conformal fixed points. In general, these will involve more moduli parameters, and one might expect that in these cases we will have a mixture of the above results: along some directions the probability for conformal fixed point is maximized and along others it is minimized. In other words, it is an extremum for the entropy functional, but not a definite maximum or minimum. In fact we will discuss an explicit example which corresponds to a multi-parameter model where one finds that it is an extremum but not a pure maximum or a minimum (see sec. 3.3.4 below).

3.3.2 Large Complex Structure Limit

Below we will argue that near the large complex structure limit in the one-parameter Calabi-Yau models there is an infinite family of solutions to the extremum equations (3.18), labeled by an integer number. Namely, for the values of the complexified Kähler parameter t, such that $\mathrm{Re}t = 0$ and $\mathrm{Im}t \gg 1$, there are infinitely many points where exactly three of the periods are aligned. In a sense, these are the points where the Calabi-Yau admits a deformed complex multiplication. Indeed, at these points, to the leading approximation, τ satisfies a quadratic equation with integer coefficients. The entropy functional (3.8) has a maximum for all such points.

Let us start with a simple model with the cubic prepotential

$$\mathcal{F}_0 = -\frac{1}{3}\frac{(X^1)^3}{X^0} \tag{3.49}$$

This model captures the leading behavior of all one-parameter Calabi-Yau models in the large complex structure limit. It can also be viewed as the exact prepotential for some parts of the CY moduli which receive no quantum corrections (such as the Kähler moduli of T^2's inside CY threefolds). The vector of periods is given by

$$\begin{pmatrix} X^0 \\ X^1 \\ F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} X^0 \\ X^1 \\ -(X^1)^2/X^0 \\ \frac{1}{3}(X^1)^3/(X^0)^2 \end{pmatrix} \tag{3.50}$$

Let us consider these periods on the subspace $\mathrm{Im}X^0 = 0$ and $\mathrm{Re}X^1 = 0$. In other words, let us set the B-field to zero[6]. Then, two of the periods (namely, X^1 and F_0) are purely imaginary, and two (X^0 and F_1) are real. However, at the points where

$$\left(\frac{X^1}{X^0}\right)^2 = n \tag{3.51}$$

for some (negative) integer n, the linear combination of the periods $F^1 + nX^0$ vanishes! This means that at these points the following three periods are aligned: $(X^1, F_0, F^1 + nX^0)$; they take purely imaginary values, while X_0 is real. According to our general principle, these points are extremum points of the entropy functional. The main question now is whether this extremum is a minimum or a maximum.

In order to answer this question we need to identify the 3-cycle A_0, to find the resulting prepotential, and to calculate the imaginary part of the coupling constant.

[6]We use conventions, where $t = a = \frac{X^1}{X^0} = B + iJ$.

We use $SL(4,\mathbb{Z})$ transformation to bring the periods into the following form

$$\begin{pmatrix} X^0 \\ X^1 \\ F_1+nX^0 \\ F_0+nX^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ n & 0 & 1 & 0 \\ 0 & n & 0 & 1 \end{pmatrix} \begin{pmatrix} X^0 \\ X^1 \\ F_1 \\ F_0 \end{pmatrix} \tag{3.52}$$

As we discussed in subsection 3.2.3, the cycle A_0 corresponds to the null direction in the space spanned by the aligned periods. In the present case, it is easy to see that the period corresponding to this 3-cycle is F_0+nX^1. Therefore, normalizing the value of the period of Ω over this 3-cycle to unity, as in (3.6), is equivalent to rescaling of all the periods by $(F_0+nX^1)^{-1}$. Then, the period vector takes the form:

$$\begin{pmatrix} -2\widetilde{\mathcal{F}}_0+\widetilde{X}^1\widetilde{F}_1 \\ -\widetilde{F}_1 \\ \widetilde{X}^1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{nt+t^3/3} \\ \frac{t}{nt+t^3/3} \\ \frac{-t^2+n}{nt+t^3/3} \\ 1 \end{pmatrix} \tag{3.53}$$

where $t=\frac{X^1}{X^0}$ and $\widetilde{\mathcal{F}}_0$ is the prepotential in the new basis. In these variables, the coupling constant is given by $\tau=\frac{\partial\widetilde{F}_1}{\partial\widetilde{X}^1}$. Straightforward calculation gives:

$$\tau=\frac{i}{4\sqrt{|n|}} \tag{3.54}$$

at the critical point (3.51), where $t=i\sqrt{|n|}$. Thus, all the solutions from the infinite family (3.51) have $\mathrm{Im}\tau>0$ and correspond to the maxima of the entropy functional. This simple example illustrates the behavior of a one-parameter Calabi-Yau near the large complex structure limit, $\mathrm{Im}t\gg 1$. In this limit, prepotential receives small instanton corrections and contains subleading quadratic and linear terms. However, in principle one could still find the infinite family of aligned periods by solving appropriately modified equation (3.51):

$$\mathrm{Re}F_1 + nX^0 = 0, \tag{3.55}$$

since the periods X^1 and F_0 in general case are imaginary along the zero B-field line $\mathrm{Re}X^1 = 0$, $\mathrm{Im}X^0 = 0$. These will correspond, in the large $\mathrm{Im}t$ limit to small perturbations of the solutions we have already discussed above.

However, we do not get interesting effective field theories at such points, and the relative weight ("probability") of such points is much smaller than that of the maxima where new massless degrees of freedom appear, because of the logarithmic behavior (3.46) of the coupling constant at the singular points.

3.3.3 The Quintic

Let us consider, following [30], the type IIA superstrings on the well studied compact Calabi-Yau manifold with $h^{2,1} = 1$, which is the mirror of the quintic hypersurface in $\mathbf{CP}^4$. It can be obtained as a $(\mathbb{Z}_5)^3$ quotient of the special quintic

$$\sum_{i=1}^{5} z_i^5 - 5\psi \prod_{i=1}^{5} z_i = 0 \tag{3.56}$$

The complex moduli space is $\mathbf{CP}^1$, parameterized by $z = \psi^{-5}$, with three special points:

$$\begin{aligned} z = 0 :& \quad \text{large complex structure limit} \\ z = 1 :& \quad \text{conifold point} \\ z = \infty :& \quad \text{Gepner point} \end{aligned} \tag{3.57}$$

The four periods undergo monodromy about these three points. It is convenient to use the basis of the periods corresponding to the BPS state in the mirror A-model labeled by the $(D6, D4, D2, D0)$-brane charges: $(\Pi_{D0}, \Pi_{D2}, \Pi_{D4}, \Pi_{D6})$. These periods provide corresponding D-brane tensions. The general expression for the prepotential is

$$F(t) = -\frac{5}{6}t^3 - \frac{11}{4}t^2 + \frac{25}{12}t - \frac{25i}{2\pi^3}\zeta(3) + i\sum_{k=1}^{\infty}\frac{d_k}{(2\pi k)^3}e^{2\pi ikt} \tag{3.58}$$

where

$$t = \frac{\Pi_{D2}}{\Pi_{D0}} = \frac{1}{2\pi i}\log z + \dots \tag{3.59}$$

is a mirror map, and d_k are instanton amplitudes (Gromov-Witten invariants), related to the number n_k of rational curves of degree k embedded in the quintic, as

$$\sum_{k=0}^{\infty} d_k e^{2\pi ikt} = 5 + \sum_{k=1}^{\infty}\frac{n_k k^3}{1 - e^{2\pi it}}e^{2\pi ikt} \tag{3.60}$$

We are interested in the solutions to the extremum equations (3.18). As was discussed before, we expect an infinite family of such solutions at zero B-field in the large complex structure limit. Indeed, the periods in the basis (3.9)-(3.10) are given by

$$\begin{pmatrix} X^0 \\ X^1 \\ F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} 1 \\ t \\ -\frac{5}{2}t^2 - \frac{11}{2}t + \frac{25}{12} - \sum_{k=1}^{\infty}\frac{d_k}{(2\pi k)^2}e^{2\pi ikt} \\ \frac{5}{6}t^3 + \frac{25}{12}t - \frac{25i}{\pi^3}\zeta(3) + \sum_{k=1}^{\infty}(2i + 2\pi kt)\frac{d_k}{(2\pi k)^3}e^{2\pi ikt} \end{pmatrix} \tag{3.61}$$

Therefore, at the special set of points, where $\mathrm{Re}t = 0$ and a deformed CM-type equation holds:

$$\frac{5}{2}t^2 - \frac{25}{12} + \sum_{k=1}^{\infty}\frac{d_k}{(2\pi k)^2}e^{2\pi ikt} = n \tag{3.62}$$

the following three periods are aligned: $(X^1, F_0, F^1 + nX^0)$, where n is an integer. It is clear that when $n \gg 1$ the instanton corrections in (3.62) are small and general behavior is similar to the cubic prepotential case. Therefore, we conclude that this infinite set of solutions describes local maxima of the entropy functional.

As we discussed earlier, the conifold point $z = 1$ is a potential candidate for a critical point of the entropy functional. Let us then look at the periods and try to find out which of them are aligned. The periods satisfy corresponding Picard-Fuchs differential equation of hypergeometric type:

$$\Big[\theta_z^4 - (\theta_z + \frac{1}{5})(\theta_z + \frac{2}{5})(\theta_z + \frac{3}{5})(\theta_z + \frac{4}{5})\Big]\Pi_i = 0 \tag{3.63}$$

where $\theta_z = z\frac{d}{dz}$. We use the conventions of ([52]) to write down the basis of the solutions to (3.63) as follows:

$$\begin{aligned}
&\Pi_0(z) = U_0(z) \\
&\Pi_1(z) = U_1(z) \text{ if } \mathrm{Im} z < 0, \quad U_1(z) + U_0(z) \text{ if } \mathrm{Im}\ z > 0 \\
&\Pi_2(z) = U_2(z) \\
&\Pi_3(z) = U_3(z) \text{ if } \mathrm{Im} z < 0, \quad U_3(z) + U_2(z) \text{ if } \mathrm{Im}\ z > 0
\end{aligned} \tag{3.64}$$

where U_i are given in terms of the Meijer G-function [70]

$$\begin{aligned}
U_0(z) &= c\ G^{1,4}_{0,3}\left(-z \middle| \begin{matrix} \frac{4}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{matrix}\right) \\
U_1(z) &= \tfrac{c}{2\pi i} G^{2,4}_{1,2}\left(z \middle| \begin{matrix} \frac{4}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{matrix}\right) \\
U_2(z) &= \tfrac{c}{(2\pi i)^2} G^{3,4}_{1,1}\left(-z \middle| \begin{matrix} \frac{4}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{matrix}\right) \\
U_3(z) &= \tfrac{c}{(2\pi i)^3} G^{4,4}_{1,0}\left(z \middle| \begin{matrix} \frac{4}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{matrix}\right)
\end{aligned} \tag{3.65}$$

and

$$c = \frac{1}{\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})} \tag{3.66}$$

Near $z = 0$ the periods Π_j behave as $(\log z)^j$. One can go to another natural basis, corresponding to $(D6, D4, D2, D0)$-brane state with the help of the following transformation matrix

$$\begin{pmatrix} \Pi_{D6} \\ \Pi_{D4} \\ \Pi_{D2} \\ \Pi_{D0} \end{pmatrix} = \begin{pmatrix} 0 & 5 & 0 & 5 \\ 0 & 1 & -5 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Pi_0 \\ \Pi_1 \\ \Pi_2 \\ \Pi_3 \end{pmatrix} \tag{3.67}$$

The intersection form in the new basis is defined by

$$\#(D6 \cap D0) = 1, \qquad \#(D2 \cap D4) = 1.$$

Straightforward calculation at the conifold point[7] $z = e^{-i0}$ with the help of the *Mathematica* package gives:

$$\begin{pmatrix} \Pi_{D6} \\ \Pi_{D4} \\ \Pi_{D2} \\ \Pi_{D0} \end{pmatrix} = \begin{pmatrix} 0 \\ 5\alpha - 7i\beta \\ 2i\beta \\ \gamma \end{pmatrix} \tag{3.68}$$

where (α, β, γ) are real constants:

$$\begin{aligned} \alpha &= -\tfrac{\sqrt{5}}{16\pi^4} \operatorname{Re} G^{3,4}_{0,1}\left(-1 \,\middle|\, \begin{matrix} \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ 0 & 0 & 0 & 0 \end{matrix}\right) \approx -1.239 \\ \beta &= \tfrac{\sqrt{5}}{16\pi^3} G^{2,4}_{0,2}\left(1 \,\middle|\, \begin{matrix} \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ 0 & 0 & 0 & 0 \end{matrix}\right) \approx 0.646787 \\ \gamma &= {}_4F_3(\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}; 1, 1, 1; 1) \approx 1.07073 \end{aligned} \tag{3.69}$$

From (3.68) we see that it is possible to align three periods by choosing appropriate

[7]We fix this choice of the branch cut by requiring that D6 brane become massless at the conifold point.

linear transformation. For example, we can take

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Pi_{D6} \\ \Pi_{D4} \\ \Pi_{D2} \\ \Pi_{D0} \end{pmatrix} = \begin{pmatrix} 2\Pi_{D6} \\ 2\Pi_{D4} + 7\Pi_{D2} \\ \Pi_{D2} \\ \Pi_{D0} \end{pmatrix} = \begin{pmatrix} 0 \\ 10\gamma \\ 2i\beta \\ \alpha \end{pmatrix} \tag{3.70}$$

Therefore, the conifold point in the quintic is a solution to the extremum equations (3.18) if we fix appropriate 3-cycle. In particular, the choice (3.70) corresponds to fixing $X^0 = 2\Pi_{D4} + 7\Pi_{D2}$.

3.3.4 A Multi-parameter Model

Finally, we consider an example of a non-compact Calabi-Yau manifold with several moduli fields. Such models exhibit some new phenomena. For example, there can be points in the moduli space where several different periods vanish, and the corresponding BPS states become massless. In general, one might expect such points to be saddle points for the entropy functional (neither maxima nor minima). This is indeed what we find in a specific example considered below.

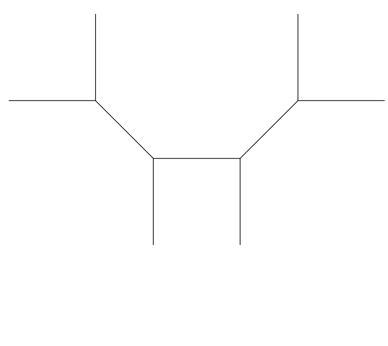


Figure 3.2: The toric diagram of the 3-parameter model.

Consider a 3-parameter model studied *e.g.* in [3]. It has three 2-cycles, whose

Kähler parameters we denote[8] by t_1, t_2, and r. The prepotential has the form:

$$F^0_{top} = \sum_n \frac{e^{-nt_1}}{n^3} \sum_n \frac{e^{-nt_2}}{n^3} + \sum_n \frac{e^{-nr}(1-e^{-nt_1})(1-e^{-nt_2})}{n^3} \tag{3.71}$$

and the dual variables are

$$t_1^D = \frac{1}{4\pi^2} \sum_n \frac{e^{-nt_1}(1-e^{-nr}(1-e^{-nt_2}))}{n^2} \tag{3.72}$$

$$t_2^D = \frac{1}{4\pi^2} \sum_n \frac{e^{-nt_2}(1-e^{-nr}(1-e^{-nt_1}))}{n^2} \tag{3.73}$$

$$r^D = \frac{1}{4\pi^2} \sum_n \frac{e^{-nr}(1-e^{-nt_1})(1-e^{-nt_2})}{n^2} \tag{3.74}$$

The coupling constant matrix τ_{ij} is given by (3.45) and has the following entries (symmetric in $i, j = 1, 2, 3$):

$$\begin{aligned}
\tau_{11} &= -\tfrac{i}{2\pi} \log \tfrac{(1-e^{-t_1})(1-e^{-(r+t_1+t_2)})}{(1-e^{-(r+t_1)})} \\
\tau_{22} &= -\tfrac{i}{2\pi} \log \tfrac{(1-e^{-t_2})(1-e^{-(r+t_1+t_2)})}{(1-e^{-(r+t_2)})} \\
\tau_{12} &= -\tfrac{i}{2\pi} \log(1-e^{-(r+t_1+t_2)}) \\
\tau_{13} &= -\tfrac{i}{2\pi} \log \tfrac{(1-e^{-(r+t_1+t_2)})}{(1-e^{-(r+t_1)})} \\
\tau_{23} &= -\tfrac{i}{2\pi} \log \tfrac{(1-e^{-(r+t_1+t_2)})}{(1-e^{-(r+t_2)})} \\
\tau_{33} &= -\tfrac{i}{2\pi} \log \tfrac{(1-e^{-r})(1-e^{-(r+t_1+t_2)})}{(1-e^{-(r+t_1)})(1-e^{-(r+t_2)})}
\end{aligned} \tag{3.75}$$

Consider taking the limit $(t_1, t_2, r) \to 0$ along the imaginary line, such that the ratios t_1/r and t_2/r are kept fixed. Then imaginary parts of the dual variables (3.72) are zero. Therefore, it is a particular solution to the extremum equations (3.25).

In order to determine the behavior of the entropy functional near this extremum, we should diagonalize imaginary part of the matrix τ_{ij} and look at the eigenvalues.

[8]In notations (3.13), they are given by $t_1 = -2\pi i a^1, t_2 = -2\pi i a^2, r = -2\pi i a^3$.

It is easy to see that in this limit it is given by:

$$\mathrm{Im}\tau = -\frac{\log|\,r|}{2\pi}\begin{pmatrix} 1+\mathcal{O}(x) & 1+\mathcal{O}(x) & \mathcal{O}(x) \\ 1+\mathcal{O}(x) & 1+\mathcal{O}(x) & \mathcal{O}(x) \\ \mathcal{O}(x) & \mathcal{O}(x) & -1+\mathcal{O}(x) \end{pmatrix} \tag{3.76}$$

where $x \sim \log^{-1}|\,r|$. To solve the diagonalization problem to the leading order in x, it is enough to consider the matrix of the form

$$\begin{pmatrix} 1 & 1 & a \\ 1 & 1 & b \\ a & b & -1 \end{pmatrix} \tag{3.77}$$

where $a \ll 1$ and $b \sim a$. The eigenvalues of this matrix are given by $(1, \frac{1}{2}(a-b)^2, -2)$ to the leading order. Therefore, the imaginary part of the coupling matrix (3.76) near the extremum point has one large positive eigenvalue of order $\log|\,r|$, one positive eigenvalue of order 1, and one large negative eigenvalue of order $\log|\,r|$. Notice that $\mathrm{sign}(\mathrm{Im}\tau) = (2,1)$ and therefore, this is an example of a signature of type $(h^{2,1}-1, 1)$. In this case we are having a saddle point of the entropy functional.

3.4 Conclusions and Further Issues

In this chapter we discussed the behavior of the stringy wave function on the moduli space of a Calabi-Yau manifold. It became a meaningful quantity once we fixed a particular combination of charge/chemical potential for one of the magnetic/electric charges of the black hole. The square of this wave function can be interpreted as a measure for string compactifications. As we discussed, the solution to finding maxima/minima of this function has a nice geometric meaning: they correspond to points on the moduli space where all but one period of the holomorphic 3-form Ω have equal phase. The formulation of this geometric problem involves a choice of a 3-cycle $A_0 \in H_3(M,\mathbb{Z})$, whose period we denote X^0 (or, a choice of $A_0 \in H^{\text{even}}(M,\mathbb{Z})$ in type IIA theory).

While it appears to be a rather challenging problem to obtain a complete solution to these equations, we managed to find a certain class of solutions. They fall into two families: They either correspond to 'quantum deformed' complex multiplication points on the moduli space of a Calabi-Yau manifold, or to points with extra massless particles. Moreover, for the examples with extra massless degrees of freedom the maxima that we found correspond to the points where the effective field theory is asymptotically free.

As discussed above, in order to write down our wave function, we need to choose a particular direction in the charge lattice. In the type IIB case, this corresponds to choosing an integral 3-cycle. It is natural to ask how our conclusions depend on this choice (for the type IIA on non-compact CY there is a natural direction of the charge lattice which corresponds to D0 brane charge). For a compact manifold M there is no natural choice of $A_0 \in H_3(M,\mathbb{Z})$. In fact, even for a particular choice of A_0, there is an ambiguity related to the monodromy action on $H_3(M,\mathbb{Z})$. To classify these choices we need to study the monodromy group action on $H_3(M,\mathbb{Z})$. For a Calabi-Yau manifold M, the monodromy group H is a subgroup of $G = Sp(2h^{2,1}+2,\mathbb{Z})$, so that the number of distinct choices of a 3-cycle is given by the index $[G : H]$. The calculation of $[G : H]$ for a compact Calabi-Yau space is an interesting and challenging problem. By analogy with the mapping class group of a genus-g Riemann surface [164], we may expect that $[G : H]$ is finite. In fact, the monodromy group H can be generated by two elements, which correspond to monodromies around the conifold point and infinity. For example, for the quintic threefold, we have

$$M_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad M_\infty = \begin{pmatrix} 1 & -1 & 5 & -3 \\ 0 & 1 & -8 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

In this case, it is easy to see that $[G : H] > 1$.

Another important question that deserves further study is classification of the extremum points on the moduli space, as solutions to the equations (3.25). In mathematical terms, the problem is to find all the points on the moduli space where all but one of the Calabi-Yau periods are aligned.

Finally it would be interesting to understand more physically what it means to fix a charge/chemical potential, and why that is natural. It is conceivable that this becomes natural in the context of decoupling gravity from gauge theory. In particular a preferred direction may be the direction corresponding to the graviphoton charge. It is worthwhile trying to dynamically explain such a formulation of the problem.

Chapter 4

Abelian Varieties, RCFTs, Attractors, and Hitchin Functional in Two Dimensions

In this chapter we study a two-dimensional toy model of the topological M-theory [58, 133] in order to gain some insights on its plausible quantum description. The analysis of this model supports the idea that quantum partition function of the topological M-theory is given by a generalized index theorem for the moduli space. In particular, this implies that the OSV conjecture [142] should be viewed as a higher dimensional analog of the E. Verlinde's formula for the number of conformal blocks in a two-dimensional conformal field theory.

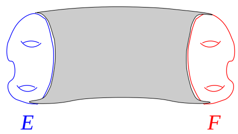
4.1 Universal Partition Function and Universal Index Theorem

Below we briefly sketch some relatively old ideas that provide a motivation for this picture. This introductory section is inspired by the talks of R. Dijkgraaf [56] and A. Gerasimov [80].

It is well known that many generic features of topological theories can be nicely described using the category theory. Roughly speaking, the translation between the category theory and quantum mechanics language goes as follows (for more details and references see, *e.g.* [55]). One starts with associating a wave-function $|\Psi_0(M)\rangle$ to a d-dimensional manifold M:

$$= |\Psi_0(M)\rangle. \quad (4.1)$$

A natural generalization is assigning some additional structures E (bundles, sheaves, gerbes, etc.) to M: $|\Psi_0(M)\rangle \to |\Psi_E(M)\rangle$. In the language of physics, this is equivalent to putting some branes and/or fluxes on M. The morphisms in the category of $(d+1)$-dimensional manifolds with extra structures are bordisms $E \to F$:

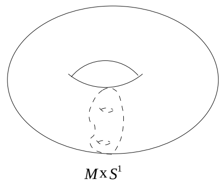


$$= \langle\Psi_E(M)|\Psi_F(M)\rangle, \quad (4.2)$$

interpreted as quantum mechanical propagators between the states E and F. The composition law of two bordisms given by "gluing" two boundaries is a basic feature of the functional integral:

$$\langle\Psi_E|\Psi_G\rangle = \int \mathcal{D}F \langle\Psi_E|\Psi_F\rangle\langle\Psi_F|\Psi_G\rangle. \quad (4.3)$$

These pictures, of course, mimic the well-known operations with the world-sheets in string theory. The universal partition function $Z_{M\times\mathbf{S}^1}$ is assigned to the manifold of the form $M \times \mathbf{S}^1$:



$$= \ \mathrm{Tr}_E\langle\Psi_E(M)|\Psi_E(M)\rangle. \tag{4.4}$$

Roughly speaking, it counts the number of topological (massless) degrees of freedom, or (super) dimension of the corresponding Hilbert space. Relation (4.4) is a manifestation of the equivalence between the Lagrangian and Hamiltonian formulation of the path integral. In the framework of geometric quantization, the Hilbert space is given by the cohomology groups of the moduli space $\mathcal{M}_E$ of E-structures on M, with coefficients in the (prequantum) line bundle $\mathcal{L}$. Therefore, the universal partition function is associated with the corresponding index: $Z_{M\times\mathbf{S}^1} = \mathrm{Ind}\mathcal{D}_{\mathrm{E}}$, where[1]

$$\mathrm{Ind}\mathcal{D}_{\mathrm{E}} = \sum_n (-1)^n \mathrm{dim} H^n(\mathcal{M}_E, \mathcal{L}). \tag{4.5}$$

In many interesting cases higher cohomology groups vanish, and the partition function computes the dimension of the Hilbert space: $\mathrm{dim}Hilb_{\mathcal{M}_E} = \mathrm{dim}H^0(\mathcal{M}_E, \mathcal{L})$. On the other hand, the partition function can also be computed via the universal index theorem:

$$\mathrm{Ind}\mathcal{D}_{\mathrm{E}} = \int_{\mathcal{M}_E} \mathrm{ch}(\mathcal{L})\mathrm{Td}(T\mathcal{M}_E), \tag{4.6}$$

where the integral over the moduli space arises after localization in the functional integral[2].

[1] In a more general setup $\mathrm{dim}H^\bullet$ is substituted by $\mathrm{Tr}_{H^\bullet}\mathcal{D}_E$.

[2] This formula is very schematic, and its exact form depends on the details of the problem. For example, twisting by $K^{1/2}$ will result in appearance of $\widehat{A}$ instead of the Todd class.

Moreover, since one can think of the wave-function (4.1) as of a partition function itself: $Z_M = |\Psi(M)\rangle$, the definitions (4.2)-(4.4) imply the quadratic relation of the form

$$Z_{M\times\mathbf{S}^1} \sim Z_M Z_M^*. \tag{4.7}$$

If M is a (generalized) complex manifold, this can be true even at the level of the Lagrangian for the local (massive) degrees of freedom. Indeed, in the functional integral formalism we are dealing with the generalized Laplacian operator $\Delta_E = \mathcal{D}_E^\dagger \mathcal{D}_E$ constructed from the generalized Dirac operator $\mathcal{D}_E$. The square factor for the local degrees of freedom arises from the Quillen theorem:

$$\det{}'\Delta_E = e^{-\mathcal{A}(\mathcal{D}_E)}|\det{}'\mathcal{D}_E|^2. \tag{4.8}$$

Here $\mathcal{A}(\mathcal{D}_E)$ is the holomorphic anomaly: $\partial\overline{\partial}\mathcal{A}(\mathcal{D}_E) \neq 0$. It is natural to assign this anomaly to the integration measure over the moduli space, and then interpret the deviation from the quadratic relation (4.7) as a quantum correction.

Below we list some examples that illustrate these phenomena, which sometimes is referred to as the bulk/boundary correspondence (for more examples, see, *e.g.*, [58, 133, 172, 128]).

Dimension	Correspondence	Index Theorem
$5_{\mathbb{C}}+1$	M-theory/Type IIA	$Z_{\rm M} \sim Z^L_{\rm IIA} Z^R_{\rm IIA}$
$4_{\mathbb{C}}+1$	?/8d Donaldson-like theory	?
$3_{\mathbb{C}}+1$	G_2/CY$_3$ in topological M-theory	$Z_{\rm BH} \sim \lvert Z_{\rm top}\rvert^2$
$2_{\mathbb{C}}+1$	5d SYM/Donaldson-Witten theory	$Z_{\rm SYM} \sim Z_{\rm DW} Z^*_{\rm DW}$
$1_{\mathbb{C}}+1$	Chern-Simons theory/CFT	$Z_{\rm CS} \sim \lvert Z_{\rm CFT}\rvert^2$
$0_{\mathbb{C}}+1$	2d quantum gravity/2d topological gravity	$Z_{\rm qg} \sim Z^2_{\rm tg}$

Table 1: *Examples of the bulk/boundary correspondence.*

Relation of the type (4.7) is known in the context of the matrix models ($0_{\mathbb{C}}+1$ dimensions) as a manifestation of the correspondence between the quantum gravity and topological gravity in two dimensions (see, *e.g.* [166, 122]). Another form of this relation is $\tau = \sqrt{Z_{qg}}$, where τ is a tau-function of the KdV hierarchy [63]. The index formula (4.6) represents computation of the Euler characteristic of $\mathcal{M}_{g,h}$ via the Penner matrix model.

In $1_{\mathbb{C}}+1$ dimensions (4.7) is the famous relation between the Chern-Simons theory and two-dimensional conformal field theory [165]. The index theorem in this case gives the E. Verlinde's formula for the number of conformal blocks [161]. It will be the subject of primary interest of this chapter.

In $2_{\mathbb{C}} + 1$ dimensions (4.6) and (4.7) express computation of the Gromov-Witten invariants from the counting of the BPS states in the five dimensional supersymmetric gauge theory [132, 117].

In $3_{\mathbb{C}} + 1$ dimensions relation (4.7) is known as the OSV conjecture [142]. The exact formulation of the index theorem (4.6) in this case is not known, and the question about the non-pertubative (quantum) corrections to (4.7) is very important for clarifying the relation between the topological strings and the black holes entropy. It is expected that the answer can be given in the framework of the topological M-theory [58] (which was called Z-theory in [133], see also [81, 136, 86, 16, 50, 9, 25, 61] for a discussion on the related issues).

Not much is known about the $4_{\mathbb{C}} + 1$ dimensional example, apart from its relation to the Donaldson-like theory in eight dimensions [133].

The M-theory/Type IIA relation (topological version of which is the $5_{\mathbb{C}} + 1$ dimensional example) was a source of tremendous progress in string theory over the last decade. Needless to say, there are many subtle details involved in this correspondence (see, *e.g.* [54]).

The $5_{\mathbb{C}} + 1$ dimensions is not the end of the story, it probably continues to higher dimensions (F-theory, etc.). Also, it is worth mentioning that there are many signs for the (hidden) integrability in these theories, which is intimately related to the free fermion representation. It allows for a tau-function interpretation of the partition function and is responsible for the appearance of the integrable hierarchies.

Finally, let us note that the theories in different dimensions from Table 1 are connected (apart from the obvious dimensional reduction) via the generalized transgression and descent equations (at least at the classical level), which allow for going one complex dimension up or down. From the geometry-categoric viewpoint this is related to the sequence of d-manifolds serving as a boundaries for $d+1$-manifolds:

$$0 \to M_1 \to \ldots \to M_d \to M_{d+1} \to \ldots$$

By analogy with [133], one can call the theory unifying these theories in different dimensions the Z-theory.

4.1.1 The Entropic Principle and Quantum Mechanics on the Moduli Space

So far we discussed theories describing topological invariants of some structures living on a fixed d-manifold M. It is interesting to ask how these invariants, captured by the universal partition function (4.4), change if we vary M within its topological class. For example, we can talk about transport on the moduli space of genus g Riemann surfaces $\mathcal{M}_g$ or even fantasize about the moduli space of "all" Calabi-Yau threefolds $\mathcal{M}_{CY_3}$.

In this chapter we take a modest step in this direction by applying this idea to the two-dimensional toy model, that has many interesting features which are expected to survive in higher dimensions. The advantage of taking digression to the two dimensions is that in this case (almost) everything becomes solvable. Our goal is to find a two-dimensional sibling of the Hitchin functional, formulate an analog of the entropic principle in $1_{\mathbb{C}}+1$ dimensions, and describe corresponding quantum theory. It turns out that this way we find a unified description of all two-dimensional topologies. Moreover, study of the two-dimensional model leads to a natural generalization of the six-dimensional Hitchin functional, which may be useful for understanding of the topological M-theory at quantum level.

The rest of this chapter is organized as follows: In section 4.2 we review the Hitchin construction for Calabi-Yau threefolds and formulate the problem of describing the moduli space of Riemann surfaces in terms of the cohomology classes of 1-forms, in the spirit of Hitchin. In section 4.3 we construct a two-dimensional analog of the Hitchin functional and comment on quantization of the corresponding theory. In section 4.4 we show that this functional can be related to the gauged WZW model with a target space an abelian variety. In section 4.5 we describe corresponding quantum theory and interpret its partition function (which as a generating function for the number of conformal blocks in $c = 2g$ RCFTs) as an entropy functional on the moduli space of complex structures. The non-perturbative coupling to two-dimensional gravity generates an effective potential on the moduli space, critical points of which are attractive and correspond to Jacobian varieties admitting complex multiplication. We end in section 4.6 with conclusions and discussion on the possible directions for future research.

4.2 The Hitchin Construction

The problem of characterizing a complex manifold in terms of the data associated with closed 1-forms on it goes back to Calabi [28]. In the context of Riemann surfaces, the non-trivial information encoded in a closed 1-forms reveals itself in the ergodicity and integrability of the associated Hamiltonian systems, which have been extensively studied since 1980s by the Novikov school (see, *e.g.* [141] and the references therein).

Kontsevich and Zorich observed an interesting relation between these systems and c=1 topological strings [115]. A new twist to the story became possible after Hitchin [97, 98] discovered some diffemorphism-invariant functionals on stable p-forms.

The critical points of these topological functionals yield special geometric structures. For example, the Hitchin functional on 3-forms defines a complex structure and holomorphic 3-form in 6 dimensions, and G_2 holonomy metrics in 7 dimensions. Hitchin's construction provides an explicit realization of the idea that geometrical structures on a manifold can be described via the cohomology class of a closed form on this manifold. In this approach, geometric structures arise as solutions to the equations obtained by extremizing canonical topological action.

In this section we review Hitchin's approach to parameterizing complex structures on a Calabi-Yau threefold, and formulate the problem of describing the moduli space of genus g Riemann surfaces in a similar manner.

4.2.1 Stable forms in Six Dimensions

Let us describe the Hitchin construction, using a Calabi-Yau threefold M as an example. We present below a Polyakov-like version of the Hitchin functional [133] (see also [58, 81]), although originally it was written in a Nambu-Goto-like form [97, 98]. The reason why we need the Polyakov-like version is that it is quadratic in fields, and therefore is more suitable for quantization. We will also extend the construction of [133] in order to incorporate the generalized geometric structures [99].

Let us introduce a (stable) closed poliform ρ, which is a formal sum of the odd differential forms $\rho = \rho^{(1)} + \rho^{(3)} + \rho^{(5)}$ on a compact oriented six-dimensional manifold

M. If we fix the cohomology class $[\rho]$ of this poliform, it defines a generalized Calabi-Yau structure on M as follows. Consider the functional

$$S = -\frac{\pi}{2}\int_M \Big(\sigma(\rho)\wedge \mathcal{J}^{\varsigma\upsilon}\Gamma_{\varsigma\upsilon}\rho + \sqrt{-1}\lambda\,\mathrm{tr}(\mathcal{J}^2+\mathrm{Id})\Big), \tag{4.9}$$

where $\sigma(\rho^{(k)}) = (-)^{[k/2]}\rho^{(k)}$ transforms the standard wedge pairing between the differential forms into the Mukai pairing [88], the 6-form λ serves as a Lagrange multiplier, $\varsigma, \upsilon = 1, \ldots, 12$ are indices in $TM \oplus T^*M$, the matrix $\Gamma_{\varsigma\upsilon} = [\Gamma_\varsigma, \Gamma_\upsilon]$ is defined by the gamma-matrices Γ_ς of Clifford$(6,6)$, and the tensor field $\mathcal{J} \in \mathrm{End}(TM \oplus T^*M)$. After solving the equations of motion and using the constraint imposed by λ, this field becomes a generalized almost complex structure on M: $\mathcal{J}^2 = -\mathrm{Id}$. Hitchin [99] proved that this almost complex structure is integrable and can be used to reduce the structure group of $TM \oplus T^*M$ to $SU(3,3)$. This endows M with a generalized Calabi-Yau structure.

It is perhaps more illuminating to see how this construction gives rise to the ordinary Calabi-Yau structure, when ρ is a stable closed 3-form: $\rho = \rho^{(3)}$. The Polyakov-like version [133] of the Hitchin functional has the form[3]

$$S = -\frac{\pi}{2}\int_M \Big(\rho\wedge \imath_{\mathcal{K}}\rho + \sqrt{-1}\lambda\,\mathrm{tr}(\mathcal{K}^2+\mathrm{Id})\Big). \tag{4.10}$$

Here $\mathcal{K} \in \mathrm{End}T_{\mathbb{R}}M$ is a traceless vector valued 1-form. We denote it as $\mathcal{K}$ in order to distinguish it from the generalized complex structure $\mathcal{J}$. Also, $\rho = \rho^{(3)}$ is a closed 3-form in a fixed de Rham cohomology class. It can be decomposed as $\rho = [\rho] + d\beta$, where $[\rho] \in H^3(M, \mathbb{R})$ and $\beta \in \Lambda^2 T^*M$. The equations of motion, obtained by

[3]The coefficient $-\pi/2$ in front of the integral can be fixed after comparing the Hitchin action with the black hole entropy functional (see, *e.g.*, [58, 91]). It is tempting to speculate that this normalization factor can also be determined from the topological considerations, similarly to the way the coefficient $-1/8\pi$ in front of the WZW functional is fixed.

varying β in the functional (4.10), accompanied by the closeness condition for ρ, take the form

$$d\rho = 0, \qquad d\,\imath_{\mathcal{K}}\rho = 0. \tag{4.11}$$

The Lagrange multiplier λ imposes the constraint $\mathrm{tr}\mathcal{K}^2(\rho) = -6$ on the solution of the equation of motion for the field $\mathcal{K}$, which in some local coordinates on M can be written as $\mathcal{K}^b{}_a \sim \epsilon^{ba_1a_2a_3a_4a_5}\rho_{a_1a_2a_3}\rho_{a_4a_5a}$. This allows to identify $\mathcal{K}(\rho)$ as an almost complex structure: $\mathcal{K}^2(\rho) = -\mathrm{Id}$. Moreover, it can be shown [97], that this almost complex structure is integrable. Therefore, solutions of (4.11), parameterized by the cohomology class $[\rho]$, define a unique holomorphic 3-form on the Calabi-Yau manifold M, according to

$$\Omega = \rho + \sqrt{-1}\imath_{\mathcal{K}(\rho)}\rho. \tag{4.12}$$

We can use the periods of (4.12) to introduce local coordinates on the complex moduli space of Calabi-Yau. Then, a holomorphic 3-form Ω, viewed as a function of the cohomology class $[\rho]$, gives a map between an open set in $H^3(M, \mathbb{R})$ and a local Calabi-Yau moduli space [97]. We will call it the Hitchin map. After integrating out the field $\mathcal{K}$ we arrive at the original Hitchin functional [97], written in the Nambu-Goto-like form:

$$S = -\frac{\pi}{2}\int_M \rho \wedge *_\rho\, \rho. \tag{4.13}$$

Here $*_\rho$ denotes the Hodge star-operator for the Ricci-flat Kähler metric on M, compatible with the complex structure $\mathcal{K}(\rho)$. Finally, the value of the Hitchin functional (4.10) calculated at the critical point (4.11) can also be written in terms of the holomorphic 3-form (4.12) as follows:

$$S = -i\frac{\pi}{4}\int_M \Omega \wedge \overline{\Omega}. \tag{4.14}$$

4.2.2 Riemann Surfaces and Cohomologies of 1-forms

We want to find an analog of the Hitchin construction for the two-dimensional surfaces. It is natural to expect that the role that was played by the closed 3-forms in $3_{\mathbb{C}}$ dimensions, in $1_{\mathbb{C}}$ dimensions will be played by the closed 1-forms. Therefore, we want to construct a functional, depending on closed 1-forms in a fixed cohomology class, critical points of which will determine the complex structure on a genus g Riemann surface Σ_g. In fact, a two-dimensional version of the Hitchin functional was already discussed in [133, 81]. There, it was pointed out that it is very similar to Polyakov's formulation of the bosonic string as a sigma model coupled to the two-dimensional gravity.

However, before discussing the explicit form of this functional, we want to explain why a naive carry-over of the Hitchin idea from six to two dimensions will not work. First, the very existence of the Hitchin map is based on the fact that in the case of Calabi-Yau threefold M the dimension of the intermediate cohomology space $\dim H^3(M, \mathbb{R})$ coincide with the dimension of the moduli space of calibrated Calabi-Yau manifolds[4], which is equal to $2 + 2h^{2,1}$. In the case of a genus g Riemann surface $\dim H^1(\Sigma_g, \mathbb{R}) = 2g$, but dimension of the moduli space $\mathcal{M}_g$ for $g > 1$ is $\dim \mathcal{M}_g = 6g - 6$. Therefore, the cohomology class of a closed 1-form on Σ_g does not

[4]The calibrated Calabi-Yau manifold is a pair: (M, Ω), where M Calabi-Yau threefold and Ω is a fixed non-vanishing holomorphic 3-form on M. Hitchin construction naturally gives calibrated Calabi-Yau manifolds.

contain enough data to describe the moduli space. This is, of course, not surprising, as it is well known that natural parameterization of the moduli space $\mathcal{M}_g$ is given in terms of the Beltrami differentials μ, which are dual to the holomorphic quadratic differentials $\chi \in H^0(\Sigma_g, \Omega^{\otimes 2})$. In particular, $\dim H^0(\Sigma_g, \Omega^{\otimes 2}) = 6g - 6$, as it should be. One could then try to use $H^0(\Sigma_g, \Omega^{\otimes 2})$ instead of $H^1(\Sigma_g, \mathbb{R})$, but if we go by this route, we will lose the "background independence" on the complex structure on Σ_g, which is a nice feature of the Hitchin construction. The relevant set-up in this case seems to be provided by the theory of beta-gamma systems [118, 134]. However, it turns out that it is hard to write down an analog of the Hitchin functional for (μ, χ) system with decoupled conformal factor.

The only exception, when the dimension of the moduli space coincides with the dimension of the first cohomology space, is the elliptic curve Σ_1, which is in fact a direct one-dimensional analog of the Calabi-Yau threefold. In this case, $\dim H^1(\Sigma_1, \mathbb{R}) = 2 = \dim \mathcal{M}_1$ and therefore we might expect that one closed 1-form can play the role of ρ in two dimensions. However, as it was noted in [133], one needs at least two closed 1-forms in order to write down a two-dimensional analog of the Hitchin functional. In a certain sense, it is a lower dimensional artefact, as there just don't happen to be enough indices to write down a non-zero expression.

Clearly, some modification of the Hitchin construction is needed in the two-dimensional case. We suggest the following extension that preserves the spirit of the original construction. First, we will use complex cohomologies instead of the real ones:

$$H^1(\Sigma_g, \mathbb{R}) \to H^1(\Sigma_g, \mathbb{C}). \tag{4.15}$$

Second, for a genus g surface we will consider g closed 1-forms:

$$H^1(\Sigma_g,\mathbb{C}) \to (H^1(\Sigma_g,\mathbb{C}))^{\otimes g}. \tag{4.16}$$

As we will see, this will allow us to define close analog of the Hitchin functional.

4.3 Construction of the Lagrangian

The case of interest for us in this section is a complex valued 1-forms on a two dimensional compact surface Σ_g of genus g. We will not assume that Σ_g is endowed with any additional structures, such as a metric or complex structure. Instead, in the spirit of Hitchin, we would like to construct a functional, critical points of which will define a complex structure on Σ_g, making it a Riemann surface. In order to keep the presentation self-contained and to fix the notations, we start with a brief review of the basics of Riemann surfaces. Then we proceed to the construction of the functional on the space of closed 1-forms, the critical points of which in a fixed cohomology class are harmonic 1-forms. The complex structure on Σ_g will arise from the cohomology classes of these 1-forms. We will also briefly discuss the quantization of the corresponding theory.

4.3.1 Mathematical Background on Riemann Surfaces

We summarize below some basic facts from the theory of compact Riemann surfaces [11, 73]. Let Σ_g be a topological surface with g handles, that is a compact connected oriented differentiable manifold of real dimension 2. The number of handles g is the genus of Σ_g. Topologically, Σ_g is completely specified by the Euler number $\chi(\Sigma_g) = 2 - 2g$. In particular, the dimensions of the homology groups are

$$\mathrm{dim}H_0(\Sigma_g) = 1, \quad \mathrm{dim}H_1(\Sigma_g) = 2g, \quad \mathrm{dim}H_2(\Sigma_g) = 1. \tag{4.17}$$

On Σ_g one can choose the canonical symplectic basis of 1-cycles $\{A_I, B_I\},\ I = 1, \dots, g$ for $H_1(\Sigma_g)$, with the intersection numbers

$$\#(A_I, A_J) = 0, \quad \#(A_I, B_J) = \delta_{IJ}, \quad \#(B_I, B_J) = 0. \tag{4.18}$$

This basis, however, is not unique. The ambiguity is controlled by the Siegel modular group $\Gamma_g = \mathrm{Sp}(2g, \mathbb{Z})$ preserving symplectic pairing (4.18):

$$\begin{pmatrix} B \\ A \end{pmatrix} \rightarrow \begin{pmatrix} B \\ A \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} B \\ A \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_g \tag{4.19}$$

Once we have chosen a homology basis $\{A_I, B_I\}$, or "marking" for Σ_g, we can cut surface along $2g$ curves homologous to the canonical basis and get a $4g$-sided polygon with appropriate boundary identifications. This representation of Σ_g in terms of the polygon is very helpful in deriving some important identities. For example, one can show that for any closed 1-forms η and θ on Σ_g

$$\int_{\Sigma_g} \eta \wedge \theta = \sum_{I=1}^{g} \Big(\oint_{A_I} \eta \oint_{B_I} \theta - \oint_{A_I} \theta \oint_{B_I} \eta \Big), \tag{4.20}$$

which is called the Riemann bilinear identity. The scalar product of two closed 1-forms η and θ on Σ_g is given by the Petersson inner product:

$$\langle \eta, \theta \rangle = \frac{i}{2} \int_{\Sigma_g} \eta \wedge \overline{\theta}. \tag{4.21}$$

As follows from the Riemann bilinear identity, this scalar product depends only on the cohomology class of the closed forms: $\langle \eta, \theta \rangle = \langle [\eta], [\theta] \rangle$. The canonical symplectic form

$$\mathcal{S}(\eta,\theta) = \int_{\Sigma_g} \eta \wedge \theta \tag{4.22}$$

for closed 1-forms also depends only on the cohomology class: $\mathcal{S}(\eta,\theta) = \mathcal{S}([\eta],[\theta])$.

Let us introduce the basis $\{\alpha^I, \beta^I\}$, $I = 1,\ldots,g$ for $H^1(\Sigma_g, \mathbb{R})$, which is dual to the canonical homology basis (4.18):

$$\oint_{A_I} \alpha^J = \delta^{IJ}, \qquad \oint_{B_I} \alpha^J = 0 \tag{4.23}$$

$$\oint_{B_I} \beta^J = \delta^{IJ}, \qquad \oint_{A_I} \beta^J = 0. \tag{4.24}$$

The ambiguity of this basis is controlled by an exact 1-forms on Σ_g. Therefore, we can think of $\{\alpha^I, \beta^I\}$ as of some fixed representatives in the de Rham cohomology class. A natural way to fix this ambiguity is to pick some Riemann metric h on Σ_g and require $\{\alpha^I, \beta^I\}$ to be harmonic:

$$d *_h \alpha^I = 0, \quad d *_h \beta^I = 0 \tag{4.25}$$

where $*_h$ is a Hodge $*$-operator defined by h. This choice provides a canonical basis for $H^1(\Sigma_g, \mathbb{R})$, associated with the metric h. We will always use Euclidean signature on Σ_g.

Topological surface Σ_g endowed with a complex structure is called a Riemann surface. Let us recall that an almost complex structure on Σ_g is a section J of a vector bundle $\mathrm{End}(T_{\mathbb{R}}\Sigma_g)$ such that $\mathrm{J}^2 = -1$. Here $T_{\mathbb{R}}\Sigma_g$ is a real tangent bundle of Σ_g. If we pick some (real) local coordinates $\{x^a\}$, $a = 1,2$ on Σ_g, then J can be represented by a real tensor field which components $\mathrm{J}^a{}_b$ obey

$$\mathrm{J}^a{}_b \mathrm{J}^b{}_c = -\delta^a{}_c \tag{4.26}$$

Here and in what follows, a sum over the repeating indices is always assumed. We reserve the indices $\{a, b, c, \ldots\}$ that range from 1 to 2, for the world-sheet (Riemann surface), and indices $\{I, J, K, \ldots\}$ that range from 1 to g, for the complex coordinates on the target (first cohomology) space. The indices $\{i, j, k, \ldots\}$ label real coordinates on the target space and range from 1 to $2g$. We do not distinguish between the upper and lower indices. In particular, we do not use any metric to contract it. We will also sometimes omit indices and use matrix notations in the target space for shortness.

According to the Newlander-Nirenberg theorem J, is an integrable complex structure if it is covariantly constant:

$$\nabla_a \mathrm{J}^b{}_c = 0. \tag{4.27}$$

In fact, any almost complex structure on a topological surface is integrable, and therefore below we will just call it a complex structure. In particular, we will be interested in a complex structure compatible with the metric h. In local coordinates the metric has the form

$$h = h_{ab} dx^a \otimes dx^b, \tag{4.28}$$

and the corresponding complex structure is given by

$$\mathrm{J}(h)^a{}_b = \sqrt{\det \|h_{df}\|}\, \epsilon_{bc} h^{ca}, \tag{4.29}$$

where $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = -\epsilon_{21} = 1$. It is straightforward to check that this complex structure indeed obeys (4.26)-(4.27). Notice that complex structure (4.29) depends only on the conformal class of the metric, since it is invariant under the conformal transformations:

$$h \to e^{\varphi} h, \quad \mathrm{J}(h) \to \mathrm{J}(h). \tag{4.30}$$

The complex coordinates $z, \overline{z}$ on Σ_g associated with (4.29) are determined from the solution of the Beltrami equation

$$\mathrm{J}^a{}_b \frac{\partial z}{\partial x^a} = i \frac{\partial z}{\partial x^b}. \tag{4.31}$$

Given a marking for Σ_g, there is a unique basis of holomorphic abelian differentials of the first kind $\omega^I \in H^0(\Sigma_g, \Omega)$, normalized as follows

$$\oint_{A_I} \omega^J = \delta^{IJ}. \tag{4.32}$$

Here $\omega^I = \omega^I_z dz$. Holomorphic 1-differentials span $-i$ eigenspace of the Hodge $*$-operator for the metric compatible with the complex structure:

$$*\omega = -i\omega, \tag{4.33}$$

$$\overline{\omega} = +i\overline{\omega}. \tag{4.34}$$

The period matrix of Σ_g is defined by

$$\tau^{IJ} = \oint_{B_I} \omega^J. \tag{4.35}$$

If we apply the Riemann identity (4.20) to the trivial 2-form $\omega^I \wedge \omega^J = 0$, we find that the period matrix is symmetric:

$$\tau^{IJ} = \tau^{JI}. \tag{4.36}$$

The imaginary part of the period matrix can be represented as follows:

$$\operatorname{Im} \tau^{IJ} = \frac{i}{2} \int_{\Sigma_g} \omega^I \wedge \overline{\omega}^J. \tag{4.37}$$

If we use the fact that the norm (4.21) of the non-zero holomorphic differentials of the form $\nu = \nu_I \omega^I$ is positive: $\langle \nu, \nu \rangle > 0$, we find that the period matrix has a positive definite imaginary part:

$$\mathrm{Im}\,\tau > 0. \tag{4.38}$$

Now we can express the holomorphic abelian differentials (4.32)-(4.35) via the canonical cohomology basis (4.23) of harmonic 1-forms (4.25) on Σ_g as follows

$$\omega = \alpha + \tau\beta, \tag{4.39}$$

where we used the matrix notations. Under the modular transformations (4.19) the period matrix transforms as

$$\tau \;\rightarrow\; \tau' = (a\tau + b)(c\tau + d)^{-1}, \tag{4.40}$$

while the basis of abelian differentials transforms as

$$\omega \;\rightarrow\; \omega' = (\tau c^T + d^T)^{-1}\omega. \tag{4.41}$$

The space of a complex $g \times g$ matrices obeying (4.36), (4.38) is the Siegel upper half-space $\mathcal{H}_g$. We will call it the Siegel space, for short. Torelli's theorem states that a complex structure of Σ_g is uniquely defined by the period matrix up to a diffeomorphism. Moreover, to each complex structure there corresponds a unique point in the fundamental domain of the modular group

$$\mathcal{A}_g = \mathcal{H}_g/\Gamma_g. \tag{4.42}$$

Unfortunately, for higher genus surfaces the converse is not true (Schottky's problem). This is easy to see, since for $g > 3$ the dimension of (4.42) $\dim_{\mathbb{C}}\mathcal{A}_g = \frac{g(g+1)}{2}$ is bigger than the dimension of the complex structures moduli space $\dim_{\mathbb{C}}\mathcal{M}_g = 3g - 3$.

4.3.2 The Canonical Metric

There is a canonical Kähler metric on a Riemann surface, the so-called Bergmann metric. Sometimes it is also called the Arakelov metric in literature. It can be written in terms of the abelian differentials (4.32) as

$$h^{\mathrm{B}}_{z\bar{z}} = (\operatorname{Im}\tau)^{-1}_{IJ}\, \omega^I_z \overline{\omega}^J_{\bar{z}}. \tag{4.43}$$

This metric has a nonpositive curvature. If $g \geq 2$, the curvature vanishes at most in a finite number of points, and by an appropriate conformal transformation (4.43) can be brought into a metric of constant negative curvature (see, *e.g.* [105]). The Kähler form corresponding to the Bergmann metric is given by

$$\varpi^{\mathrm{B}} = \frac{i}{2}(\operatorname{Im}\tau)^{-1}_{IJ}\, \omega^I \wedge \overline{\omega}^J. \tag{4.44}$$

It is easy to see that the volume of the Riemann surface in this metric is independent of the complex structure and is equal to the genus:

$$\int_{\Sigma_g} \varpi^{\mathrm{B}} = g. \tag{4.45}$$

The special role of the Bergmann metric will become clear if we consider the period map $z \to \xi^I$ from the Riemann surface Σ_g into its Jacobian variety $\mathrm{Jac}(\Sigma_g) = \mathbb{C}^n/(\mathbb{Z}^n \oplus \tau\mathbb{Z}^n)$:

$$\xi^I = \int_{z_0}^{z} \omega^I. \tag{4.46}$$

Here z_0 is some fixed point on Σ_g, the exact choice of which is usually not important. Jacobian variety, being a flat complex torus, is endowed with a canonical metric, which is induced from the Euclidian metric on $\mathbb{C}^n$. The Bergmann metric (4.43) is nothing but a pull-back of this canonical metric from $\mathrm{Jac}(\Sigma_g)$ to Σ_g under the period map (4.46).

The metric (4.43) does not depend on the choice of a basis[5] in a space of holomorpic differentials $H^0(\Sigma_g, \Omega)$. In particular, it is invariant under the modular transformations (4.40)-(4.41). If we consider ω^I as a set of g closed 1-forms on Σ_g in a fixed cohomology class, parameterized by the period matrix $\tau \in \mathcal{A}_g$, then (4.43) combined with (4.29) gives an explicit realization of the Torelli's theorem, by providing the map $\mathcal{A}_g \to \mathcal{M}_g$. This should be viewed as a two-dimensional analog of the Hitchin map [97, 98] from the cohomology space of the stable forms on a compact six-dimensional manifolds to the moduli space of the calibrated Calabi-Yau threefolds.

Indeed, let us recall that the space of all metrics on a genus g surface Σ_g is factorized as follows

$$\mathrm{Met}(\Sigma_g) = \mathcal{M}_g \times \mathrm{Diff}(\Sigma_g) \times \mathrm{Conf}(\Sigma_g). \tag{4.47}$$

Once we fixed the cohomology class of ω^I, we are not allowed to do the conformal transformations, since this will spoil the closeness of ω^I. Therefore, expression (4.43) provides a unique representative among the conformal structures on Σ_g. This takes care of the $\mathrm{Conf}(\Sigma_g)$ factor in (4.47). Moreover, since diffeomorphisms do not change the cohomology class, the Bergmann metric (4.43) is invariant under the action of the $\mathrm{Diff}(\Sigma_g)$ group, and we end up on the moduli space of genus g Riemann surfaces $\mathcal{M}_g$.

[5]In the orthonormal basis $\{\omega_o^I = \omega_{oz}^I dz : \langle \omega_o^I, \omega_o^J \rangle = \delta^{IJ}\}$, the metric takes the canonical form: $h_{z\bar{z}} = \sum_{I=1}^{g} \left|\omega_{oz}^I\right|^2$.

4.3.3 An analog of the Hitchin Functional in Two Dimensions

As we discussed earlier, the problem in defining an analog of the Hitchin functional in two dimensions is that the cohomology class of only one 1-form is not enough to parameterize the moduli space of complex structures. However, if we take g closed 1-forms on a genus g surface, this can be done. In fact, this will give us even more degrees of freedom than we need (g^2 complex parameters instead of $3g-3$), but it is a minimal set of data that we can start with, because of the Schottky problem. The functional that we will use is a direct generalization of [133, 81]. The fields of the theory are

- ζ^I : g closed complex valued 1-forms, $d\zeta^I = 0$
- $\mathcal{K}$: real traceless vector valued 1-form, $\mathcal{K} \in \mathrm{End}(T_{\mathbb{R}}\Sigma_g)$
- λ : imaginary 2-form

The Lagrangian has the form

$$L = \frac{k\pi}{4}\langle\zeta^I,\zeta^J\rangle^{-1}\int_{\Sigma_g}(\zeta^I\wedge\imath_{\mathcal{K}}\,\overline{\zeta}^J + \overline{\zeta}^J\wedge\imath_{\mathcal{K}}\,\zeta^I) - i\frac{k\pi}{4}\int_{\Sigma_g}\lambda\,\mathrm{tr}(\mathcal{K}^2+\mathrm{Id}), \qquad (4.48)$$

where k is a coupling constant, 2-form λ serves as a Lagrange multiplier, and Id is a unit 2×2 matrix. Hermitian $g\times g$ matrix $\langle\zeta^I,\zeta^J\rangle^{-1}$ is an inverse of the scalar product (4.21) for 1-forms. We assume that cohomology classes of 1-forms $[\zeta^I]$ are linear independent. In order to discuss classical equations of motion for the action (4.48) and their solutions it is useful to write is explicitly in components:

$$L = \frac{k\pi}{4}\int_{\Sigma_g}\Big(\langle\zeta^I,\zeta^J\rangle^{-1}(\zeta^I_a\overline{\zeta}^J_c + \zeta^I_c\overline{\zeta}^J_a)\mathcal{K}^c{}_b - \frac{i}{2}\lambda(x)\epsilon_{ab}(\mathcal{K}^d{}_c\mathcal{K}^c{}_d + 2)\Big)dx^a\wedge dx^b, \ (4.49)$$

where $\zeta^I = \zeta^I_a dx^a$, $\mathcal{K} = \mathcal{K}^a{}_b \frac{\partial}{\partial x^a} \otimes dx^b$, $\lambda = \frac{1}{2}\lambda(x)\, \epsilon_{ab} dx^a \wedge dx^b$, and $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = -\epsilon_{21} = 1$. The equations of motion for $\mathcal{K}$ give

$$\mathcal{K}^a{}_b = \frac{i}{2\lambda(x)} \langle \zeta^I, \zeta^J \rangle^{-1} (\zeta^I_b \overline{\zeta}^J_c + \zeta^I_c \overline{\zeta}^J_b) \epsilon^{ca}, \tag{4.50}$$

where $\epsilon^{11} = \epsilon^{22} = 0$, $\epsilon^{12} = -\epsilon^{21} = -1$, such that $\epsilon^{ac}\epsilon_{cb} = \delta^a{}_b$. Variation with respect to λ imposes the constraint

$$\mathcal{K}^a{}_b \mathcal{K}^b{}_a = -2, \tag{4.51}$$

which is solved by setting[6]

$$\lambda(x) = \pm i \sqrt{\det \| h_{ab} \|}, \tag{4.52}$$

where, by definition, $\det\|h_{ab}\| = \frac{1}{2} h_{ab} h_{cd} \epsilon^{ac} \epsilon^{bd}$, and h_{ab} is the metric induced on Σ_g:

$$h_{ab} = \frac{1}{2} \langle \zeta^I, \zeta^J \rangle^{-1} (\zeta^I_a \overline{\zeta}^J_b + \zeta^I_b \overline{\zeta}^J_a). \tag{4.53}$$

We choose the "+" sign in (4.52) by requiring positivity of the Lagrangian (4.49) after solving for $\mathcal{K}$ and λ:

$$L = \frac{k\pi}{2} \int_{\Sigma_g} \sqrt{\det \| h_{ab} \|}. \tag{4.54}$$

The corresponding solution

$$\mathcal{K}^a{}_b = \frac{1}{\sqrt{\det \| h_{df} \|}} h_{bc} \epsilon^{ca} = -\sqrt{\det \| h_{df} \|}\, \epsilon_{bc} h^{ca} \tag{4.55}$$

represents the action of the complex structure, compatible with the metric (4.53), on the cotangent bundle $T^*_{\mathbb{R}}\Sigma_g$:

$$\mathcal{K} = \mathrm{J}^{-1} = -\mathrm{J}. \tag{4.56}$$

[6] Notice that it is possible for λ to vanish at some points, if the determinant of the induced metric becomes zero. In this case, expression (4.50) is not well defined, and potentially is singular. Later we will see, that in order to give well-defined complex structure, the cohomologies $[\zeta^I]$ should lie in the Jacobian locus in the Siegel upper half-space.

This should be compared with (4.29). Let us introduce the notation $*_\zeta$ for a Hodge star operator, defined by the metric (4.53). For example, the Hodge dual of a 1-form $\theta = \theta_a dx^a$ is given by

$$*_\zeta \theta = \theta_a \sqrt{\det \| h_{df} \|}\, h^{ab} \epsilon_{bc} dx^c. \tag{4.57}$$

This Hodge $*$-operator acts on 1-forms exactly as the field $\mathcal{K}$ (4.50):

$$*_\zeta \theta = \imath_{\mathcal{K}} \theta. \tag{4.58}$$

Therefore, we can rewrite the action (4.48) in yet another form:

$$L = \frac{k\pi}{4} \langle \zeta^I, \zeta^J \rangle^{-1} \int_{\Sigma_g} \zeta^I \wedge *_\zeta \overline{\zeta}^J \ + \ \text{h.c.} \tag{4.59}$$

This expression is similar to the Hitchin functional (4.13) and relation between (4.48) and (4.59) is very much like relation between the Polyakov and Nambu-Goto actions in string theory, as was noted in [133, 81].

Following the idea of Hitchin, we should restrict this functional to the closed forms on Σ_g in a given de Rham cohomology class, and look for the critical points. In order to parameterize variations of ζ^I in a fixed cohomology class $[\zeta^I] \in H^1(\Sigma_g, \mathbb{C})$, we decompose it as

$$\zeta^I = [\zeta^I] + d\xi^I, \tag{4.60}$$

where ξ^I is a proper function $\Sigma_g \to \mathbb{C}^g$. By varying ξ^I in (4.48), we get

$$d *_\zeta \zeta^I = 0. \tag{4.61}$$

Thus, the critical points of the functional (4.59) correspond to the harmonic forms on Σ_g. The complex dimension of the space of harmonic 1-forms on Σ_g is equal to

g. Since initial conditions (4.60) are parameterized by g linear independent vectors $[\zeta^I]$, solution to (4.61) will give us a basis in the space of harmonic 1-forms. We can parameterize cohomology classes $[\zeta^I]$ using their periods over the A and B-cycles:

$$[\zeta^I] = A^{IJ}\alpha^J + B^{IJ}\beta^J, \tag{4.62}$$

where A^{IJ} and B^{IJ} are $g \times g$ complex matrices. We impose some natural restrictions on the form of these matrices. First, since the action (4.48) is invariant under the linear transformations

$$\zeta^I \to M^{IJ}\zeta^J, \quad M^{IJ} \in \mathrm{GL}(g,\mathbb{C}), \tag{4.63}$$

we can always set $A^{IJ} = \delta^{IJ}$ by using this transformation with $M = A^{-1}$. Then, (4.62) becomes

$$[\zeta^I] = \alpha^I + \Pi^{IJ}\beta^J, \tag{4.64}$$

where $\Pi = A^{-1}B$. The fact that all cohomology classes $[\zeta^I]$ are linear independent means that

$$\mathrm{rank}\,\Pi = g. \tag{4.65}$$

The second restriction comes from the fact that the matrix of the scalar products of 1-forms

$$\langle\zeta^I, \zeta^J\rangle = \frac{i}{2}(\Pi^\dagger - \Pi)^{IJ} \tag{4.66}$$

should be invertible and positive definite for the theory, based on the action (4.48), to be well-defined. Moreover, it is natural to require that cohomology classes $[\zeta^I]$ do not intersect[7]

$$\int_{\Sigma_g} \zeta^I \wedge \zeta^J = \Pi^{IJ} - \Pi^{JI} = 0. \tag{4.67}$$

Let us recall that this intersection number is essentially the canonical symplectic form $\mathcal{S}(\zeta^I, \zeta^J)$ on $H^1(\Sigma_g, \mathbb{C})$, defined in (4.22). Therefore, from the perspective of future quantization of the cohomology space, it is necessary to require that the points $[\zeta^I]$ in the configuration space commute. This requirement is similar to considering only commuting set of the periods in quantum mechanics of the self-dual form (see, *e.g.*, [173]).

Therefore, instead of dealing with all non-degenerate matrices $\Pi \in \mathrm{GL}(g, \mathbb{C})$, we can concentrate only on the matrices that obey

$$\Pi^T = \Pi, \quad \mathrm{Im}\Pi > 0. \tag{4.68}$$

In other words, we parameterize cohomology classes $[\zeta^I]$ by the points on the Siegel upper half-space $\mathcal{H}_g$. In fact, $\mathcal{H}_g$ is the smallest linear space where we can embed Jacobian variety $\mathrm{Jac}(\Sigma_g)$ without knowing its detailed description, which is unavailable for $g > 4$ because of the Schottky problem. There is a natural action of the symplectic group on $\mathcal{H}_g$. We will denote this "target" modular group as $\mathrm{Sp}(2g, \mathbb{Z})_{\mathrm{t}}$, in order to distinguish it from the "world-sheet" modular group $\mathrm{Sp}(2g, \mathbb{Z})_{\mathrm{ws}}$ acting on the cover of the moduli space of Riemann surfaces.

Given the solution to (4.61), the complex structure on Σ_g is uniquely determined by the corresponding cohomology class via (4.53)-(4.50), very much in the spirit of

[7] This condition can be imposed, for example, by adding a term of the form $i\mathrm{A}_{IJ} \int_{\Sigma_g} \zeta^I \wedge \zeta^J$ to the action (4.48) and integrating out antisymmetric matrix A_{IJ}. This term is purely topological (it is not coupled to $\mathcal{K}$ and depends only on the cohomology classes), so it does not affect the ordinary Hitchin story.

Hitchin. The map $\zeta^I \to *_\zeta \zeta^I$ globally defines a decomposition of ζ^I on components of type $(1,0)$ and $(0,1)$ with respect to this complex structure. For example, the $(1,0)$ component of the solution to (4.61)

$$\zeta^I_{(1,0)} = \zeta^I + i *_\zeta \zeta^I, \tag{4.69}$$

being a harmonic, must be a linear combination of the abelian differentials (4.32). This observation allows us to express the period matrix as a function of the cohomology classes:

$$\tau^{IJ} = \sum_K \Big(\oint_{A_K} \zeta^I_{(1,0)} \Big)^{-1} \oint_{B_J} \zeta^K_{(1,0)}. \tag{4.70}$$

In practice, however, we will have to solve the equation (4.61) in order to compute corresponding period matrix via (4.70). This should be as hard to do as to solve the Schottky problem. Furthermore, the complex structure (4.50), that we will get, will in general be different from the background complex structure on the abelian variety $\mathcal{T}(\Pi)$, that we use to parameterize the cohomologies. Only if we start from a point[8] on the Siegel space that corresponds to the Jacobian variety $\mathcal{T}(\Pi) = \mathrm{Jac}(\Sigma_g(\tau))$, the critical point of the functional (4.48) will give us the same complex structure on the world-sheet as on the target space. In this case harmonic maps (4.61) are promoted to the holomorphic maps, and (4.70) gives $\tau = \Pi$. The metric (4.53) then is the Bergmann metric (4.43). This will happen on a very rare occasion, since Jacobian locus has measure zero in the Siegel space. However, in general there is no obstruction for the map $\mathcal{H}_g \to \mathcal{M}_g$ defined by (4.70), since all almost complex structures on Σ_g are integrable.

[8]To be precise, we can also use any point that can be obtained from this one by the action of the modular group $\mathrm{Sp}(2g, \mathbb{Z})_{\mathrm{t}}$.

Formally, this is the end of the ordinary Hitchin story in two dimensions. However, a new interesting direction for study emerges if we allow the cohomology classes $[\zeta^I]$ to vary. In this case we will be dealing with the effective quantum mechanics of g points on the Siegel space $\mathcal{H}_g$ defined by the functional (4.48).

Let us discuss the dependence of this functional on the "massless" degrees of freedom encoded in Π and $\mathcal{K}$. We choose some complex structure on Σ_g, which is equivalent to fixing the corresponding value of the field $\mathcal{K}$. Modulo diffeomorphisms, it is defined by the corresponding period matrix τ. This is equivalent[9] to choosing a set of the abelian differentials of the first kind (4.32). Then, we can express the cohomology class (4.64) as follows:

$$[\zeta] = (\Pi - \overline{\tau})\frac{1}{\tau - \overline{\tau}}\omega - (\Pi - \tau)\frac{1}{\tau - \overline{\tau}}\overline{\omega}. \tag{4.71}$$

The background dependence on the complex structure on Σ_g is encoded in the period matrix τ. Let $*$ be the Hodge star-operator compatible with this complex structure: $\imath_{\mathcal{K}} \to *$. Using the identity

$$\zeta^I \wedge *\overline{\zeta}^J = i\zeta^I \wedge \overline{\zeta}^J - \frac{i}{2}(\zeta^I - i * \zeta^I) \wedge (\overline{\zeta}^J + i * \overline{\zeta}^J) \tag{4.72}$$

and assuming that the classical equations of motion (4.61) are satisfied, we get the following expression for the functional (4.48):

$$L(\Pi, \tau) = kg\pi + \frac{k\pi}{4}\mathrm{Tr}\frac{1}{\mathrm{Im}\Pi}(\Pi - \tau)\frac{1}{\mathrm{Im}\tau}\overline{(\Pi - \tau)}. \tag{4.73}$$

It is clear that this expression has a maximum at the point $\Pi = \tau$ on the Siegel plane. Moreover, it is straightforward to check, using the symmetry of the matrices Π and

[9] We assume that some marking for Σ_g is fixed, and discuss the modular group $\mathrm{Sp}(2g, \mathbb{Z})_{\mathrm{ws}}$ issues later.

τ, and the positivity of Imτ, that

$$\Pi = \tau \tag{4.74}$$

is the only solution[10] of the corresponding equation of motion

$$\frac{\partial L(\Pi, \tau)}{\partial \Pi} = 0. \tag{4.75}$$

Therefore, if we allow the cohomology classes in the theory with Lagrangian (4.48) to fluctuate, we find the following picture. For the generic period matrix Π, parameterizing the cohomology classes, solution to the equation of motion (4.61) for the "massive" degrees of freedom (scalars ξ^I in (4.60)) give harmonic maps $\zeta : \Sigma_g \to \mathcal{H}_g$. Further extremization with respect to Π picks up only the holomorphic maps that correspond to the Jacobian variety Jac$(\Sigma_g(\tau)) \in \mathcal{H}_g$ of the Riemann surface $\Sigma_g(\tau)$ with the period matrix τ.

Another important feature of the expression (4.73) is that it is invariant under the diagonal subgroup of the group $\mathrm{Sp}(2g, \mathbb{Z})_{\mathrm{t}} \times \mathrm{Sp}(2g, \mathbb{Z})_{\mathrm{ws}}$, which acts as follows

$$\Pi \to \Pi' = (a\Pi + b)(c\Pi + d)^{-1} \tag{4.76}$$

$$\tau \ \to \tau' = (a\tau + b)(c\tau + d)^{-1}. \tag{4.77}$$

This can be easily checked using the basic relations for the $\mathrm{Sp}(2g, \mathbb{Z})$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$\begin{aligned} ad^T - bc^T &= a^T d - c^T b = \mathbb{I}_{g\times g} \\ ab^T - ba^T &= cd^T - dc^T = 0 \\ a^T c - c^T a &= b^T d - d^T b = 0. \end{aligned} \tag{4.78}$$

[10] There is also a nonphysical solution $\Pi = \overline{\tau}$, that does not lie on the Siegel upper-space, since Im$\Pi < 0$ in this case.

This, in particular, implies that after integrating $e^{-L(\Pi,\tau)}$ over the Siegel space $\mathcal{H}_g$ we will get a modular invariant function of τ. As we will see shortly, this gives an interesting topological quantum mechanical toy model on $\mathcal{M}_g$. However, this toy model can hardly be interpreted from the entropic principle perspective. Therefore, further refinement of the functional (4.48) will be needed.

4.3.4 Towards the Quantum Theory

Consider the following partition function defined by the functional(4.48):

$$Z_g(k,\tau) = \int \frac{\mathcal{D}\Pi\mathcal{D}\xi}{\det\mathrm{Im}\Pi}\, e^{-L}, \tag{4.79}$$

where the canonical modular invariant measure is used. It is assumed that we have fixed the value of the field $\mathcal{K}$, corresponding to the complex structure on a Riemann surface $\Sigma_g(\tau)$ with the period matrix τ. After performing the Gaussian integral over ξ and using (4.73), we find

$$Z_g(k,\tau) = e^{-kg\pi} \int \mathcal{D}\Pi \, \exp\Big(-\frac{k\pi}{4}\mathrm{Tr}\frac{1}{\mathrm{Im}\Pi}(\Pi-\tau)\frac{1}{\mathrm{Im}\tau}\overline{(\Pi-\tau)}\Big). \tag{4.80}$$

The canonical modular invariant measure on $\mathcal{H}_g$ is

$$\mathcal{D}\Pi = (\det\mathrm{Im}\Pi)^{-g-1}\prod_{I\leq J}^{g}|d\Pi_{IJ}|^2 \tag{4.81}$$

Since we know that the exponent in (4.80) has only one minimum (4.74), for large k we can study the perturbative expansion of the matrix integral (4.80) near $\Pi = \tau$. It is convenient to describe the fluctuations by introducing the matrix H as follows

$$\Pi = \tau - \mathrm{Im}\tau H. \tag{4.82}$$

Then (4.80) becomes

$$Z_g(k,\tau) = e^{-kg\pi} \int \mathcal{D}H \, \exp\Big(- \frac{k\pi}{4} \mathrm{Tr}\, H\overline{H} \frac{1}{1 - \mathrm{Im}H} \Big). \tag{4.83}$$

As we discussed earlier, because of the invariance of the exponent (4.73) in (4.80) under the diagonal modular action (4.76), the partition function $Z_g(k,\tau)$ is a modular invariant function of τ. Therefore, it descends to a function on the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces. Notice that expression (4.83) does not depend on τ at all. Therefore, the modular invariant function that we will get is actually a constant[11].

The integral in (4.83), as a function of k, can be expressed in terms of the 1-matrix model. Let us split the matrix H into its real and imaginary parts

$$H = H_1 + iH_2. \tag{4.84}$$

Then we can rewrite (4.83) as

$$Z_g(k,\tau) = e^{-kg\pi} \int \mathcal{D}H_2 \int \mathcal{D}H_1 \, \exp\Big(- \frac{k\pi}{4} \mathrm{Tr}\,(H_1^2 + H_2^2) \frac{1}{1 - H_2} \Big), \tag{4.85}$$

where we integrate over the symmetric matrices, and the matrix measures are defined in accordance with (4.81)

$$\mathcal{D}H_1 = \prod_{I \le J}^{g} dH_1^{IJ}, \qquad \mathcal{D}H_2 = (\mathrm{det}H_2)^{-g-1} \prod_{I \le J}^{g} dH_2^{IJ} \tag{4.86}$$

The integral over H_1 is Gaussian, and gives $(\frac{2}{k\pi})^{\frac{g(g+1)}{2}} \det(1-H_2)^{\frac{g+1}{2}}$, up to a numerical constant. Then, (4.85) becomes

$$Z_g(k,\tau) = (\frac{2}{k\pi})^{\frac{g(g+1)}{2}} e^{-kg\pi} \int \prod_{I \le J}^{g} dH_2^{IJ} \Big(\det \frac{1 - H_2}{H_2^2} \Big)^{\frac{g+1}{2}} e^{-\frac{k\pi}{4} \mathrm{Tr} \frac{H_2^2}{1 - H_2}}. \tag{4.87}$$

[11] Here we ignored possible contribution from the boundary terms. On the boundary of the moduli space, when $\det\mathrm{Im}\Pi = 0$ and $\det\mathrm{Im}\tau = 0$, the integral (4.80) needs to be carefully regularised. This could result in a non-trivial τ-dependence, but such effects are beyond the scope of this chapter.

It is unclear, however, whether this expression has any interesting interpretation. In principle, we could use an alternative definition of the partition function, where the canonical measure is multiplied by some function of Π, instead of (4.79). Morally, this is equivalent to adding corresponding topological terms (depending only on the cohomology classes $[\zeta]$, without coupling to $\mathcal{K}$) to the action (4.48). This will bring us into the realm of the matrix models. However, it looks like just a digression to $0_{\mathbb{C}} + 1$ theory, and we are looking for the links with the higher dimensional theories.

Thus, we suggest a natural generalization of the theory, that comes from the following observation. If we concentrate only on the massive modes in the functional (4.48), described by the free non-compact fields ξ, it looks very much like the action for the string propagating on a complex torus $\mathcal{T}^g_{\mathbb{C}} = \mathbb{C}^g/\mathbb{Z}^g \oplus \Pi\mathbb{Z}^g$ with the period matrix Π. Indeed, as we discussed earlier, the metric $(\mathrm{Im}\Pi)^{-1}$ is a canonical metric induced on the torus from the flat Euclidian metric on $\mathbb{C}^g$. Therefore, from the stringy point of view the non-compact scalars $\xi : \Sigma_g \to \mathbb{C}^g$ can be promoted to the maps $\phi : \Sigma_g \to \mathcal{T}^g_{\mathbb{C}}$ with non-trivial winding numbers. Then, the cohomology class $[\zeta]$ in the combination $\zeta = [\zeta] + d\xi$ can be interpreted as a background abelian gauge field on the torus: $[\zeta] \to \mathcal{A}$. Therefore, we want to substitute

$$\zeta \to d\phi + \mathcal{A} \tag{4.88}$$

in the functional (4.48) and study resulting quantum theory. This modification will give us a new insight on the gauged WZW model for abelian varieties, coupled to the complex structure on Σ_g in the specific way (4.48).

4.4 Gauged WZW Model for Abelian Varieties and the Hitchin Functional

In this section we argue that in order to use quantum theory based on the Hitchin functional for computing topological invariants, one has to incorporate stringy effects into it. In particular, the target space has to be compactified, and in accordance with that one has to consider topologically non-trivial maps $\Sigma_g \to \mathcal{T}^g_{\mathbb{C}}$, instead of $\Sigma_g \to \mathbb{C}^g$. Here $\mathcal{T}^g_{\mathbb{C}}$ is a complex g-torus, viewed as a principally polarized abelian variety. Moreover, the translation group of the target space has to be gauged, so that a two-dimensional stringy version of the Hitchin functional becomes the gauged WZW model with an abelian gauge group. It is well known that the partition function of this model, representing the number of conformal blocks in corresponding toroidal CFT, is independent of the complex structure on Σ_g. However, as we will see, in the Hitchin extension of the model the coupling to the two-dimensional gravity appears non-perturbatively via the instanton effects.

Before describing the topological extension of Hitchin functional in two dimensions, we will recall some general aspects of the gauged Wess-Zumino-(Novikov)-Witten model. This theory was extensively studied in the literature, see *e.g.* [78, 77, 154, 168, 24, 79, 170, 23], therefore below we just summarize basic features of the model, following [168, 170, 23]. We will also use some facts about the abelian Chern-Simons theories with the gauge group $U(1)^d$, which has been discussed recently in great details in [89, 21]. We will be particularly interested in the case $d = 2g$, and focus on viewing gauge group as a complex algebraic variety.

4.4.1 Review of the Gauged WZW Model

Let G be a compact Lie group. The group G acts on itself by left and right multiplication, which is convenient to view as the action of $G_L \times G_R$. For any subgroup $H_L \times H_R \subseteq G_L \times G_R$ consider a principal $H_L \times H_R$ bundle X over Riemann surface Σ_g, with connection $(\mathcal{A}_L, \mathcal{A}_R)$. Let J be a complex structure on Σ_g. As we discussed earlier, it determines the action of the Hodge $*$-operator, corresponding to the Riemann metric compatible with this complex structure, on 1-forms. Consider the functional:

$$I(\mathcal{A}_L, \mathcal{A}_R; g) = -\frac{1}{8\pi}\int_{\Sigma_g} \mathrm{Tr}(g^{-1}d_{\mathcal{A}}g \wedge * g^{-1}d_{\mathcal{A}}g) - i\Gamma(g) + \tag{4.89}$$

$$+\frac{i}{4\pi}\int_{\Sigma_g} \mathrm{Tr}(\mathcal{A}_L \wedge dgg^{-1} + \mathcal{A}_R \wedge g^{-1}dg + \mathcal{A}_R \wedge g^{-1}\mathcal{A}_L g), \tag{4.90}$$

where $d_{\mathcal{A}}$ is the gauge-covariant extension of the exterior derivative:

$$d_{\mathcal{A}}g = dg + \mathcal{A}_L g - g\mathcal{A}_R, \tag{4.91}$$

and $\Gamma(g)$ is the topological WZNW term:

$$\Gamma(g) = \frac{1}{12\pi}\int_{B:\,\partial B=\Sigma_g} \mathrm{Tr}(g^{-1}dg)^{\wedge 3}. \tag{4.92}$$

Here Tr is an invariant quadratic form on the Lie algebra Lie$G$ of the group G, normalized so that $\Gamma(g)$ is well-defined modulo $2\pi\mathbb{Z}$. The field[12] g is promoted from the map $g : \Sigma_g \to G$ to a section of the bundle $X \times_{H_L\times H_R} G$, where G is understood as a trivial principal G bundle over Σ_g. We are interested in the non-anomalous gauging, which is only possible if for all $t, t' \in \mathrm{Lie}(H_L \times H_R)$

$$\mathrm{Tr}_L tt' - \mathrm{Tr}_R tt' = 0 \tag{4.93}$$

[12]not to be confused with the genus g of Σ_g.

where Tr_L and Tr_R are traces on LieH_L and LieH_R. The standard choice for a non-abelian group is $H_L = G_L$ and $H_R = G_R$, with diagonal action $g \to h^{-1}gh$. This gives the G/G gauged WZW model.

Consider the following propagator

$$\langle \Psi_{\mathcal{A}_L}(\Sigma_g)|\Psi_{\mathcal{A}_R}(\Sigma_g)\rangle = \int \mathcal{D}g e^{-kI(\mathcal{A}_L,\mathcal{A}_R;g)}. \tag{4.94}$$

This should be compared to (4.2). In order to simplify the notations, we will often write $\Psi_{\mathcal{A}}$ instead of $\Psi_{\mathcal{A}}(\Sigma_g)$, when the dependence on the complex structure of Σ_g is not essential. We will use the notation $\Psi_{\mathcal{A}}(\Sigma_g(\tau))$ if we want to stress dependence on the complex structure, parameterized by the period matrix τ of Σ_g.

By performing the change of variables $g \to g^{-1}$ in the functional integral (4.94), we find that the propagator has the necessary property

$$\overline{\langle \Psi_{\mathcal{A}_L}|\Psi_{\mathcal{A}_R}\rangle} = \langle \Psi_{\mathcal{A}_R}|\Psi_{\mathcal{A}_L}\rangle. \tag{4.95}$$

Furthermore, using the Gaussian integration and the Polyakov-Wiegmann formula

$$I(0,0,gh) = I(0,0,g) + I(0,0,h) - \frac{1}{4\pi}\int_{\Sigma_g} \mathrm{Tr}\, g^{-1}dg \wedge dhh^{-1}, \tag{4.96}$$

it is easy to check that propagator (4.94) satisfies the "gluing" condition (4.3). It is also straightforward to obtain the relation

$$I(\mathcal{A}_L^h, \mathcal{A}_R^{\widetilde{h}}; h^{-1}g\widetilde{h}) = I(\mathcal{A}_L,\mathcal{A}_R;g) - i\Phi(\mathcal{A}_L;h) + i\Phi(\mathcal{A}_R;\widetilde{h}), \tag{4.97}$$

where the gauge transformed connection is

$$\mathcal{A}_L^h = h^{-1}\mathcal{A}_L h + h^{-1}dh, \qquad \mathcal{A}_R^{\widetilde{h}} = \widetilde{h}^{-1}\mathcal{A}_R\widetilde{h} + \widetilde{h}^{-1}d\widetilde{h}, \tag{4.98}$$

and the cocycles

$$\Phi(\mathcal{A};h) = \frac{1}{4\pi}\int_{\Sigma_g} \mathrm{Tr}\mathcal{A}\wedge dhh^{-1} - \Gamma(h) \tag{4.99}$$

are independent of the complex structure (metric) on Σ_g, and satisfy

$$\Phi(\mathcal{A};hh') = \Phi(\mathcal{A}^h;h') + \Phi(\mathcal{A};h). \tag{4.100}$$

Infinitesimal form of these global gauge transformations, combined with a direct variation over $\mathcal{A}_{Lz}$ in the functional integral (4.94), leads to the following set of the equations for the propagator

$$\frac{D}{D\mathcal{A}_{Lz}}\left\langle\Psi_{\mathcal{A}_L}|\Psi_{\mathcal{A}_R}\right\rangle = 0 \tag{4.101}$$

$$\Big(D^L_a\frac{D}{D\mathcal{A}_{La}} + \frac{ik}{4\pi}\epsilon^{ab}\mathcal{F}^L_{ab}\Big)\left\langle\Psi_{\mathcal{A}_L}|\Psi_{\mathcal{A}_R}\right\rangle = 0 \tag{4.102}$$

where we introduced a connection $\frac{D}{D\mathcal{A}_L}$ on the line bundle $\mathcal{L}^k$ over the space of H_L-valued connections

$$\frac{D}{D\mathcal{A}_{Lz}} = \frac{\delta}{\delta\mathcal{A}_{Lz}} + \frac{k}{4\pi}\mathcal{A}_{L\bar{z}} \tag{4.103}$$

$$\frac{D}{D\mathcal{A}_{L\bar{z}}} = \frac{\delta}{\delta\mathcal{A}_{L\bar{z}}} - \frac{k}{4\pi}\mathcal{A}_{Lz}, \tag{4.104}$$

and covariant derivatives on the principal H_L bundle over Σ_g

$$D^L_a = \partial_a + [\mathcal{A}_L, \,.\,], \tag{4.105}$$

with the curvature form

$$\mathcal{F}^L = [D^L, D^L] = d\mathcal{A}_L + \mathcal{A}_L\wedge\mathcal{A}_L. \tag{4.106}$$

Connections (4.103) obey canonical commutation relation

$$\Big[\frac{D}{D\mathcal{A}_{Lz}(z)}, \frac{D}{D\mathcal{A}_{L\overline{w}}(w)}\Big] = +\frac{k}{2\pi}\delta(z,w) \tag{4.107}$$

The propagator (4.94) also satisfies a set of conjugate equations that describe its dependence on $\mathcal{A}_R$. These equations are obtained from (4.101)-(4.107) by a change of the indices and signs, according to

$$L \leftrightarrow R, \qquad +k \leftrightarrow -k. \tag{4.108}$$

Geometrically, it means that the propagator $\langle \Psi_{\mathcal{A}_L} | \Psi_{\mathcal{A}_R} \rangle$ is an (equivariant) holomorphic section of the line bundle $\boldsymbol{\mathcal{L}} = \mathcal{L}^k \otimes \mathcal{L}^{-k}$.

The quantum field theory with Lagrangian $L = kI(\mathcal{A}, \mathcal{A}; g)$, $k \in \mathbb{Z}_+$ is conformal and gauge invariant and is called the G/G gauged WZW model for the non-abelian group G at level k:

$$Z_k(G/G; \Sigma_g) = \int \frac{\mathcal{D}g\mathcal{D}\mathcal{A}}{\mathrm{vol(Gauge)}} e^{-kI(\mathcal{A},\mathcal{A};g)}. \tag{4.109}$$

It is a two-dimensional sigma model with target space G gauged by a non-anomalous subgroup diag$(G_R \times G_R)$. The partition function of the G/G gauged WZW model can be also written as

$$Z_k(G/G; \Sigma_g) = \mathrm{Tr}_{\mathcal{A}} \langle \Psi_{\mathcal{A}}(\Sigma_g) | \Psi_{\mathcal{A}}(\Sigma_g) \rangle = \int \frac{\mathcal{D}\mathcal{A}_L \mathcal{D}\mathcal{A}_R}{\mathrm{vol}^2\mathrm{(Gauge)}} \left| \langle \Psi_{\mathcal{A}_L} | \Psi_{\mathcal{A}_R} \rangle \right|^2, \tag{4.110}$$

where we used (4.95). This should be compared to (4.4), with the identification $Z_k(G/G; \Sigma_g) = Z_{\Sigma_g \times \mathbf{S}^1}$. After performing the Gaussian integration, applying the Polyakov-Wiegmann formula and relation (4.97), we indeed get (4.109).

The gauged WZW functional (4.89) allows one to connect three-dimensional Chern-Simons theory and its dual two-dimensional rational conformal field theory in a simple and effective way. The partition function of the WZW model is

$$Z_k(G; \Sigma_g) = \int \mathcal{D}g e^{-kI(0,0;g)}. \tag{4.111}$$

The holomorphic factorization of the WZW model into the conformal blocks can be explained by observing [168] that (4.111) can also be written as

$$Z_k(G;\Sigma_g) = \langle \Psi_0(\Sigma_g)|\Psi_0(\Sigma_g)\rangle = ||\, \Psi_0(\Sigma_g)\rangle|^2 = \int \frac{\mathcal{DA}}{\text{vol(Gauge)}}\, |\langle \Psi_0|\Psi_{\mathcal{A}}\rangle|^2. \quad (4.112)$$

The WZW model is a rational conformal field theory if it is constructed from a finite number of conformal blocks[13]. In this case the conformal blocks of the WZW model are in one-to-one correspondence with the states in a Hilbert space[14], obtained from canonical quantization of the Chern-Simons theory on $\Sigma_g \times \mathbb{R}$ [165, 68]. A geometrical interpretation of this Hilbert space (achieved in the framework of the geometrical quantization [14]) is that it is a space $V_{g,k}(G)$ of (equivariant) holomorphic sections of k-th power of the determinant line bundle $\mathcal{L}$ over the moduli space of (semistable) holomorphic $G_{\mathbb{C}}$-connections on Σ_g, which by the Narasimhan-Seshadri theorem is the same as the moduli space $\mathcal{M}_G$ of flat connections on the principal G-bundle over Σ_g. Thus, the Hilbert space is given exactly by the holomorphic sections that satisfy (4.101).

Let $s_\gamma(\mathcal{A};\tau)$, $\gamma = 1, \ldots \text{dim} H^0(\mathcal{M}_G, \mathcal{L}^k)$ be an orthonormal basis in the space $H^0(\mathcal{M}_G, \mathcal{L}^k)$ of holomorphic sections. Then we can write the propagator in (4.112) as

$$\langle \Psi_0(\Sigma_g(\tau))|\Psi_{\mathcal{A}}(\Sigma_g(\tau))\rangle = \sum_{\gamma=1}^{\text{dim} H^0(\mathcal{M}_G,\mathcal{L}^k)} \overline{F_\gamma(\tau)}\, s_\gamma(\mathcal{A};\tau). \quad (4.113)$$

The coefficients $F_\gamma(\tau)$ in (4.113) are the conformal blocks of the WZW model. Of course, the dimensions of the space $V_{g,k}(G)$ of conformal blocks and the space $H^0(\mathcal{M}_G, \mathcal{L}^k)$ of holomorphic sections coincide. After plugging (4.113) into (4.112) and using the

[13]There are many definitions of RCFT, but this one is the most convenient for our purposes.
[14]For simplicity we are not considering marked points on Σ_g.

orthonormality of the basis $s_\gamma(\mathcal{A};\tau)$, we obtain

$$Z_k(G;\Sigma_g(\tau)) = \sum_{\gamma=1}^{\dim V_{g,k}} |F_\gamma(\tau)|^2. \tag{4.114}$$

The propagator (4.94) can now be written as

$$\langle \Psi_{\mathcal{A}_L}(\Sigma_g(\tau))|\Psi_{\mathcal{A}_R}(\Sigma_g(\tau))\rangle = \sum_{\gamma=1}^{\dim V_{g,k}} s_\gamma(\mathcal{A}_L;\tau)\overline{s_\gamma(\mathcal{A}_R;\tau)}, \tag{4.115}$$

which is a unique solution to the equations (4.101) (and conjugate equations (4.108)) obeying the "gluing" condition (4.3). After plugging this into (4.139) we find

$$Z_k(G/G;\Sigma_g) = \dim V_{g,k}(G) = \dim H^0(\mathcal{M}_G,\mathcal{L}^k). \tag{4.116}$$

Therefore, the partition function of the gauged WZW model computes the dimension of the Chern-Simons Hilbert space $V_{g,k}(G)$, which coincides with the number of conformal blocks in the corresponding RCFT. This can be viewed as an example of the universal index theorem (4.5) for the universal partition function (4.4) of $\Sigma_g \times \mathbf{S}^1$, which in this case is equal to $Z_k(G/G;\Sigma_g)$. The higher cohomology groups vanish since we are dealing with the integrable representations of RCFT.

Another way to explain (4.116) is to observe [79, 170] that the propagator (4.94) is exactly the free propagator of the Chern-Simons theory multiplied by the projector on the gauge invariant subspace, enforcing the Gauss law. In other words, equation (4.109) is equivalent to $Z_k(G/G;\Sigma_g) = \mathrm{Tr}\ 1$, which yields (4.116).

¿From the CFT algebra viewpoint, the number of conformal blocks $\dim V_{g,k}(G)$ is given by the E. Verlinde's formula [161]. For example, when $G = SU(2)$,

$$\dim V_{g,k}(SU(2)) = \Big(\frac{k+2}{2}\Big)^{g-1} \sum_{j=0}^{k} \sin^{2-2g} \frac{(j+1)}{k+2}\pi. \tag{4.117}$$

The gauged WZW model provides a constructive method of computing the dimension of the Verlinde algebra via the localization of the functional integral [24, 79].

4.4.2 Abelian Case

We are particularly interested in the case when G is an abelian group: $G \sim U(1)^{2g}$. Moreover, we want to view it as an algebraic complex variety with a fixed complex structure. Therefore, we will describe G as a g-dimensional complex torus $\mathcal{T}$, which is a principally polarized abelian variety: $\mathcal{T} = \mathbb{C}^g/\Lambda$, $\Lambda = \mathbb{Z}^g \oplus \Pi\mathbb{Z}^g$. Sometimes we will use the notation $\mathcal{T}(\Pi)$ to show the explicit dependence of $\mathcal{T}$ on the defining period matrix Π.

Let us first describe an analog of the functional (4.89) for the case of abelian group $\mathcal{T} \sim U(1)^{2g}$. In complex coordinates, it has the form[15]

$$\begin{aligned} &I(\mathcal{A}_L, \mathcal{A}_R; \phi) = \\ &= \tfrac{\pi\mathcal{G}_{IJ}}{2} \int\limits_{\Sigma_g} d_{\mathcal{A}}\phi^I \wedge * d_{\overline{\mathcal{A}}}\overline{\phi}^J + i(\mathcal{A}_L^I + \mathcal{A}_R^I) \wedge d\overline{\phi}^J + i(\overline{\mathcal{A}}_L^I + \overline{\mathcal{A}}_R^I) \wedge d\phi^J - \\ &-i\tfrac{\pi\mathcal{G}_{IJ}}{2} \int\limits_{\Sigma_g} (\mathcal{A}_L^I \wedge \overline{\mathcal{A}}_R^J + \overline{\mathcal{A}}_L^I \wedge \mathcal{A}_R^J) \; - \; i\Gamma(\phi), \end{aligned} \tag{4.118}$$

where $(\mathcal{A}_L, \mathcal{A}_R)$ are connections on a principal bundle $\mathcal{T}_L \times \mathcal{T}_R$ over Σ_g, and the scalar fields $\phi \sim \phi + \mathbb{Z} + \Pi\mathbb{Z}$ describe the maps $\Sigma_g \to \mathcal{T}$, which, after coupling to the gauge fields, are promoted to the corresponding sections, with the covariant derivative defined as

$$d_{\mathcal{A}}\phi^I = d\phi^I + \mathcal{A}_L^I - \mathcal{A}_R^I. \tag{4.119}$$

The role of the trace operator Tr in (4.89) now is played by the matrix $\mathcal{G}_{IJ} = (\frac{1}{\mathrm{Im}\Pi})_{IJ}$,

[15]We should be careful with the expression (4.89) in the case of $U(1)^{2g}$ abelian group. Naively, it looks like we can take $2g$ copies of the WZNW term (4.92) for $U(1)$ group, but this expression vanishes for abelian group element $g = e^{i\varphi}$. The analog of this term in the abelian group case is $\Gamma(\phi) = \pi\mathcal{G}_{IJ}\int_{\Sigma_g} d\phi^I \wedge d\overline{\phi}^J$, which can be interpreted a B-field. It is crucial for a global identification with the corresponding three-dimensional Chern-Simons theory.

that defines a canonical metric on $\mathcal{T}$:

$$\mathcal{G}(\phi, \phi) = \frac{1}{2}\mathcal{G}_{IJ}(\phi^I \otimes \overline{\phi}^J + \overline{\phi}^I \otimes \phi^J). \tag{4.120}$$

The analog of the topological WZNW term is

$$\Gamma(\phi) = \pi \mathcal{G}_{IJ} \int_{\Sigma_g} d\phi^I \wedge d\overline{\phi}^J, \tag{4.121}$$

which obey the corresponding Polyakov-Wiegmann formula

$$\Gamma(\phi + \psi) = \Gamma(\phi) + \Gamma(\psi) + \pi \mathcal{G}_{IJ} \int_{\Sigma_g} d\phi^I \wedge d\overline{\psi}^J - d\overline{\phi}^I \wedge d\psi^J. \tag{4.122}$$

Under the small gauge transformations

$$\mathcal{A}_L^{\psi} = \mathcal{A}_L + d\psi, \qquad \mathcal{A}_R^{\widetilde{\psi}} = \mathcal{A}_R + d\widetilde{\psi}, \tag{4.123}$$

the change of the functional (4.118) depends on $\mathcal{A}_{L,R}$ but not on ϕ or the complex structure of Σ_g:

$$I(\mathcal{A}_L^{\psi}, \mathcal{A}_R^{\widetilde{\psi}}; \phi + \widetilde{\psi} - \psi) - I(\mathcal{A}_L, \mathcal{A}_R; \phi) = \tag{4.124}$$

$$= i\frac{\mathcal{G}_{IJ}\pi}{2} \int_{\Sigma_g} \mathcal{A}_R^I \wedge d\overline{\widetilde{\psi}}^J + \overline{\mathcal{A}}_R^I \wedge d\widetilde{\psi}^J - \mathcal{A}_L^I \wedge d\overline{\psi}^J - \overline{\mathcal{A}}_L^I \wedge d\psi^J. \tag{4.125}$$

This should be compared to (4.97). There are certain restrictions [68, 27] on the possible choice of the period matrix Π of the torus $\mathcal{T}(\Pi)$. First, in order for the functional (4.118) to be a well defined modulo $2\pi i\mathbb{Z}$, the lattice $\Lambda = \mathbb{Z}^g \oplus \Pi\mathbb{Z}^g$ has to be integral. Second, modular invariance requires Λ to be even lattice. Therefore,

$$\mathbb{Z}^g \oplus \Pi\mathbb{Z}^g \in \Gamma_{2g}^{2\mathbb{Z}}. \tag{4.126}$$

where $\Gamma_{2g}^{2\mathbb{Z}}$ denotes the moduli space of even integral $2g$-dimensional lattices. The dual conformal field theory in this case is rational.

Let us define the propagator as

$$\langle \Psi_{\mathcal{A}_L} | \Psi_{\mathcal{A}_R} \rangle = \int \mathcal{D}\phi \, e^{-kI(\mathcal{A}_L, \mathcal{A}_R; \phi)}. \tag{4.127}$$

Notice that interactions in (4.118) are such that ϕ is coupled only to $\mathcal{A}_{L\bar{z}}$, via the term $d\phi^I \wedge (i - *)\overline{\mathcal{A}}^J_L$, and to $\mathcal{A}_{Rz}$, via the term $d\phi^I \wedge (i + *)\overline{\mathcal{A}}^J_R$. Moreover, "left" and "right" gauge fields interact only via the coupling $\mathcal{A}^I_{Rz} \wedge \overline{\mathcal{A}}^J_{L\bar{z}}$, and its complex conjugate. This observation, combined with the Ward identity, that follows from (4.124), leads to the following set of equations[16], which the propagator obeys:

$$\frac{D}{D\mathcal{A}^I_{Lz}} \langle \Psi_{\mathcal{A}_L} | \Psi_{\mathcal{A}_R} \rangle = 0 \tag{4.128}$$

$$\frac{D}{D\mathcal{A}^I_{R\bar{z}}} \langle \Psi_{\mathcal{A}_L} | \Psi_{\mathcal{A}_R} \rangle = 0 \tag{4.129}$$

$$\Big(\partial_a \frac{D}{D\mathcal{A}^I_{La}} + ik\pi \mathcal{G}_{IJ} \epsilon^{ab} \partial_a \mathcal{A}^J_{Lb}\Big) \langle \Psi_{\mathcal{A}_L} | \Psi_{\mathcal{A}_R} \rangle = 0 \tag{4.130}$$

$$\Big(\partial_a \frac{D}{D\mathcal{A}^I_{Ra}} - ik\pi \mathcal{G}_{IJ} \epsilon^{ab} \partial_a \mathcal{A}^J_{Rb}\Big) \langle \Psi_{\mathcal{A}_L} | \Psi_{\mathcal{A}_R} \rangle = 0 \tag{4.131}$$

Here we introduced a connection

$$\frac{D}{D\mathcal{A}^I_{Lz}} = \frac{\delta}{\delta\mathcal{A}^I_{Lz}} + k\pi \mathcal{G}_{IJ} \mathcal{A}^J_{L\bar{z}}, \qquad \frac{D}{D\mathcal{A}^I_{L\bar{z}}} = \frac{\delta}{\delta\mathcal{A}^I_{L\bar{z}}} - k\pi \mathcal{G}_{IJ} \mathcal{A}^J_{L\bar{z}}, \tag{4.132}$$

$$\frac{D}{D\mathcal{A}^I_{Rz}} = \frac{\delta}{\delta\mathcal{A}^I_{Rz}} - k\pi \mathcal{G}_{IJ} \mathcal{A}^J_{L\bar{z}}, \qquad \frac{D}{D\mathcal{A}^I_{R\bar{z}}} = \frac{\delta}{\delta\mathcal{A}^I_{R\bar{z}}} + k\pi \mathcal{G}_{IJ} \mathcal{A}^J_{L\bar{z}}, \tag{4.133}$$

on the line bundle $\mathcal{L}^k \times \mathcal{L}^{-k}$ over the space $\mathbf{A}$ of $\mathcal{T}_L \times \mathcal{T}_R$-valued connections on Σ_g.

The geometrical interpretation of the equations (4.128) is very simple. We pick a standard complex structure on the space $\mathbf{A}$ of connections induced from the complex structure on Σ_g. In this complex structure, $\mathcal{A}_{L\bar{z}}$ and $\mathcal{A}_{Rz}$ are holomorphic, and $\mathcal{A}_{Lz}$ and $\mathcal{A}_{R\bar{z}}$ are antiholomorphic. Then the propagator $\langle \Psi_{\mathcal{A}_L} | \Psi_{\mathcal{A}_R} \rangle$ is a holomorphic

[16] We treat $\mathcal{A}_{L,R}$ and $\overline{\mathcal{A}}_{L,R}$ as independent variables, and there is also a corresponding set of equations with $\mathcal{A}_{L,R} \to \overline{\mathcal{A}}_{L,R}$.

section of the line bundle $\boldsymbol{\mathcal{L}} = \mathcal{L}^k \otimes \mathcal{L}^{-k}$, equivariant with respect to the action of the abelian group $\mathcal{T}_L \times \mathcal{T}_R$.

It is well-known (see, *e.g.*, [14, 158]) that the basis in the corresponding space $H^0(\mathcal{M}_{\mathcal{T}}, \mathcal{L}^k)$ of the gauge invariant holomorphic sections of $\mathcal{L}^k$ is provided by the level k Narain-Siegel theta-functions $\Theta_\gamma(\mathcal{A};\tau|\Lambda,k)$, associated with the lattice Λ, that defines the torus $\mathcal{T} = \mathbb{C}^g/\Lambda$. We will not need an explicit expression for $\Theta_\gamma(\mathcal{A},\tau|\Lambda,k)$ (it can be found, for example, in [89, 21]). What is important for us is that the linear independent Narain-Siegel theta-functions are labelled by the index $\gamma \in (\Lambda^*/k\Lambda)^{\otimes g}$, where Λ^* is the dual lattice. ¿From the viewpoint of the three-dimensional abelian Chern-Simons theory, these theta-functions are exactly the wave-functions: $\Psi_\gamma(\mathcal{A};\tau) \sim \Theta_\gamma(\mathcal{A};\tau|\Lambda,k)$. Therefore, the dimension of the corresponding Hilbert space is

$$\dim Hilb_{\mathrm{CS}}(\Lambda,k) = |\Lambda^*/k\Lambda|^g. \tag{4.134}$$

We can repeat the steps that we did in the non-abelian case, and connect abelian Chern-Simons theory and its dual CFT via the functional (4.118) and the propagator (4.143. The property of the functional (4.118)

$$\overline{I(\mathcal{A}_L,\mathcal{A}_R;\phi)} = I(\mathcal{A}_R,\mathcal{A}_L;-\phi) \tag{4.135}$$

guarantees that the propagator (4.143) is hermitian:

$$\overline{\langle\Psi_{\mathcal{A}_L}|\Psi_{\mathcal{A}_R}\rangle} = \langle\Psi_{\mathcal{A}_R}|\Psi_{\mathcal{A}_L}\rangle, \tag{4.136}$$

since we can always change the variables $\phi \to -\phi$ in the functional integral (4.143). Moreover, it is straightforward to show that the propagator obeys the gluing condition

(4.3

$$\langle \Psi_{\mathcal{A}_L} | \Psi_{\mathcal{A}_R} \rangle = \int \frac{\mathcal{D}\mathcal{A}}{\text{vol(Gauge)}} \langle \Psi_{\mathcal{A}_L} | \Psi_{\mathcal{A}} \rangle \langle \Psi_{\mathcal{A}} | \Psi_{\mathcal{A}_R} \rangle, \tag{4.137}$$

by performing the Gaussian integral over $\mathcal{A}$, using the Polyakov-Wiegmann formula, and the fact that $\int \mathcal{D}\phi = \text{vol(Gauge)}$. This allows us to write down the following expression for the propagator in terms of the Narain-Siegel theta-functions:

$$\langle \Psi_{\mathcal{A}_L} | \Psi_{\mathcal{A}_R} \rangle = \sum_{\gamma \in (\Lambda^*/k\Lambda)^{\otimes g}} \Theta_\gamma(\mathcal{A}_L; \tau | \Lambda, k) \overline{\Theta_\gamma(\mathcal{A}_R; \tau | \Lambda, k)} \tag{4.138}$$

The partition function of the gauged WZW model for abelian group $\mathcal{T}$ at level k is defined as

$$Z_k(\mathcal{T}/\mathcal{T}; \Sigma_g) = \int \frac{\mathcal{D}\phi \mathcal{D}\mathcal{A}}{\text{vol(Gauge)}} \, e^{-kI(\mathcal{A}, \mathcal{A}; \phi)} = \text{Tr}_{\mathcal{A}} \langle \Psi_{\mathcal{A}} | \Psi_{\mathcal{A}} \rangle \tag{4.139}$$

Using (4.138) and orthonormality of the Narain-Siegel theta-functions

$$\int \frac{\mathcal{D}\mathcal{A}}{\text{vol(Gauge)}} \, \Theta_\gamma(\mathcal{A}; \tau | \Lambda, k) \, \overline{\Theta_{\gamma'}(\mathcal{A}; \tau | \Lambda, k)} = \delta_{\gamma\gamma'}, \tag{4.140}$$

it is easy to see that the partition function (4.139) indeed computes the dimension of the Chern-Simons theory Hilbert space

$$Z_k(\mathcal{T}/\mathcal{T}; \Sigma_g) = |\Lambda^*/k\Lambda|^g. \tag{4.141}$$

4.4.3 Hitchin Extension of the Abelian GWZW Model

Now we are ready to discuss the Hitchin extension of the gauged WZW functional (4.118) for abelian group. We want to introduce non-trivial coupling to the complex structure on Σ_g by using the operator $\imath_{\mathcal{K}}$ instead of the Hodge $*$-operator, and adding the term $i\lambda\,\mathrm{tr}(\mathcal{K}^2+\mathbb{1})$ to the action. This leads to the following functional

$$\begin{aligned} I(\mathcal{A}_L,\mathcal{A}_R;\phi|\lambda,\mathcal{K}) \;=\;& \tfrac{\mathcal{G}_{IJ}\pi}{4}\int_{\Sigma_g} d_{\mathcal{A}}\phi^I\wedge\imath_{\mathcal{K}}\, d_{\overline{\mathcal{A}}}\overline{\phi}^J + d_{\overline{\mathcal{A}}}\overline{\phi}^I\wedge\imath_{\mathcal{K}}\, d_{\mathcal{A}}\phi^J - 4\sqrt{-1}d\phi^I\wedge d\overline{\phi}^J + \\ & + \tfrac{\sqrt{-1}\pi}{2}\mathcal{G}_{IJ}\int_{\Sigma_g}(\mathcal{A}_L^I+\mathcal{A}_R^I)\wedge d\overline{\phi}^J + (\overline{\mathcal{A}}_L^I+\overline{\mathcal{A}}_R^I)\wedge d\phi^J - \\ & - \tfrac{\sqrt{-1}\pi}{2}\mathcal{G}_{IJ}\int_{\Sigma_g}(\mathcal{A}_L^I\wedge\overline{\mathcal{A}}_R^J+\overline{\mathcal{A}}_L^I\wedge\mathcal{A}_R^J) - \tfrac{\sqrt{-1}\pi}{4}\int_{\Sigma_g}\lambda\,\mathrm{tr}(\mathcal{K}^2+\mathrm{Id}). \end{aligned} \tag{4.142}$$

The Hitchin extension of the propagator (4.143) formally is given by

$$\langle\Psi(\mathcal{A}_L|\lambda,\mathcal{K})|\Psi(\mathcal{A}_R|\lambda,\mathcal{K})\rangle = \int \mathcal{D}\phi\, e^{-kI(\mathcal{A}_L,\mathcal{A}_R;\phi|\lambda,\mathcal{K})}. \tag{4.143}$$

However, this expression can be interpreted as a propagator only for the "on-shell" values of the field $\mathcal{K}$, such that

$$\mathcal{K}^2 = -\mathrm{Id}. \tag{4.144}$$

In this case $\mathcal{K}$ defines a complex structure on Σ_g, and we can glue together two propagators defined in the same complex structure, according to the gluing rule (4.137). Moreover, if we define a "partition function" as

$$Z_k(\mathcal{T}|\lambda,\mathcal{K}) = \int \frac{\mathcal{D}\phi\mathcal{D}\mathcal{A}}{\mathrm{vol(Gauge)}}\, e^{-kI(\mathcal{A},\mathcal{A};\phi|\lambda,\mathcal{K})}, \tag{4.145}$$

then formally we can write

$$\int \mathcal{D}\lambda\; Z_k(\mathcal{T}|\lambda,\mathcal{K}) = |\Lambda^*/k\Lambda|^g\;\delta(\mathrm{tr}\mathcal{K}^2+2). \tag{4.146}$$

The meaning of this expression is that perturbatively the Hitchin extension (4.142) is equivalent to the ordinary gauged WZW model (4.118). There is no non-trivial $\mathcal{K}$ dependence in (4.146), and after performing the integration over $\mathcal{D}\mathcal{K}$ we will just get some multiplicative constant, depending on g. This should not be surprising. After all, the gauged WZW model computes the number of conformal blocks (the dimension of the corresponding Hilbert space), and this number does not depend on the choice of the complex structure on Σ_g, which is controlled by the field $\mathcal{K}$.

However, the very new feature of the Hitchin extension is that dependence on $\mathcal{K}$ can be restored non-perturbatively. Indeed, if the action $kI(\mathcal{A}, \mathcal{A}; \phi|\lambda, \mathcal{K})$ has non-trivial critical point, we have to do expansion around this point in the functional integral. In this case, the answer will depend on the value $\mathcal{K}_{min}$ of the complex structure tensor at the minimal point of the action.

4.4.4 Attractor Points and Complex Multiplication

Therefore, we have to study the critical points of the functional

$$\begin{aligned} I(\mathcal{A}, \mathcal{A}; \phi|\lambda, \mathcal{K}) \quad &= \tfrac{\mathcal{G}_{IJ}\pi}{4} \int\limits_{\Sigma_g} (d\phi^I \wedge \imath_{\mathcal{K}}\, d\overline{\phi}^J + d\overline{\phi}^I \wedge \imath_{\mathcal{K}}\, d\phi^J - 4\sqrt{-1} d\phi^I \wedge d\overline{\phi}^J) + \\ &+ \sqrt{-1}\pi \mathcal{G}_{IJ} \int\limits_{\Sigma_g} (\mathcal{A}^I \wedge d\overline{\phi}^J + \overline{\mathcal{A}}^I \wedge d\phi^J) - \tfrac{\sqrt{-1}\pi}{4} \int\limits_{\Sigma_g} \lambda\, \mathrm{tr}(\mathcal{K}^2 + \mathrm{Id}). \end{aligned} \tag{4.147}$$

Let us recall that in the functional integral (4.145) we integrate over the exact parts φ^I of the fields $d\phi^I = [d\phi^I] + d\varphi^I$ and sum over non-trivial maps $[d\phi^I] \in H^1(\Sigma_g, \Lambda)$. After dividing by the gauge transformations, we need to integrate only over the space of gauge inequivalent flat gauge fields $\mathcal{A}^I \in H^1(\Sigma_g, \mathbb{C}^g)/H^1(\Sigma_g, \Lambda)$. Therefore, in the functional (4.147) the exact part of $d\phi$ couples only to the term $\imath_{\mathcal{K}}\, d\overline{\phi}$. By varying $\overline{\varphi}$, we get a classical equation of motion, analogous to (4.61):

$$d\imath_{\mathcal{K}}\, d\phi^I = 0. \tag{4.148}$$

After solving the constraint $\mathrm{tr}\mathcal{K}^2 = -2$, imposed by the Lagrange multiplier λ, the equations of motion for $\mathcal{K}$ give

$$\mathcal{K}^a{}_b = \frac{\mathcal{G}_{IJ}}{2\sqrt{\det\|h\|}}\,(d\phi^I_b d\overline{\phi}^J_c + d\overline{\phi}^I_c d\phi^J_b)\epsilon^{ca}, \tag{4.149}$$

where h is the metric induced on Σ_g. If we recall that $\mathcal{G}_{IJ} = (\frac{1}{\mathrm{Im}\Pi})_{IJ}$, this metric takes the form

$$h_{ab} = \Big(\frac{1}{2\mathrm{Im}\Pi}\Big)_{IJ}(d\phi^I_a d\overline{\phi}^J_b + d\overline{\phi}^I_b d\phi^J_a). \tag{4.150}$$

Expressions (4.149 and (4.150) should be compared with (4.50) and (4.53).

For generic choice of the matrix $\Pi \in \mathcal{H}_g$ and cohomology vectors $[d\phi^I] \in H^1(\Sigma_g, \Lambda)$ expression (4.149) for the complex structure can be singular at some points on Σ_g. Those are the points where the determinant of the metric (4.150 vanishes[17]. However, it is easy to find a family of non-singular solutions (4.149). Let us compare(4.150) with the expression (4.43) for the canonical Bergmann metric on the Riemann surface $\Sigma_g(\tau)$

$$h^{\mathrm{B}}_{ab} = \Big(\frac{1}{2\mathrm{Im}\tau}\Big)_{IJ}(\omega^I_a \overline{\omega}^J_b + \overline{\omega}^I_b \omega^J_a). \tag{4.151}$$

The complex structure on $\Sigma_g(\tau)$ is defined by the period matrix τ, and is such that the differentials ω^I are holomorphic. If we set

$$d\phi^I = \omega^I, \tag{4.152}$$

and choose the torus $\mathcal{T}$, for which

$$\Pi = \tau, \tag{4.153}$$

[17]For example, the metric $h_{z\bar{z}} = |\omega^1_z|^2$ vanishes at zeroes of the abelian differential ω^1. Strictly speaking, even in this singular case it is possible to define complex structure globally on Σ_g via appropriate conformal transformation and analytical continuation, but the resulting complex structure will not be given by (4.149).

then the metric (4.150) coincides with the Bergmann metric: $h = h^{\rm B}$, and therefore the complex structure defined by $\mathcal{K}$ coincides with the complex structure defined by τ.

In order to parameterize general non-singular complex structure solutions (4.149 we proceed as follows. Suppose that some $\mathcal{K}$, given by (4.149, provides a globally well-defined complex structure on Σ_g. All complex structures on Σ_g are parameterized by the period matrices. Therefore, there is the period matrix τ that defines the same complex structure on Σ_g as $\mathcal{K}$. Then $\mathcal{K}$ must be equal to the corresponding Bergmann complex structure: $\mathcal{K} = \mathcal{K}^{\rm B}(\tau)$, which is a canonical complex structure compatible with the metric $h^{\rm B}$ (4.151). This gives

$$\sqrt{\det\|h\|^{\rm B}}\Big(\frac{1}{{\rm Im}\Pi}\Big)_{IJ}(d\phi^I_b d\overline{\phi}^J_c + d\overline{\phi}^I_c d\phi^J_b) = \sqrt{\det\|h\|}\Big(\frac{1}{{\rm Im}\tau}\Big)_{IJ}(\omega^I_b\overline{\omega}^J_c + \overline{\omega}^I_c\omega^J_b). \quad (4.154)$$

Since the 1-forms $d\phi$ are harmonic, we can express them in terms of the abelian differentials:

$$d\phi = M\omega + N\overline{\omega}, \quad (4.155)$$

where M and N are certain $g\times g$ complex matrices representing non-trivial mappings $\Sigma_g(\tau) \to \mathcal{T}(\Pi)$. If A and B are the period matrices of the 1-forms:

$$A^{IJ} = \oint_{A_J} d\phi^I, \qquad B^{IJ} = \oint_{B_J} d\phi^I, \quad (4.156)$$

then

$$M = (B - A\overline{\tau})\frac{1}{\tau - \overline{\tau}}, \qquad N = -(B - A\tau)\frac{1}{\tau - \overline{\tau}}. \quad (4.157)$$

We stress that (4.155) is an exact expression for the 1-forms $d\phi$, that solves classical equations of motion, as opposed to (4.71), that captures only the cohomology class.

Once the cohomology class $[d\phi]$ of the 1-forms is fixed, the exact part $d\phi - [d\phi]$ is uniquely determined by (4.148), which states that $d\phi$ is a linear combination of the harmonic representatives.

Combining the $\omega_b^I \omega_c^J$ terms in (4.154), we find

$$M^T \frac{1}{\mathrm{Im}\Pi} \overline{N} = 0, \tag{4.158}$$

which means that either $N = 0$ or $M = 0$, since $\mathrm{Im}\Pi$ is non-degenerate. The terms of the form $\omega_b^I \overline{\omega}_c^J$ give

$$\mathrm{Tr}\Big(\omega_b^T M^T \frac{1}{\mathrm{Im}\Pi} \overline{M} \overline{\omega}_c\Big) + \mathrm{Tr}\Big(\overline{\omega}_b^T N^T \frac{1}{\mathrm{Im}\Pi} \overline{N} \omega_c\Big) = \sqrt{\frac{\det\|h\|}{\det\|h^{\mathrm{B}}\|}}\, \mathrm{Tr}\Big(\omega_b^T \frac{1}{\mathrm{Im}\tau} \overline{\omega}_c\Big). \tag{4.159}$$

Thus, the only way to satisfy (4.158)-(4.159) is to set $N = 0$, and

$$M^T \frac{1}{\mathrm{Im}\Pi} \overline{M} = \frac{1}{\mathrm{Im}\tau}. \tag{4.160}$$

This equation implies that $\det M \neq 0$, since the matrices $\mathrm{Im}\Pi$ and $\mathrm{Im}\tau$ are not degenerate. Moreover, in this case we also have $h = h^{\mathrm{B}}$. ¿From (4.157) we see, that the condition $N = 0$ is equivalent to

$$B = A\tau, \tag{4.161}$$

so that

$$M = A. \tag{4.162}$$

The columns of the matrix A (4.156) are the vectors of the lattice $\Lambda = \mathbb{Z}^g \oplus \Pi\mathbb{Z}^g$. We can write it as $A = P_{\mathbb{Z}} + \Pi Q_{\mathbb{Z}}$, where $P_{\mathbb{Z}}$ and $Q_{\mathbb{Z}}$ are integral $g \times g$ matrices. Therefore, the complex structures $\mathcal{K}_*$ corresponding to the critical points of the functional (4.147) can be parameterized by the period matrices τ, obeying

$$\frac{1}{\mathrm{Im}\tau} = (P_{\mathbb{Z}}^T + Q_{\mathbb{Z}}^T\Pi)\frac{1}{\mathrm{Im}\Pi}(P_{\mathbb{Z}} + \overline{\Pi}Q_{\mathbb{Z}}). \tag{4.163}$$

This equation puts additional constraint on the period matrix, which according to (4.161) can be written as $\tau = A^{-1}B$. Since the columns of the matrix B are also the vectors of the lattice Λ, we can write it as $B = P'_{\mathbb{Z}} + \Pi Q'_{\mathbb{Z}}$, where $P'_{\mathbb{Z}}$ and $Q'_{\mathbb{Z}}$ are integral $g \times g$ matrices. Then (4.161) takes the form

$$\tau = \frac{1}{P_{\mathbb{Z}} + \Pi Q_{\mathbb{Z}}}(P'_{\mathbb{Z}} + \Pi Q'_{\mathbb{Z}}). \tag{4.164}$$

Equations (4.163)-(4.164) can be interpreted as a two-dimensional analog of the attractor equations [76, 155, 35, 124]. In the $3_{\mathbb{C}}$-dimensional case attractor equations define complex structure of the Calabi-Yau threefold in terms of the integral cohomology class, given by the magnetic and electric charges of the associated black hole. In $1_{\mathbb{C}}$-dimensional case equations (4.163) - (4.164) define the complex structure of the Riemann surface $\Sigma_g(\tau)$ in terms of the integral matrices $P_{\mathbb{Z}}, Q_{\mathbb{Z}}, P'_{\mathbb{Z}}$, and $Q'_{\mathbb{Z}}$.

The critical points (4.149) minimize the value of the functional (4.147), viewed as a function on the moduli space of complex structures. Indeed, the second variation of the functional at the critical point is

$$\frac{\delta^2 I(\mathcal{A},\mathcal{A};\phi|\lambda,\mathcal{K})}{\delta\mathcal{K}^2}\Big|_* = -\frac{i\pi}{2}\lambda_* = \frac{\pi}{2}\sqrt{\det\|h^{\mathrm{B}}\|} > 0. \tag{4.165}$$

If we perform functional integration over $\mathcal{DK}$ with the weight $e^{-kI(\mathcal{A},\mathcal{A};\phi|\lambda,\mathcal{K})}$, the main contribution will come from these critical points. Therefore, from the point of view of the corresponding quantum mechanical problem on the moduli space of complex structures, these points are attractive. We will denote a set of these points on the moduli space of genus g Riemann surfaces as Attr_g.

For the particular choice $P_{\mathbb{Z}} = Q'_{\mathbb{Z}} = \mathbb{I}$, and $Q_{\mathbb{Z}} = P'_{\mathbb{Z}} = 0$ the attractor equations (4.163)-(4.164) reduce to (4.152)-(4.153). This allows us to generate all solutions to (4.163)-(4.164) from (4.152)-(4.153) by an appropriate symplectic transformation. Indeed, a compatibility of (4.163) and (4.164), combined with the symmetry requirement $\tau^T = \tau$ imposes certain restrictions on the possible choice of the integer matrices. After some algebra one finds that these restrictions are equivalent to relations (4.78) for the symplectic group, with the identification: $a = Q'^T_{\mathbb{Z}},\ b = P'^T_{\mathbb{Z}},\ c = Q^T_{\mathbb{Z}},\ d = P^T_{\mathbb{Z}}$. Therefore,

$$\begin{pmatrix} Q'^T_{\mathbb{Z}} & P'^T_{\mathbb{Z}} \\ Q^T_{\mathbb{Z}} & P^T_{\mathbb{Z}} \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z}), \tag{4.166}$$

and all solutions to (4.163)-(4.164) for a given Π correspond to the same Riemann surface, with a different choices of the symplectic basis. To summarize, we find that the critical points of the functional (4.147) on the moduli space of complex structures $\mathcal{M}_g$ are given by the intersection of the Jacobian locus $\mathrm{Jac}(\Sigma_g) \subset \mathcal{H}_g$ with a set $\Gamma^{2\mathbb{Z}}_{2g}$ of abelian varieties generated by the even integer $2g$-dimensional lattices:

$$\mathrm{Attr}_g = \mathrm{Jac}(\Sigma_g) \cap \Gamma^{2\mathbb{Z}}_{2g}. \tag{4.167}$$

There is another interesting property of the critical points defined by (4.161) and (4.163): the corresponding Riemann surface $\Sigma_g(\tau)$ admits a non-trivial endomorphism, known as the complex multiplication (CM). The notion of the complex multiplication appears in the study of black hole attractors and rational conformal field theories (see, *e.g.*, [124, 92] for more details and references). In particular, it was shown in [91], that the critical attractor points of the Calabi-Yau holomorpic volume functional (4.14) (which is morally the higher-dimensional analog of the functional

(4.147)) lead to the abelian varieties (associated with the coupling constant matrix) admitting complex multiplication.

In order to illustrate the CM-property of the critical points (4.149), we will use a simple criterion [92]) that says that an abelian variety defined by the period matrix τ admits complex multiplication, if τ obeys a second order matrix equation

$$\tau n \tau + \tau m - n' \tau - m' = 0, \tag{4.168}$$

for some integer $g \times g$ matrices m, n, m', n', with $\mathrm{rank}(n) = g$. It is straightforward to show that any solution to the attractor equations (4.163) -(4.164) also obeys the CM-type equation (4.168). After using (4.157), the equation (4.160) takes the form

$$(B^T - \overline{\tau} A^T)\frac{1}{\mathrm{Im}\Pi}(\overline{B} - \overline{A}\tau) = 4\mathrm{Im}\tau. \tag{4.169}$$

By substituting $\overline{\tau} = \tau - 2i\mathrm{Im}\tau$ into the real part of (4.169), we find

$$\tau \mathrm{Re}(A^T \frac{1}{\mathrm{Im}\Pi}\overline{A})\tau - \tau \mathrm{Re}(A^T \frac{1}{\mathrm{Im}\Pi}\overline{B}) - \mathrm{Re}(B^T \frac{1}{\mathrm{Im}\Pi}\overline{A})\tau + \mathrm{Re}(B^T \frac{1}{\mathrm{Im}\Pi}\overline{B}) = 0, \tag{4.170}$$

where we used (4.161) and the attractor equation (4.163) in the form $A^T \frac{1}{\mathrm{Im}\Pi}\overline{A} = \frac{1}{\mathrm{Im}\tau}$. Let us now recall that $\Lambda = \mathbb{Z}^g \oplus \Pi \mathbb{Z}^g$ is an even integral lattice. This guarantees that the corresponding three-dimensional abelian Chern-Simons theory is well-defined, and the associated two-dimensional conformal field theory is rational. Therefore, for any two vectors $\mathbf{a}, \mathbf{b} \in \Lambda$ a scalar product, defined as

$$(\mathbf{a}, \mathbf{b}) = \mathrm{Re}(\mathbf{a}^I (\mathrm{Im}\Pi)^{-1}_{IJ} \overline{\mathbf{b}}^J) \tag{4.171}$$

is an integer, and the norm of any vector of the lattice Λ is an even number:

$$(\mathbf{a}, \mathbf{b}) \in \mathbb{Z},\ \mathbf{a} \neq \mathbf{b}; \qquad (\mathbf{a}, \mathbf{a}) \in 2\mathbb{Z}. \tag{4.172}$$

We can write the period matrices (4.156) in terms of the lattice vectors, as $A = (\mathbf{a}_1, \dots \mathbf{a}_g)$, and $B = (\mathbf{b}_1, \dots \mathbf{b}_g)$. Then all elements of the matrices

$$\begin{aligned} n &= \mathrm{Re}(A^T \frac{1}{\mathrm{Im}\Pi} \overline{A}), \qquad & m &= -\mathrm{Re}(A^T \frac{1}{\mathrm{Im}\Pi} \overline{B}), \\ n' &= \mathrm{Re}(B^T \frac{1}{\mathrm{Im}\Pi} \overline{A}), \qquad & m' &= -\mathrm{Re}(B^T \frac{1}{\mathrm{Im}\Pi} \overline{B}), \end{aligned} \tag{4.173}$$

according to (4.172) are integral, and thus equation (4.170) is indeed of the CM-type (4.168). This fact should not be surprising. As was proven recently in [42], the complex multiplication on abelian variety is equivalent to the existence of the rational Kähler metric. This is, of course, true in our case, since we consider abelian varieties generated by the even integer lattices. The fact that the associated CFT in this case is rational, fits nicely with the observations of Gukov and Vafa [92].

4.5 Quantization and the Partition Function

In this section, we define a generating function for the dimension of the space of conformal blocks in a family of toroidal $c = 2g$ RCFTs on a genus g Riemann surface. We use Hitchin construction to introduce coupling to two-dimensional gravity. The universal index theorem in the context of the Chern-Simons/CFT correspondence is a computation of the number of conformal blocks via the gauged WZW model. After coupling to two-dimensional gravity it gives, according to the entropic principle, the effective entropy functional on the moduli space of complex structures. The functional is peaked at the attractor points. We will be interested in the fluctuation of the complex geometry around the gravitational instanton solution corresponding to these points. It gives some version of the two-dimensional Kodaira-Spencer theory of gravity.

4.5.1 Generating Function for the Number of Conformal Blocks and Attractors

We learned that the Hitchin extension $I(\mathcal{A},\mathcal{A};\phi|\lambda,\mathcal{K})$ of the abelian gauged WZW model gives rise to effective potential on the moduli space of the complex structures, whose critical points (4.167) correspond to Jacobians of Riemann surfaces admitting complex multiplication. In order to describe all such points we have to sum over all even integer lattices $\{\Lambda(\Pi) = \mathbb{Z}^g \oplus \Pi\mathbb{Z}^g : \Lambda(\Pi) \in \Gamma_{2g}^{2\mathbb{Z}}\}$. This discrete sum is basically a sum over the moduli space[18] of the toroidal rational two-dimensional conformal field theories:

$$Z_{g,k}(\Theta|\lambda,\mathcal{K}) = \sum_{\Pi:\,\Lambda(\Pi)\in\Gamma_{2g}^{2\mathbb{Z}}} e^{i\mathrm{Tr}\Theta\Pi} \int \frac{\mathcal{D}\phi\mathcal{D}\mathcal{A}}{\mathrm{vol(Gauge)}} e^{-kI(\mathcal{A},\mathcal{A};\phi|\lambda,\mathcal{K})}. \tag{4.174}$$

We perform a sum with the weight factor $\exp(i\mathrm{Tr}\Theta\Pi)$, where Θ is an auxiliary symmetric matrix, so that $Z_k(\Theta|\lambda,\mathcal{K})$ can be interpreted as a generating function capturing all the relevant information about the theory. In principle, we can go one step further and sum over the theories at different levels k as well:

$$Z_g(q,\Theta|\lambda,\mathcal{K}) = \sum_{k=1}^{\infty} q^k Z_{g,k}(\Theta|\lambda,\mathcal{K}). \tag{4.175}$$

If we compute (4.175) at any classical value $\mathcal{K}_* : \mathcal{K}_*^2 = -\mathrm{Id}$, the functional $I(\mathcal{A},\mathcal{A};\phi|\lambda,\mathcal{K}_*)$ describes the ordinary gauged WZW model, and therefore (4.175) becomes a gener-

[18]To be more precise, the moduli space of a $2g$-dimensional torus is $\frac{SO(2g,2g)}{SO(2g)\times SO(2g)\times SO(2g,2g,\mathbb{Z})}$. We are interested in the subspace of the complex algebraic tori $\frac{\mathrm{Sp}(2g)}{U(g)\times \mathrm{Sp}(2g,\mathbb{Z})}$ in this moduli space, and moreover, consider only the tori generated by the even integral lattices.

ating function[19] for the number of conformal blocks in $c = 2g$ RCFTs

$$Z_k(q,\Theta|\lambda,\mathcal{K}_*) = \sum_k \sum_{\Pi:\,\Lambda(\Pi)\in\Gamma_{2g}^{2\mathbb{Z}}} q^k e^{i\mathrm{Tr}\Theta\Pi} |\Lambda^*(\Pi)/k\Lambda(\Pi)|^g. \tag{4.176}$$

Let us discuss the quantum aspects of the theory on the moduli space of the complex structures that arises after averaging the generating function (4.176) over the fluctuations of the fields $\mathcal{K}$ and λ, according to

$$Z_{g,k}(\Theta) = \int \frac{\mathcal{D}\mathcal{K}\mathcal{D}\lambda}{\mathrm{vol}(\mathrm{Diff}(\Sigma_g))} Z_{g,k}(\Theta|\lambda,\mathcal{K}). \tag{4.177}$$

The measure for the vector-valued 1-form $\mathcal{K}$ can be defined as follows. Let us first notice that for any 1-form θ on Σ_g

$$\theta \wedge \imath_{\mathcal{K}-\mathrm{tr}\mathcal{K}}\theta = \theta \wedge \imath_{\mathcal{K}}\theta - (\mathrm{tr}\mathcal{K})\,\theta \wedge \imath_{\mathrm{Id}}\theta = \theta \wedge \imath_{\mathcal{K}}\theta, \tag{4.178}$$

since $\theta \wedge \imath_{\mathrm{Id}}\theta = \theta \wedge \theta = 0$. Therefore, $\mathrm{tr}\mathcal{K}$ does not couple to the scalars ϕ in the action (4.147). This is the reason why we can integrate only over the traceless tensor fields $\mathrm{tr}\mathcal{K} = 0$. In this case the measure on the space of the fields $\mathcal{K}$ is induced from the following metric:

$$\|\delta\mathcal{K}\|^2 = \int_{\Sigma_g} d^2x\,(\mathrm{tr}\mathcal{K}^2)^{-\frac{3}{2}}\,\mathrm{tr}(\imath_{\mathcal{K}}\delta\mathcal{K})^2. \tag{4.179}$$

In order to motivate this choice of the metric, we note that on-shell, $\mathcal{K}$ is linearly related to the Riemann metric h on Σ_g: $\mathcal{K}^a{}_b \sim h_{bc}\epsilon^{ca}$. Traceless vector-valued 1-form $\mathcal{K}^a{}_b$ contains 3 local degrees of freedom, the same amount as the symmetric metric

[19]The simplest example of such generating function corresponds to the $U(1)_k$ theory, describing the free boson at k times the self-dual radius. The holomorphic wave-functions of the dual Chern-Simons theory are the level k Jacobi theta-functions. The dimension of the corresponding Hilbert space is k^g. Therefore, on this case $Z_g(q) = \sum_{k=1}^{\infty} q^k k^g = Li_{-g}(q)$.

tensor h_{ab}. However, $\mathcal{K}$ and h scale differently under the conformal transformations. This can be taken care of by introducing a conformal factor σ, such that

$$\mathcal{K}^a_{\ b} = \frac{h_{bc}}{\sqrt{\det h}} e^{\sigma} \epsilon^{ca}. \tag{4.180}$$

Then it is easy to see that the metric (4.179) for the variations of $\mathcal{K}$ that does not involve change of $\mathrm{tr}\mathcal{K}^2$, coincides with the standard metric [147] on the space of Riemann metrics

$$\|\delta h\|^2 = \int_{\Sigma_g} d^2x \sqrt{\det h}\, h^{ac} h^{bd} \delta h_{ab} \delta h_{cd}, \tag{4.181}$$

for the variations of h that does not involve conformal transformations. In order to parameterize general variations, we follow the standard procedure [17], and introduce complex coordinates on Σ_g, in terms of which the metric takes the conformal form $h = h_{z\overline{z}} dz \otimes d\overline{z}$. The group $\mathrm{Diff}(\Sigma_g)$ is generated by the coordinate transformations $z \to z + \varepsilon(z, \overline{z})$. Then the metric (4.179) takes the form

$$\|\delta \mathcal{K}\|^2 = \int_{\Sigma_g} d^2x e^{\sigma} ((\delta\sigma)^2 + \partial\overline{\varepsilon}\overline{\partial}\varepsilon) + \sum_{i,j=1}^{3g-g} \delta m_i (N_2^{-1})^{ij} \delta \overline{m}_j \tag{4.182}$$

where m_i are coordinates on the moduli space of Riemann surfaces $\mathcal{M}_g$, and N_2 is the matrix of scalar products of the quadratic holomorphic differentials on Σ_g. Therefore, the measure in the functional integral (4.177) is given by

$$\mathcal{D}\mathcal{K} = \mathrm{vol}(\mathrm{Diff}(\Sigma_g)) \frac{\det' \Delta_{-1}}{\det N_2} d\sigma dm \tag{4.183}$$

where Δ_j denotes the Laplace-Beltrami operator acting on the space of the holomorphic j-differentials, N_j is the matrix of scalar products of holomorphic j-differentials, and the volume form on the moduli space is $dm = \prod_{i=1}^{3g-g} dm_i \wedge d\overline{m}_i$. We see that σ

plays the role of the Liouville field (the conformal factor of the metric). In particular, we can compute the σ-dependence of the determinant in (4.183) using the standard formula

$$\frac{\det'\Delta_j}{\det N_j \, \det N_{1-j}} = |\det\overline{\partial}_j|^2 \, e^{-\frac{c_j}{12\pi}S_L[\sigma]}, \tag{4.184}$$

where $c_j = 6j^2 - 6j + 1$ and $S_L[\sigma]$ is the Liouville action. However, because of the choice of the parameterizaton (4.180), the conformal field σ enters the Hitchin extension of the gauged WZW model (4.147) in a special way. The relevant terms of the functional (4.147) have the form:

$$\mathcal{G}_{IJ}\pi \int_{\Sigma_g} d^2x e^{\sigma} \partial\phi^I \overline{\partial}\,\overline{\phi}^J + \frac{i\pi}{2}\int_{\Sigma_g} \lambda(e^{\sigma} - 1) \tag{4.185}$$

The additional factor e^{σ} makes this theory at the quantum level very different from Polyakov's bosonic string. Let us recall that the quantum theory (4.177) is defined as an expansion around the attractor point

$$\mathcal{K} = \mathcal{K}_* + \delta\mathcal{K}, \quad \lambda = \lambda_* + \delta\lambda. \tag{4.186}$$

This means that we should expand (4.185) around $\sigma = 0$. If we formally do this expansion, in perturbation series we will encounter terms of the form

$$\sum_{n>0} \frac{1}{n!}\int_{\Sigma_g} d^2x \sigma^n \langle \partial\phi^I \overline{\partial}\,\overline{\phi}^J \rangle. \tag{4.187}$$

These terms are singular, since $\langle \phi(z)\overline{\phi}(w)\rangle \sim \log|z-w|$, and we are taking the limit $z \to w$, $\sigma \to 0$. Therefore, for this theory to make sense, (4.187) has to be regularized in some way.

However, in the classical (weak coupling) limit $k \to \infty$ we can ignore this regularization ambiguity. If we neglect possible contributions from the boundary of the moduli space, in this limit the main contribution to (4.177) comes from the attractor points (4.167):

$$Z_{g,k}(\Theta)|_{k\to\infty} = \sum_{\Pi\in \mathrm{Attr}_g} e^{i\mathrm{Tr}\Theta\Pi}\Big(|\Lambda^*(\Pi)/k\Lambda(\Pi)|^g + \ldots\Big). \tag{4.188}$$

¿From the viewpoint of the entropic principle (1.1), it means that the wave-function (4.1) on the moduli space of the complex structures $\mathcal{M}_g$ is peaked at the attractor points (4.167).

There is one physically natural way to resolve the regularization ambiguity in (4.185). We would like to think about the corresponding theory as of a $1_{\mathbb{C}}$-dimensional analog of the Kodaira-Spencer theory of gravity [22]. In the $3_{\mathbb{C}}$-dimensional case, the target space KS action [22] also suffers from the regularization ambiguities. However, there the topological string B-model provides a natural regularization. Unfortunately, the higher genus topological string amplitudes vanish if the target manifold has dimension different from the critical dimension $\widehat{c} = 3$, so we can not view $1_{\mathbb{C}}$-dimensional analog of KS theory as a topological strings on Σ_g. Instead, we can define it by requiring that a generating function (4.177) should be identified with the corresponding computation in the dimensional reduction of the Kodaira-Spencer theory of gravity [22] from six to two dimensions.

4.5.2 Dimensional Reduction of the Topological M-Theory

Let us discuss the relation between the two-dimensional Hitchin model studied above, and the dimensional reduction of the topological M-theory[20]. At the moment,

[20]We thank C. Vafa and E. Witten for raising the question about the relation between Hitchin functionals in different dimensions.

there is no consistent quantum definition of the topological M-theory [58]. However, many ingredients of the theory can be identified at the classical level. In particular, a seven-dimensional topological action

$$S_7 = \frac{1}{2\pi} \int_{M_7} H \wedge dH, \tag{4.189}$$

which is a $U(1)$ Chern-Simons theory for 3-form H plays an important role in interpreting the topological string partition function as a wave-function (see, *e.g.*, [58, 133, 81, 169] and references therein).

On the other hand, it is well-known [162], that we can get a $1_{\mathbb{C}} + 1$-dimensional abelian Chern-Simons theory from (4.189) via dimensional reduction on the manifold of the form $M_7 = M_4 \times \Sigma_g \times \mathbb{R}$. Using the ansatz $H = \sum \alpha_i A^i$, where α_i are integral harmonic 2-forms on M_4, we obtain:

$$\frac{1}{2\pi} \int\limits_{M_4 \times \Sigma_g \times \mathbb{R}} H \wedge dH \to \frac{K_{ij}}{2\pi} \int\limits_{\Sigma_g \times \mathbb{R}} A^i \wedge dA^j. \tag{4.190}$$

Here K_{ij} is an intersection form for harmonic 2-forms on M_4. If we use the spin manifold, this form is an even integral, and therefore the dual conformal field theory is rational. In this chapter, we studied a special case of such compactifications, with the form K_{ij} defining an abelian variety. In general case, K_{ij} is an integral form, and if $b_+ \neq b_-$, we get lattices of various signatures. It would be interesting to understand how these lattices can be embedded in our framework, given that the relevant abelian (spin) Chern-Simons theories has been recently classified [21].

4.6 Conclusions and Further Directions

In this chapter we studied Hitchin-like functionals in two dimensions. They lead to topological theories of a special kind: the metric is not required for constructing the theory. Instead, it arises dynamically from the topological data, characterized by particular choice of the cohomologies. We considered the cases of non-compact and compact cohomologies. In both cases the theory generates a map between the cohomologies $H^1(\Sigma_g,\mathbb{C})^{\otimes g}$ of genus g Riemann surface Σ_g and moduli space $\mathcal{M}_g$ of the complex structures on Σ_g, in the spirit of the original Hitchin construction [97, 98]. The Hitchin parameterization of the moduli space in terms of the cohomologies has several useful features. The fact that we can use simplicial complexes for the description of cohomologies is a natural source of the modular group appearance. Although explicit calculations may involve a choice of a symplectic basis, the action is modular invariant and therefore provides a laboratory for generating modular invariant objects. Moreover, the symplectic structure on the cohomology space allows one to perform canonical quantization of the moduli space via the Hitchin map.

The geometric picture that arises in this approach is shown on Fig. 4.1. The cohomology space in question is parameterized by the Seigel upper half-space $\mathcal{H}_g$. It can also be viewed as a space of a complex g-dimensional principally polarized abelian varieties. The Hitchin map classically is just the Torelli map between M_g and Jacobian locus $\mathrm{Jac}(\Sigma_g) \in \mathcal{H}_g$. In the spirit of the Kodaira-Spencer theory [22], we can start at some "background" point τ on the moduli space $\mathcal{M}_g$ and study resulting quantum mechanical problem on the Seigel upper half-space. The corresponding wave-function is then peaked at $\Pi = \tau$, and classical trajectories $\Pi \to \Pi'$ on $\mathcal{H}_g$ are obtained from

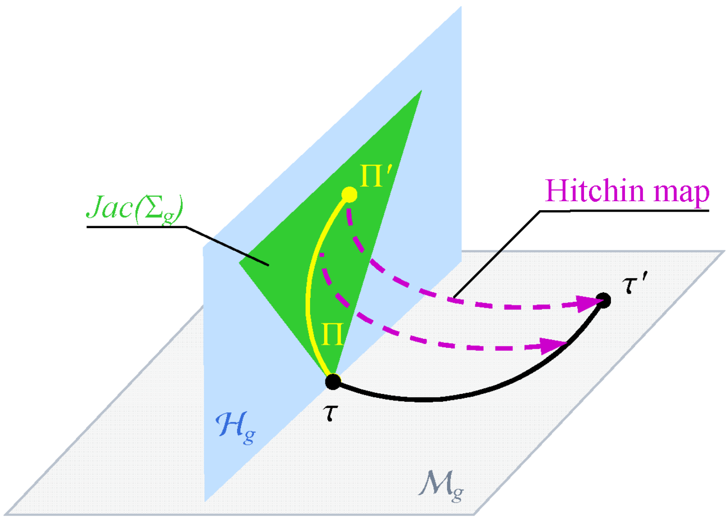


Figure 4.1: Transport on the moduli space and the Hitchin map.

trajectories $\tau \to \tau'$ on $\mathcal{M}_g$ via the Torelli map.

In the dual approach, we start at some point $\Pi \in \mathcal{H}_g$ and study resulting quantum mechanical problem on the moduli space of Riemann surfaces. We find that in this case it is convenient to integrate over the part of the cohomology space given by the complex torus $\mathcal{T}(\Pi) = \mathbb{C}^g/\mathbb{Z}^g \oplus \Pi\mathbb{Z}^g$. Then the effective theory on $\mathcal{M}_g$ can be interpreted as the abelian gauged WZW model coupled to the two-dimensional gravity in a special way. Furthermore, the choice of the classical starting points Π is then restricted to those that correspond to the even integral lattices. The classical solutions of the gauged WZW model correspond to the harmonic maps $\Sigma_g \to \mathcal{T}$. After coupling to the two-dimensional gravity, variation with respect to the complex structure implies that these maps are holomorphic with respect to both complex structures on $\Sigma_g(\tau)$ and $\mathcal{T}(\Pi)$. This is only possible if $\mathcal{T}$ is equivalent to the Jacobian of some Riemann surface, up to the modular transformation. In this special case the wave function in the corresponding quantum mechanical problem on $\mathcal{M}_g$ is peaked at the attractor point (4.167) $\tau = \Pi_*$. Otherwise, the wave function is extremized at the boundary of the moduli space $\partial\mathcal{M}_g$.

The probability/entropy function that we get by squaring the wave function has special value at the attractor point: it is equal to the dimension of the Hilbert space in the associated three-dimensional Chern-Simons theory with the abelian group $\mathcal{T}_*$. Therefore, the Hitchin construction allows us to effectively organize the moduli space of $c = 2g$ RCFTs by introducing canonical index/entropy function that weights different points on the moduli space $\mathcal{M}_g$ according to the number of conformal blocks in corresponding RCFT.

It is widely believed that there is a vast landscape of consistent theories of quantum gravity, that can be realized in string theory. It was recently suggested [160] that this landscape is surrounded by the huge area of consistent looking effective theories, that cannot be completed to a full theory, called the swampland. On the abelian varieties side, an analog of the string landscape is the Jacobian locus in the Siegel upper half-space, and the ”swampland” is a vast area of non-geometric points in Siegel space, which do not correspond to any Riemann surface. By extremizing the Hitchin functional, we land on a special set of points in the Jacobian locus, corresponding to the surfaces admitting complex multiplication. On the string theory side, similar phenomena occur [124] if the complex moduli of the compactification manifolds are fixed by the attractor mechanism [76, 155, 35]. Moreover, in both situations we have an entropy/index weight function assigned to those points on the moduli space. This gives us an interesting analogy between the moduli space of string compactifications and the moduli space of abelian varieties. Very schematically, it is shown on Fig. 4.2. It would be interesting to develop this analogy further.

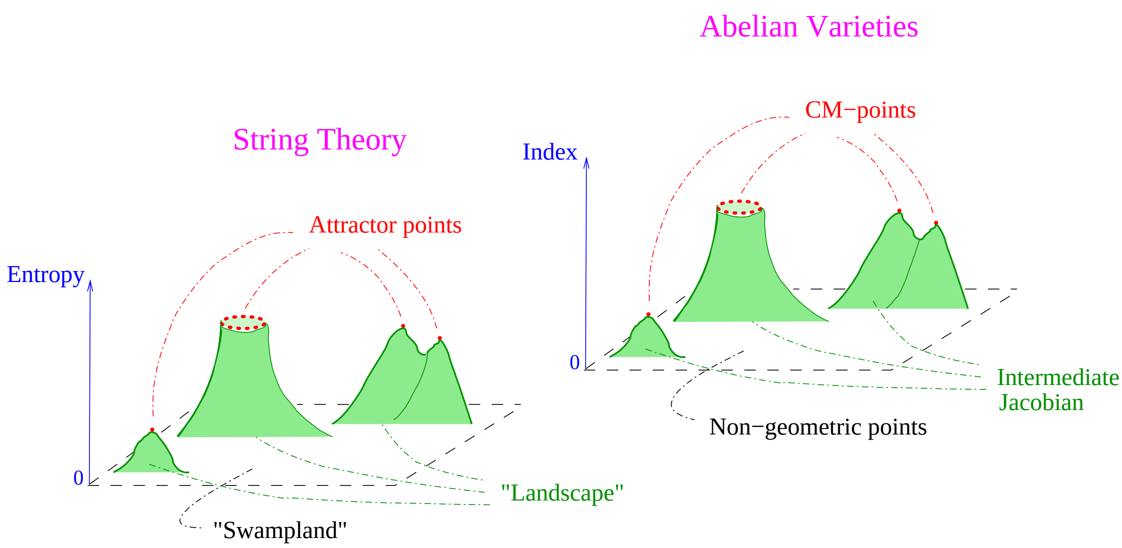


Figure 4.2: A similarity between the moduli spaces of string compactifications and abelian varieties, arising if the framework of Hitchin theory.

In particular, it is worth mentioning that there is a direct analog of the non-geometric locus $\mathcal{H}_g \backslash \mathrm{Jac}(\Sigma_g)$ in Siegel space for a Calabi-Yau threefold M described in terms of the cohomologies $H^3(M, \mathbb{R})$. The Hitchin theorem [97] states that the critical points of the functional (4.10) on a fixed cohomology class $[\rho] \in H^3(M, \mathbb{R})$ define complex structure on a Calabi-Yau three-fold only if there is a *stable* solution $\rho_* :\ \mathrm{tr} K^2(\rho_*) < 0$

everywhere[21]. From the viewpoint of the attractor equations, the boundary of the stability region in $H^3(M, \mathbb{R})$ corresponds to the black hole solutions with classically vanishing entropy. There is no classical solutions outside of the stability region, but it can be probed in quantum theory. Apart form the obvious physical importance, to describe and classify the stability regions in $H^3(M, \mathbb{R})$ is a challenging mathematical question. To the best of our knowledge, the answer to this question is not known even in the simple case of one-parametric Calabi-Yau threefolds. This can be thought of as the Schottky problem for Calabi-Yau threefolds.

In this chapter, we only considered non-degenerate Riemann surfaces and concentrated on the massless degrees of freedom. The next natural step is to incorporate punctures and holes into the story, and study contributions from the boundary of the moduli space. This way, one could control not only the fluctuations of the geometry, but also the change of topology. By concentrating on the local degrees of freedom at the punctures it should be possible to find a connection with the Kodaira-Spencer theory on a local Calabi-Yau geometries, following [1]. Another interesting direction for further study is incorporating supersymmetry and considering more general curved target spaces, for example, non-abelian groups and $\beta\gamma$-systems.

It is likely that the analysis of the Hitchin functionals performed in the chapter can be extended from two to six dimensions. The insight that we get from studying the two-dimensional toy model (4.142) is that the six-dimensional Hitchin functional (4.10) should be viewed as an analog of the gauged WZW model for the seven-dimensional Chern-Simons theory. Then the OSV conjecture [142] will have an interpretation in terms of the corresponding index theorem.

[21] The importance of this condition was stressed to us by C. Vafa.

Chapter 5

Non-supersymmetric Black Holes and Topological Strings

We start with discussion on the relation between the black hole entropy and topological strings proposed in [142]. Define a mixed partition function for a black hole with magnetic charge p^I and electric potential ϕ^I by

$$Z_{\rm BH}(p^I,\phi^I)=\sum_{q_I}\Omega(p^I,q_I)\,e^{-\phi^I q_I}, \tag{5.1}$$

where $\Omega(p^I,q_I)$ represent supersymmetric black hole degeneracies for a given set of charges (p^I,q_I). Then the OSV conjecture [142] reads

$$\Omega(p^I,q_I)=\int d\phi^I e^{q_I\phi^I}|Z_{\rm top}(p^I,\phi^I)|^2. \tag{5.2}$$

As was already mentioned in [142], expression (5.2) needs some additional refinement. In particular, rigorous definition of (5.2) requires taking care of the background dependence of the topological string partition function $Z_{\rm top}$, governed by the holomorphic anomaly [22]. Also, the integration measure, as well as the choice of a suitable integration contour needs to be specified. Some of these issues were investigated in [46, 47, 153, 163, 143], see [53] for a recent discussion of these and other subtleties.

In this chapter we will address an even more fundamental ambiguity in (5.2) that is present already at the semiclassical level (without considering higher genus topological string corrections). The problem is that although the right hand side of (5.2) can be defined for any set of charges (p^I, q_I), it is well known [124] that not for all such (p^I, q_I) a supersymmetric spherically symmetric black hole solution exists. Typically, there is a real codimension one 'discriminant' hypersurface $\mathcal{D}(p^I, q_I) = 0$ in the space of charges, such that supersymmetric black hole solutions exist only when $\mathcal{D}(p^I, q_I) < 0$. Therefore, in this case $\Omega(p^I, q_I)$ on the left hand side of (5.2), representing a suitable index of BPS states of charge (p^I, q_I), is zero.

This phenomenon can be illustrated by several examples. Consider compactification of Type IIB string theory on the diagonal $T^6 = \Sigma_\tau \times \Sigma_\tau \times \Sigma_\tau$ [124, 15], where Σ_τ is the elliptic curve with modular parameter τ, with $D3$-brane wrapping a real 3-cycle on T^6. This can be viewed as part of the Calabi-Yau moduli when we orbifold T^6. In this chapter when we refer to the diagonal T^6 we have in mind the corresponding locus in the moduli of an associated Calabi-Yau 3-fold with $\mathcal{N} = 2$ supersymmetry where part of the homology of the CY 3-cycles is identified with the charges (p^I, q_I). Let the charge configuration be invariant under the permutation symmetry of the three elliptic curves Σ_τ. Note also that the diagonal T^6 model is a good approximation to the generic behavior of Type IIB compactification on a one-modulus Calabi-Yau threefold in the large radius limit. If we label homology of 3-cycles on T^6 according to the mirror IIA D-brane charges as $(u, q, p, v) = (D0, D2, D4, D6)$, the leading contribution to the corresponding black hole degeneracy takes the form

$$\Omega_{\rm susy}(p, q, u, v) \approx \exp\left(\pi\sqrt{-\mathcal{D}(p, q, u, v)}\right), \tag{5.3}$$

where the discriminant is $\mathcal{D}(p, q, u, v) = -(3p^2q^2 + 4p^3u + 4q^3v + 6pquv - u^2v^2)$. It is clear that for some sets of charges this quartic polynomial can become positive (for example, it is always the case for $D0 - D6$ system, where $\mathcal{D}(0, 0, u, v) = u^2v^2$), and (5.3) breaks down. Similar situations occurs in $\mathcal{N} = 2$ truncation of the heterotic string on T^6, the so-called STU model, where $\mathcal{D}$ becomes Cayley's hyperdeterminant [66]) that can also be either positive or negative. Another example of this phenomenon arises in Type IIB compactification on $K3 \times T^2$. This leads to $\mathcal{N} = 4$ supergravity in four dimensions, and corresponding expression for the degeneracy [74, 45, 44]

$$\Omega_{\rm susy}(p^I, q_I) \approx \exp\left(\pi\sqrt{(P\cdot P)(Q\cdot Q) - (P\cdot Q)^2}\right). \tag{5.4}$$

breaks down when $(P \cdot Q)^2 > (P \cdot P)(Q \cdot Q)$.

Thus, the OSV formula (5.2) needs to be modified even at the semiclassical level. One remedy one may think is to sum in (5.1) only over the charges that support BPS states: $Z_{\rm BH}(p^I, \phi^I) = \sum_{q_I : \mathcal{D}(p^I, q_I) \leq 0} \Omega_{\rm susy}(p^I, q_I)\, e^{-\phi^I q_I}$. This, however, will not work because the inverse transform of the topological string partition function would have to automatically give zero when (p^I, q_I) are non-supersymmetric. This however turns out not to be the case, and one gets the naive analytic continuation of the BPS case (leading to imaginary entropy!). Instead, we can use an observation that in many examples studied recently in the literature [157, 109, 109, 75, 18, 20] there exists a non-supersymmetric extremal black hole solution for those sets of charges that do not support a BPS black hole: $\mathcal{D}(p^I, q_I) > 0$. The attractor behavior of a non-supersymmetric extremal black hole solutions [107, 84, 83, 19, 108, 32, 13, 8] is similar to the BPS black hole case, since it is a consequence of extremality rather than supersymmetry [72]. Moreover, in the simplest examples, the macroscopic entropy of a non-supersymmteric extremal black holes is proportional to the square root of the discriminant: $S_{\text{BH}}^{\text{n-susy}} \approx \pi\sqrt{\mathcal{D}}$, so that a general expression for the extremal black holes degeneracy takes the form

$$\Omega_{\text{extrm}}(p^I, q_I) \approx \exp\left(\pi\sqrt{|\mathcal{D}(p^I, q_I)|}\right), \tag{5.5}$$

which is valid both for supersymmetric $\mathcal{D} \leq 0$ and non-supersymmetric $\mathcal{D} > 0$ solutions.

Therefore, it is natural to look for an extension of the OSV formula (5.2) that can be applied *simultaneously* for both BPS and non-BPS extremal black holes and obtain corrections to their entropy due to higher derivative terms in the Lagrangian as a perturbative series in the inverse charge. Recently, several steps in this direction were taken from the supergravity side. A general method (the entropy function formalism) for computing the macroscopic entropy of extremal black holes based on $\mathcal{N} = 2$ supergravity action in the presence of higher-derivative interactions was developed in [151, 152], and applied for studying corrected attractor equations and corresponding entropy formula for non-supersymmetric black holes in [41, 148, 7, 12, 71, 149, 34, 48, 31]. A five-dimensional viewpoint on higher derivative corrections to attractor equations and entropy, based on the c-function extremization, was developed in [116, 39]. Black hole partition function for non-supersymmetric extremal black holes was discussed in [7, 144].

In this chapter we propose a generalization of (5.2) motivated by the topological string considerations as well as the work SS: It was observed in [148] that the higher order corrections to the non-supersymmetric black hole entropy needs higher derivative corrections in the $\mathcal{N} = 2$ theory which are not purely antiself-dual in the 4d sense, because unlike the BPS case, the radii of AdS_2 and S^2 factors of the near horizon geometry are not the same. Thus, more information than F-terms computed by topological strings, which only capture antiself-dual geometries, is needed. Indeed if one considers only the antiself-dual higher derivative corrections to the 4d action, there is already a contradiction with the microscopic count of the non-supersymmetric black hole at one loop [148]. Instead it is natural to look for an extension of topological string which incorporates non-antiself-dual corrections as well. Such a generalization of topological strings, in the context of geometrically engineered gauge theories have been proposed by Nekrasov [139], where the string coupling constant is replaced by a pair of parameters (ϵ_1, ϵ_2) which roughly speaking capture the strength of the graviphoton field strength in the 12 and 34 directions of the 4d non-compact spacetime respectively. In the limit when $\epsilon_1 = -\epsilon_2 = g_{\text{top}}$ one recovers back the ordinary topological string expansion. However when $\epsilon_1 \neq -\epsilon_2$ this refinement of the topological string partition function computes additional terms in the 4d effective theory, as appears to be needed for a correct accounting of the entropy for non-supersymmetric black holes. This includes a term proportional to $\mathcal{R}^2$ which as was found in [149] is needed to get the correct one loop correction which is captured by the refined topological string partition function, but not the standard one.

Motivated by this observation and identifying (ϵ_1, ϵ_2) with physical fluxes in the non-supersymmetric black hole geometry, and motivated by the computations in [148] we propose a conjecture for the partition function of an OSV-like ensemble which includes both BPS and non-supersymmetric extremal black holes. We conjecture

$$\Omega_{\text{extrm}}(p^I, q_I) = \int d\phi^I e^{q_I \phi^I} \sum_{\text{susy,n-susy}} \left| e^{\frac{i\pi}{2}\mathcal{G}(p^I,\phi^I)} \right|^2, \tag{5.6}$$

where $\mathcal{G}(p^I, \phi^I)$ is obtained from the $\mathcal{G}$-function

$$\begin{aligned} \mathcal{G} = & \frac{1}{2}(P^I_\epsilon - X^I)(P^J_\epsilon - X^J)\overline{F}_{IJ}(\overline{X}, \overline{\epsilon}) + (P^I_\epsilon - X^I)F_I(X,\epsilon) + F(X,\epsilon) + \\ & + \frac{1}{2}(\epsilon_1+\epsilon_2)\overline{X}^I F_I(X,\epsilon) - \frac{1}{2}(\epsilon_1+\epsilon_2)(\epsilon_1\partial_{\epsilon_1} - \epsilon_2\partial_{\epsilon_2})F(X,\epsilon) + \mathcal{O}(\epsilon_1+\epsilon_2)^2 \quad , \\ & P^I_\epsilon = -\epsilon_2 p^I + \frac{i}{\pi}\epsilon_1\phi^I \end{aligned} \tag{5.7}$$

by extremizing Im$\mathcal{G}$ with respect to the parameters $\epsilon_{1,2}$ and (extended) Calabi-Yau moduli X^I, and then substituting corresponding solution $\epsilon_{1,2} = \epsilon_{1,2}(p,\phi)$, $X^I = X^I(p,\phi)$ back into $\mathcal{G}$ (5.7). The sum in (5.6) is over all such solutions to the extremum equations $\partial_{\epsilon_{1,2}}\text{Im}\mathcal{G} = \partial_I \text{Im}\mathcal{G} = 0$, one of which ends up being the supersymmetric one given by $X^I(p,\phi) = p^I + \frac{i}{\pi}\phi^I$, reproducing the OSV conjecture for this case. The function $F(X,\epsilon) \equiv F(X^I, \epsilon_1, \epsilon_2)$ in (5.7) denotes Nekrasov's refinement of the topological string free energy[1]. Depending on the choice of the charges (p^I, q_I), integration over ϕ^I near the saddle point picks out supersymmetric or non-supersymmetric black hole solution. In the supersymmetric case it reduces to the OSV formula. In the non-supersymmetric case the corrections have the general structure suggested by [148] (however the exact match cannot be made because [148] only consider higher derivative terms captured by standard topological string corrections).

[1]Supersymmetric solution corresponds to $\epsilon_1 = -\epsilon_2 = 1$, in this case we use the same conventions as in [142], and find $\mathcal{G}_{\text{susy}}(p^I, \phi^I) \equiv F(p^I + \frac{i}{\pi}\phi^I, 256)$. Nekrasov's extension of the topological string is discussed in subsection 5.7.1 below.

The above conjecture is the minimal extension of OSV needed to incorporate non-supersymmetric corrections. It is conceivable that there are further $O(\epsilon_1 + \epsilon_2)^2$ corrections to this conjecture. Such corrections will not ruin the fact that supersymmetric saddle point still reproduces the OSV conjecture.

The rest of the chapter is organized as follows: In section 5.1 we review the attractor equations and entropy formula for supersymmetric and non-supersymmetric extremal black holes of $d = 4$, $\mathcal{N} = 2$ supergravity arising in the leading semiclassical approximation. In section 5.2 we discuss an alternative formulation of the attractor equations which helps us to treat supersymmetric and non-supersymmetric black holes in a unified way, suitable for using in an OSV-like formula. In section 5.3 we formulate the *inverse problem* that allows us to find magnetic and electric charges of the extremal black hole in terms of the values of the moduli in vector multiplets fixed at the horizon. We give a solution to this problem for a general one-modulus Calabi-Yau compactification. In section 5.4 we present explicit solutions of the inverse and direct problems relating the charges and corresponding attractor complex structures for the diagonal T^6 model. In section 5.5 we discuss semiclassical approximation to the generalized OSV formula for extremal black holes. In section 5.6 we review the results [148, 7, 31] for a corrected black hole entropy in $\mathcal{N} = 2$ supergravity with higher-derivative couplings, obtained using the entropy function formalizm. In section 5.7 we observe that matching with the supergravity computations requires replacing the string coupling constant with two variables on the topological string side, and identify these variables as an equivariant parameters in Nekrasov's extension of the topological string. This allows us to formulate a generalization of the OSV entropy formula which is conjectured to be valid asymptotically in the limit of large charges both for the supersymmetric and non-supersymmetric extremal black holes. We conclude in section 5.8 with a discussion of our results and directions for future research.

5.1 The Black Hole Potential and Attractors

Let us review the attractor equations for extremal black holes in $d = 4$, $\mathcal{N} = 2$ supergravity, arising in the context of type IIB compactification on a Calabi-Yau manifold M. We start by choosing a symplectic basis of 3-cycles $(A^I, B_I)_{I=0,\ldots h^{2,1}}$ on M, such that

$$X^I = \int_{A^I} \Omega, \qquad F_I = \partial_I F = \int_{B_I} \Omega, \tag{5.8}$$

where Ω is a holomorphic 3-form and F is the prepotential of the Calabi-Yau manifold. We also choose a basis of 3-forms $(\alpha_I, \beta^I) \in H^3(M, \mathbb{Z})$ dual to (A^I, B_I). The Kähler potential is given by[2]

$$K(X, \overline{X}) = -\log\Big(-i\int_M \Omega \wedge \overline{\Omega}\Big) = -\log i(\overline{X}^I F_I - X^I \overline{F}_I). \tag{5.9}$$

It defines the Kähler metric $g_{i\bar{j}} = \partial_i \overline{\partial}_{\bar{j}} K$. Let us introduce the superpotential

$$\mathcal{W} = \int_M \Omega \wedge H, \tag{5.10}$$

where

$$H = p^I \alpha_I + q_I \beta^I \tag{5.11}$$

[2]We use the Einstein convention and always sum over repeated indices in this chapter.

is the RR 3-form, parameterized by a set of (integral) magnetic and electric charges (p^I, q_I). The central charge is defined by

$$Z = e^{\frac{K}{2}}\mathcal{W}. \tag{5.12}$$

Attractor points are the solutions minimizing the so-called black hole potential [73, 74, 72, 40]

$$V_{\rm BH} = |Z|^2 + |DZ|^2. \tag{5.13}$$

Here D is a fully covariant derivative[3], and $|DZ|^2 = g^{i\bar{j}} D_i Z \overline{D}_{\bar{j}}\overline{Z}$. Notice that for a fixed complex structure on Calabi-Yau the central charge (5.12) is linear in the charges (p^I, q_I), and therefore the black hole potential (5.13) is quadratic in the charges.

We are interested in describing the extremum points of the potential (5.13). These points correspond to the solutions of the following equations [72]

$$\begin{aligned} \partial_i V_{\rm BH} &= 2\overline{Z} D_i Z + g^{k\bar{j}}(D_i D_k Z)\overline{D}_{\bar{j}}\overline{Z} = 0, \\ \overline{\partial}_{\bar{i}} V_{\rm BH} &= 2Z\overline{D}_{\bar{i}}\overline{Z} + g^{j\bar{k}}(\overline{D}_{\bar{i}}\overline{D}_{\bar{k}}\overline{Z}) D_j Z = 0. \end{aligned} \tag{5.14}$$

There are two types of the solutions, which can be identified as follows. From the second equation in (5.14) we find, assuming $Z \neq 0$

$$\overline{D}_{\bar{j}}\overline{Z} = -\frac{g^{l\bar{k}}(\overline{D}_{\bar{j}}\overline{D}_{\bar{k}}\overline{Z})}{2Z} D_l Z. \tag{5.15}$$

By substituting this into the first equation in (5.14), we obtain[4]

$$\mathrm{M}_i{}^j D_j Z = 0, \tag{5.16}$$

[3] On the objects of Kähler weight w it acts as $D = \partial + w\partial K + \Gamma$, where Γ is the Levi-Civita connection of the Kähler metric. For example, $DZ = \partial Z + \frac{1}{2} Z \partial K$.

[4] Similar expression was derived in [19], see eq. (3.5). However, the matrix M_{ij} used in [19] is not well defined when $D_i Z = 0$, which makes it difficult to analyze supersymmetric and non-supersymmetric solutions at the same time.

where

$$\mathrm{M}_i{}^j = 4|Z|^2\delta_i{}^j - (D_iD_kZ)g^{k\overline{m}}(\overline{D}_{\overline{m}}\overline{D}_{\overline{n}}\overline{Z})g^{\overline{n}j} \tag{5.17}$$

Now it is clear that there are two types of solutions to (5.16):

$$\begin{aligned} \mathrm{susy}: \qquad & \mathrm{detM} \neq 0, \quad D_iZ = 0 \\ \mathrm{non-susy}: \qquad & \mathrm{detM} = 0, \quad D_iZ = v_i, \end{aligned} \tag{5.18}$$

where v_i are the null-vectors: $\mathrm{M}_i{}^j v_j = 0$ of the matrix (5.17).

Solutions to the extremum equations (5.14) minimize the black hole potential (5.13), if the Hessian

$$Hess(V_{\mathrm{BH}}) = \begin{pmatrix} \partial_i\partial_j V_{\mathrm{BH}} & \partial_i\overline{\partial}_{\overline{j}} V_{\mathrm{BH}} \\ \overline{\partial}_{\overline{i}}\partial_j V_{\mathrm{BH}} & \overline{\partial}_{\overline{i}}\overline{\partial}_{\overline{j}} V_{\mathrm{BH}} \end{pmatrix}, \tag{5.19}$$

computed at the extremal point, is positive definite: $Hess(V_{\mathrm{BH}})|_{\partial V_{\mathrm{BH}}=0} > 0$. We will refer to such solutions as attractor points. According to the classification (5.18), these attractors can be supersymmetric or non-supersymmetric. It is easy to show that all supersymmetric solutions (5.18) minimize the black hole potential. This is, however, not true in general for the non-supersymmetric solutions, see e.g. [157, 20, 19] for some examples.

The black hole potential (5.13) is related to the Bekenstein-Hawking entropy of the corresponding black hole in a simple way. In the classical geometry approximation (at the string tree level) the entropy is just π times the value of the potential (5.13) at the attractor point

$$S_{\mathrm{BH}} = \pi V_{\mathrm{BH}}|_{\partial V_{\mathrm{BH}}=0}. \tag{5.20}$$

After appropriate modification of the black hole potential this formula gives corrections to the classical Bekenstein-Hawking entropy in the presence of higher derivative terms. This can be effectively realized using the entropy function formalism [151, 152].

5.2 An Alternative Form of the Attractor Equations

In this section we discuss an alternative form of the attractor equations describing extremal black holes in $d = 4, \mathcal{N} = 2$ supergravity coupled to n_V vector multiplets in the absence of higher derivative terms. We describe two versions of attractor equations, one involving inhomogeneous and another involving homogeneous coordinates on Calabi-Yau moduli space. A natural generalization of these equations in the presence of higher derivative corrections will be introduced later in section 5.7.

It is convenient to start with the following representation of the black hole potential [72]

$$V_{\rm BH} = -\frac{1}{2}(q_I - \mathcal{N}_{IJ}p^J)\Big(\frac{1}{{\rm Im}\mathcal{N}}\Big)^{IJ}(q^J - \overline{\mathcal{N}}^{JK}p^K), \tag{5.21}$$

where

$$\mathcal{N}_{IJ} = \overline{F}_{IJ} + 2i\frac{{\rm Im}(F_{IK})X^K{\rm Im}(F_{JL})X^L}{{\rm Im}(F_{MN})X^MX^N}, \qquad F_{IJ} = \frac{\partial^2 F}{\partial X^I\partial X^J}. \tag{5.22}$$

Notice that $\mathcal{N}_{IJ}$ is $(n_V+1)\times(n_V+1)$ symmetric complex matrix, and ${\rm Im}\mathcal{N}_{IJ}$ is a negative definite matrix, as opposed to ${\rm Im}F_{IJ}$, which is of signature $(1, n_V)$. This is clear from the following identity [40]

$$-\frac{1}{2}\Big(\frac{1}{{\rm Im}\mathcal{N}}\Big)^{IJ} = e^K(X^I\overline{X}^J + g^{i\bar{j}}D_iX^J\overline{D}_{\bar{j}}\overline{X}^J). \tag{5.23}$$

One can use (5.23) and the defining relation [40])

$$F_I = \mathcal{N}_{IJ} X^J \tag{5.24}$$

to bring (5.21) into the form (5.13). Indeed, since

$$(q_I - \mathcal{N}_{JI} p^J) X^I = q_I X^I - p^I F_I = \mathcal{W}, \tag{5.25}$$

the black hole potential (5.21) takes the form

$$V_{\mathrm{BH}} = e^K (\mathcal{W}\overline{\mathcal{W}} + g^{i\bar{j}} D_i \mathcal{W} \overline{D}_{\bar{j}} \overline{\mathcal{W}}), \tag{5.26}$$

which is equivalent to (5.13).

5.2.1 Attractor equations and inhomogeneous variables

Let us introduce an auxiliary field P^I that later will be identified with the complexified magnetic charge p^I, and consider a modified black hole potential

$$V_{\mathrm{BH}} = \frac{1}{2} P^I \mathrm{Im}(\mathcal{N}_{IJ}) \overline{P}^J - \frac{i}{2} P^I (q_I - \mathcal{N}_{IJ} p^J) + \frac{i}{2} \overline{P}^I (q_I - \overline{\mathcal{N}}_{JK} p^K), \tag{5.27}$$

where P^I serves as a Lagrange multiplier. We want to describe the extrema of V_{BH}. Variation of (5.27) with respect to $\overline{P}^I$ gives

$$P^I = -\frac{i}{\mathrm{Im}\mathcal{N}_{IJ}} (q_J - \overline{\mathcal{N}}_{JK} p^K). \tag{5.28}$$

By plugging this expression form P^I back to (5.27) we obtain the original black hole potential (5.21). It is straightforward to solve equations (5.28) in terms of the charges:

$$\boxed{\begin{aligned} p^I &= \mathrm{Re}(P^I) \\ q_I &= \mathrm{Re}(\mathcal{N}_{IJ} P^J) \end{aligned}} \tag{5.29}$$

Variation of (5.27) with respect to the Calabi-Yau moduli $\partial_i V_{\rm BH} = 0$ gives

$$P^I \overline{P}^J \partial_i {\rm Im}\mathcal{N}_{IJ} + i(P^I \partial_i \mathcal{N}_{IJ} - \overline{P}^J \partial_i \overline{\mathcal{N}}_{IJ}) p^J = 0. \tag{5.30}$$

After using the solution (5.29), we obtain

$$P^I \partial_i \mathcal{N}_{IJ} P^J - \overline{P}^I \partial_i \overline{\mathcal{N}}_{IJ} \overline{P}^J = 0. \tag{5.31}$$

This set of the extremum equations can also be written in a compact form as follows

$$\boxed{\partial_i {\rm Im}(P^I \mathcal{N}_{IJ} P^J) = 0.} \tag{5.32}$$

For a fixed set of charges (p^I, q_I), solutions to the combined system of equations (5.29) and (5.32) which minimize the modified potential (5.27) correspond to the extremal black holes.

Among these, there is always a special solution of the form

$$P^I = C X^I, \tag{5.33}$$

where C is the complex constant. Indeed, in this case extremum equations (5.32) read

$$C^2 X^I X^J \partial_i \mathcal{N}_{IJ} - \overline{C}^2 \overline{X}^I \overline{X}^J \partial_i \overline{\mathcal{N}}_{IJ} = 0. \tag{5.34}$$

The second term in (5.34) vanishes since $\overline{X}^I \partial_i \overline{\mathcal{N}}_{IJ} = \partial_i (\overline{\mathcal{N}}_{IJ} \overline{X}^J) = \partial_i \overline{F}_I = 0$ according to (5.24). The first term in (5.34) vanishes because of the special geometry relation

$$0 = \int_M \Omega \wedge \partial_i \Omega = X^I \partial_i F_I - F_I \partial_i X^I = X^I X^J \partial_i \mathcal{N}_{IJ}. \tag{5.35}$$

The solution (5.55) describes supersymmetric attractors [76, 155, 73], since (5.29) gives in this case the well-known equations

$$\begin{cases} p^I = \mathrm{Re}(CX^I) \\ q_I = \mathrm{Re}(CF_I). \end{cases} \tag{5.36}$$

5.2.2 Attractor equations and homogeneous variables

Consider the following potential:

$$V_{\mathrm{BH}} = q_I \mathrm{Im} P^I + \mathrm{Im}(F_{IJ})\mathrm{Re}((P^I - X^I)(P^J - X^J)) - \frac{1}{2}\mathrm{Im}(F_{IJ}P^I P^J). \tag{5.37}$$

We will keep P^I fixed (in particular, $\mathrm{Re}P^I = p^I$) and vary X^I. In order to get rid of the scaling of X^I let us introduce a new variable T by

$$X^I = \widehat{X}^I T, \tag{5.38}$$

and integrate out T as follows:

$$e^{\widehat{V}_{\mathrm{BH}}} \approx \int dT e^{V_{\mathrm{BH}}}. \tag{5.39}$$

The potential (5.37) is quadratic in T

$$V_{\mathrm{BH}} = q_I \mathrm{Im} P^I + \mathrm{Im}(F_{IJ})\mathrm{Re}(P^I P^J + \widehat{X}^I \widehat{X}^J T^2 - 2\widehat{X}^I P^J T) - \frac{1}{2}\mathrm{Im}(F_{IJ}P^I P^J), \tag{5.40}$$

since F_{IJ} has zero weight under the rescaling (5.38). Variation with respect to T gives:

$$T = \frac{\widehat{X}^I \mathrm{Im}(F_{IJ}) P^J}{\widehat{X}^I \mathrm{Im}(F_{IJ}) \widehat{X}^J} \tag{5.41}$$

Therefore, the semiclassical approximation to (5.39) gives

$$\widehat{V}_{\mathrm{BH}} = q_I \mathrm{Im} P^I + \frac{i}{4} P^I \mathcal{N}_{IJ} P^J - \frac{i}{4} \overline{P}^I \overline{\mathcal{N}}_{IJ} \overline{P}^J, \tag{5.42}$$

where

$$\mathcal{N}_{IJ} = \overline{F}_{IJ} + 2i\frac{\mathrm{Im}(F_{IK})\widehat{X}^K\mathrm{Im}(F_{JL})\widehat{X}^L}{\widehat{X}^K\mathrm{Im}(F_{KL})\widehat{X}^L}. \tag{5.43}$$

The expression (5.42) should be compared to the modified black hole potential (5.27), which reduces to (5.42) if we use $\mathrm{Re}P^I = p^I$.

The choice of the potential (5.37) can be motivated by looking at the $\mathcal{N} = 2$ supergravity action [51]. At tree level, the coupling of the vector fields can be described as

$$\begin{aligned} 8\pi S_{\mathrm{vec}}^{\mathrm{tree}} = \int d^4x \Big(\frac{i}{4} F_{IJ} \mathcal{F}^{-I}_{\mu\nu} \mathcal{F}^{-J\mu\nu} + \frac{1}{4}\,\mathrm{Im}(F_{IJ})\overline{X}^J \mathcal{F}^{-I}_{\mu\nu} T^{-\mu\nu} - \\ -\frac{1}{32}\mathrm{Im}(F_{IJ})\overline{X}^I\overline{X}^J T^{-}_{\mu\nu}T^{-\mu\nu} + h.c.\Big). \end{aligned} \tag{5.44}$$

Then $V_{\mathrm{BH}} - q_I \mathrm{Im}P^I$ in (5.40) can be interpreted as a zero-mode reduction of (5.44), with the following identification:

$$\begin{aligned} \mathcal{F}^{-I}_{\mu\nu} &\to i\overline{P}^I \\ X^I &\to \widehat{X}^I \\ T^{-}_{\mu\nu} &\to 4i\overline{T} \\ \int d^4x &\to 1. \end{aligned} \tag{5.45}$$

Let us now discuss the attractor equations that describe the minima of the modified black hole potential (5.37). We can derive them in two equivalent ways. First, we can vary (5.42) with respect to the Calabi-Yau moduli, which gives (5.32). Or, second, we can vary the potential (5.37) with respect to the homogeneous coordinates X^I *before* we integrate out the overall scale T. This gives $\partial_I V_{\mathrm{BH}} = 0$ and we obtain the following attractor equations:

$$-\tfrac{i}{2}C_{IJK}\mathrm{Re}((P^J - X^J)(P^K - X^K)) - \mathrm{Im}(F_{IK})(P^K - X^K) + \tfrac{i}{4}C_{IJK}P^J P^K = 0, \tag{5.46}$$

where

$$C_{IJK} = \partial_I F_{JK} = \partial_I \partial_J \partial_K F. \tag{5.47}$$

Using the identity

$$C_{IJK}X^K = 0, \tag{5.48}$$

which follows from the homogeneity relation $X^I F_I = 2F$, we can write (5.46) as

$$\boxed{C_{IJK}(\overline{P}^J - \overline{X}^J)(\overline{P}^K - \overline{X}^K) = 4i\mathrm{Im}(F_{IJ})(P^J - X^J)} \tag{5.49}$$

It is clear that $X^I = P^I$ is the solution of (5.46). If we identify $T \to C,\ X^I \to \widehat{X}^I$, we obtain $P^I = C\widehat{X}^I$, which is the supersymmetric solution (5.55),(5.36). Moreover, if we contract (5.49) with X^I and use (5.48), we get

$$\mathrm{Im}(F_{IJ})X^I(P^J - X^J) = 0. \tag{5.50}$$

In the next section will use this relation to find all other solutions $P^I(X)$ of the attractor equations (5.49) in the one-modulus Calabi-Yau case.

5.3 The Inverse Problem

For a given set of charges (p^I, q_I) solutions to the system (5.14) define the complex structure on M. However, since these equations are highly non-linear, it is hard to write down solutions explicitly for a general Calabi-Yau manifold. On the other hand, since the black hole potential (5.13) is quadratic in charges[5] (p^I, q_I), we can try to solve the inverse problem: For a given point t^i on the Calabi-Yau moduli space, find corresponding set of the charges (p^I, q_I) that satisfy (5.14). Similar techniques were used in [96] to solve the inverse problem for metastable non-supersymmetric backgrounds in the context of flux compactifications.

5.3.1 Inverse problem and inhomogeneous variables

Strictly speaking, the physical charges (p^I, q_I) are quantized, but in semiclassical approximation in the limit of large charges we can ignore this integrality problem and treat the charges as continuous coordinates. Another ambiguity in defining the inverse problem is related to the fact that all sets of charges (p^I, q_I) connected by an $Sp(2n_V+2, \mathbb{Z})$ transformations give the same point on the moduli space, since the black hole potential (5.13) and hence the extremum equations (5.14) are symplectically invariant. Therefore, we need to choose some canonical symplectic basis in $H^3(M, \mathbb{Z})$ and keep it fixed. However, even including that, the inverse problem is not well-defined, since the extremization of (5.13) gives only $2n_V$ real equations (5.14) for $2n_V + 2$ real variables (p^I, q_I). In order to fix this ambiguity, we suggest to look only at the critical points where the superpotential (5.10) takes some particular value:

$$\mathcal{W} = \omega, \tag{5.51}$$

[5]This is clear from looking at the alternative representation (5.21) of the black hole potential.

where ω is a new complex parameter. This can be viewed as a convenient gauge fixing. Therefore, we are interested in solving the system of equations

$$\partial_i V_{\rm BH} = \overline{\partial}_{\bar{i}} V_{\rm BH} = 0, \qquad W = \omega. \tag{5.52}$$

at some particular point t^i on the Calabi-Yau moduli space. Then solution of this inverse problem gives a (multivalued) map: $(t^i, \omega) \to (p^I, q_I)$.

Since $\int_M \Omega \wedge H = q_I X^I - p^I F_I$, the equation (5.51) can be written as

$$X^I (q_I - \mathcal{N}_{IJ} p^J) = \omega. \tag{5.53}$$

If we then use (5.29), this gives $X^I \mathrm{Im}(\mathcal{N}_{IJ}) \overline{P}^J = i\omega$. Therefore, the solution of the inverse problem is given by the following system of equations:

$$\begin{aligned} p^I &= \mathrm{Re}(P^I) & \qquad \partial_i \mathrm{Im}(P^I \mathcal{N}_{IJ} P^J) &= 0 \\ q_I &= \mathrm{Re}(\mathcal{N}_{IJ} P^J) & \qquad X^I \mathrm{Im}(\mathcal{N}_{IJ}) \overline{P}^J &= i\omega \end{aligned} \tag{5.54}$$

In other words, fixing Calabi-Yau moduli and the gauge (5.51) allows one to solve for P^I from the two equations on the right of (5.54). Then the charges are given by the two equations on the left of (5.54).

Among the solutions to (5.54), there always is a supersymmetric solution (5.55), that can be written as

$$P^I = 2ie^K \overline{\omega} X^I, \tag{5.55}$$

where we used $K = -\log\big(-2X \cdot \mathrm{Im}\mathcal{N} \cdot \overline{X}\big)$ to fix the constant C as

$$C = 2i\overline{\omega} e^K = 2i(q_I \overline{X}^I - p^I \overline{F}_I) e^K = 2i\overline{Z} e^{\frac{K}{2}}. \tag{5.56}$$

An example of the explicit solution of the inverse problem in the diagonal T^6 model is presented in subsection 5.4.1.

5.3.2 Inverse problem and homogeneous variables: one-modulus Calabi-Yau case

We can think of the homogeneous variables X^I as parameterizing extended space $\widetilde{\mathcal{M}}$ of the complex structures on a Calabi-Yau threefold M. This space can also be viewed as a total space $\widetilde{\mathcal{M}}$ of the line bundle $\mathcal{L} \to \mathcal{M}$ of the holomorphic 3-forms $H^{3,0}(M,\mathbb{C})$ over the Calabi-Yau moduli space (to be precise, the Teichmüller space) $\mathcal{M}$. Let us comment on the dimension of the space of solutions to the system (5.49). For a fixed extended Calabi-Yau moduli, this is a set of $n_V + 1$ complex quadratic equations for $n_V + 1$ complex variables P^I. Therefore, this system can have at most 2^{n_V+1} solutions. One of them describes supersymmetric black hole and thus there are at most $2^{n_V+1} - 1$ non-supersymmetric solutions.

Let us discuss the inverse problem for a one-modulus Calabi-Yau case, when

$$F = (X^0)^2 f(\tau), \qquad \tau = \frac{X^1}{X^0}. \tag{5.57}$$

The homogeneity relation gives $F_0 = 2X^0 f - X^1 f'$, where $f' \equiv \partial_\tau f$, and we obtain the following matrix of second derivatives

$$F_{IJ} = \begin{pmatrix} 2f - 2\tau f' + \tau^2 f'' & f' - \tau f'' \\ f' - \tau f'' & f'' \end{pmatrix}. \tag{5.58}$$

an the matrix of third derivatives

$$C_{0IJ} = -\tau C_{1IJ} = \frac{1}{X^0} \begin{pmatrix} -\tau^3 f''' & \tau^2 f''' \\ \tau^2 f''' & -\tau f''' \end{pmatrix} \tag{5.59}$$

To simplify expressions below, let us introduce the notation

$$y^I = P^I - X^I. \tag{5.60}$$

Then the attractor equations (5.46) read

$$\begin{cases} C_{0JK}\overline{y}^J\overline{y}^K = 4i\mathrm{Im}(F_{0J})y^J \\ C_{1JK}\overline{y}^J\overline{y}^K = 4i\mathrm{Im}(F_{1J})y^J. \end{cases} \tag{5.61}$$

Using the relation (5.59), we find from (5.133)

$$\mathrm{Im}(F_{0I})y^I = -\tau\mathrm{Im}(F_{1I})y^I, \tag{5.62}$$

which is equivalent to (5.50). To shorten the notations, let us define

$$X_I \equiv X^J\mathrm{Im}F_{JI}. \tag{5.63}$$

For example, $X_0 \equiv X^0\mathrm{Im}F_{00} + X^1\mathrm{Im}F_{10}$. Then we find from (5.62)

$$y^1 = -\frac{X_0}{X_1}\,y^0. \tag{5.64}$$

If we plug this back into (5.133), we obtain

$$(\overline{y}^0)^2 = \mathcal{Y}y^0, \tag{5.65}$$

where

$$\mathcal{Y} = -4iX_1\frac{(X^0)^4\det\|\mathrm{Im}F_{IJ}\|}{f'''(X^IX_I)^2} \tag{5.66}$$

For future reference, let us write down an explicit expression for the ingredients entering (5.66), in terms of the holomorphic function f defining the prepotential (5.57):

$$\begin{aligned} &X_1 = X^0(\mathrm{Im}f' - \mathrm{Im}(\tau)\overline{f''}) \\ &X^IX_I = 2(X^0)^2(\mathrm{Im}f - \mathrm{Im}(\tau)\overline{f'} - i(\mathrm{Im}\tau)^2\overline{f''}) \\ &\det\|\mathrm{Im}F_{IJ}\| = 2\mathrm{Im}(f)\mathrm{Im}(f'') - (\mathrm{Im}f')^2 + 2\mathrm{Im}(\tau)\mathrm{Im}(f'\overline{f''}) - (\mathrm{Im}\tau)^2|f''|^2. \end{aligned} \tag{5.67}$$

In order to solve (5.65), we take the square of the complex conjugate equation and then use (5.65). This gives

$$(y^0)^4 = \overline{\mathcal{Y}}^2 \mathcal{Y} y^0. \tag{5.68}$$

Therefore, in terms of the original variables (5.60) we find the following four solutions:

$$\begin{cases} P^0_{(0)} = X^0 \\ P^1_{(0)} = X^1, \end{cases} \tag{5.69}$$

and

$$\begin{cases} P^0_{(k)} = X^0 + (\overline{\mathcal{Y}}^2 \mathcal{Y})^{1/3} e^{2\pi i k/3} \\ P^1_{(k)} = X^1 - \frac{X_0}{X_1} (\overline{\mathcal{Y}}^2 \mathcal{Y})^{1/3} e^{2\pi i k/3}, \qquad k = 1, 2, 3. \end{cases} \tag{5.70}$$

where the first solution corresponds to a supersymmetric black hole and the other three are non-supersymmetric. Corresponding black hole charges are given by

$$\begin{cases} p^I = \mathrm{Re} P^I \\ q_I = \mathrm{Re}(\mathcal{N}_{IJ} P^J). \end{cases} \tag{5.71}$$

5.4 The Diagonal Torus Example

Consider the case [124] when $M = T^6$ is the so-called diagonal torus:

$$M = \Sigma_\tau \times \Sigma_\tau \times \Sigma_\tau, \tag{5.72}$$

where $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$ is the elliptic curve with modular parameter τ. Let us introduce complex coordinates $dz^i = dx^i + \tau dy^i, i = 1, 2, 3$ on each Σ_τ. As in [15] can label the relevant 3-cycles of M according to their mirror branes in IIA picture:

$$\begin{aligned} D0 &\to & -dy^1dy^2dy^3 \\ D2 &\to & dy^1dy^2dx^3+dy^1dx^2dy^3+dx^1dy^2dy^3 \\ D4 &\to & dx^1dx^2dy^3+dx^1dy^2dx^3+dy^1dx^2dx^3 \\ D6 &\to & -dx^1dx^2dx^3 \end{aligned} \tag{5.73}$$

The intersection matrix of these 3-cycles is

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{5.74}$$

We denote the brane charge vector as $(D0, D2, D4, D6) = (u, q, p, v)$. Then

$$\mathcal{W} = u + 3q\tau - 3p\tau^2 - v\tau^3. \tag{5.75}$$

The black hole potential is

$$V_{\rm BH} = e^K(|\mathcal{W}|^2 + g^{\tau\bar{\tau}}|\partial\mathcal{W} + \mathcal{W}\partial K|^2) \tag{5.76}$$

where

$$K \sim \log(\text{Im}\tau)^3, \qquad g^{\tau\bar{\tau}} = \frac{3}{4(\text{Im}\tau)^2}. \tag{5.77}$$

Therefore, we have

$$V_{\rm BH} = \tfrac{8}{(\text{Im}\tau)^3}\Big(|u+3q\tau-3p\tau^2-v\tau^3|^2 + 3|2i\text{Im}\tau(q-2p\tau-v\tau^2) - u - 3q\tau + 3p\tau^2 + v\tau^3|^2\Big). \tag{5.78}$$

5.4.1 Solution of the inverse problem

Let us decompose τ into the real and imaginary parts

$$\tau = \tau_1 + i\tau_2, \tag{5.79}$$

and introduce new variables α, β, γ that are real linear combination of the charges

$$\begin{aligned}
\alpha &= \mathcal{W}|_{\tau_2=0} = u + 3q\tau_1 - 3p\tau_1^2 - v\tau_1^3, \\
\beta &= \tfrac{1}{3}\tfrac{\partial \mathcal{W}}{\partial \tau}\Big|_{\tau_2=0} = q - 2p\tau_1 - v\tau_1^2, \\
\gamma &= -\tfrac{1}{6}\tfrac{\partial^2 \mathcal{W}}{\partial \tau^2}\Big|_{\tau_2=0} = p + v\tau_1.
\end{aligned} \tag{5.80}$$

Using (5.108), we can rewrite the superpotential (5.75) as

$$\mathcal{W} = \alpha + 3i\beta\tau_2 + 3\gamma\tau_2^2 + iv\tau_2^3. \tag{5.81}$$

Then (5.51) gives

$$\alpha + 3\gamma\tau_2^2 = \omega_1, \tag{5.82}$$

$$3\beta\tau_2 + v\tau_2^3 = \omega_2, \tag{5.83}$$

where $\omega = \omega_1 + i\omega_2$. The black hole potential (5.13) in new variables is given by

$$V_{\mathrm{BH}} = \frac{32}{\tau_2^3}(\alpha^2 + 3\beta^2\tau_2^2 + 3\gamma^2\tau_2^4 + v^2\tau_2^6) \tag{5.84}$$

The extremum equations $\frac{\partial V_{\mathrm{BH}}}{\partial \tau_1} = \frac{\partial V_{\mathrm{BH}}}{\partial \tau_2} = 0$ take the form:

$$\alpha\beta - 2\beta\gamma\tau_2^2 + v\gamma\tau_2^4 = 0, \tag{5.85}$$

$$-\alpha^2 - \beta^2\tau_2^2 + \gamma^2\tau_2^4 + v^2\tau_2^6 = 0, \tag{5.86}$$

If we express α and v in terms of β and γ using (5.82)

$$\alpha = \omega_1 - 3\gamma\tau_2^2, \qquad v = \frac{\omega_2 - 3\beta\tau_2}{\tau_2^3}, \tag{5.87}$$

assuming $\tau_2 \neq 0$, the first equation in (5.85) gives

$$\beta = \frac{\gamma\omega_2\tau_2}{8\tau_2^2\gamma - \omega_1}. \tag{5.88}$$

Here we also assumed that $8\tau_2^2\gamma \neq \omega_1$. We will discuss this special case later. The second equation in (5.85) then takes the form

$$(4\tau_2^2\gamma - \omega_1)(128\tau_2^6\gamma^3 - 96\tau_2^4\gamma^2\omega_1 + 18\tau_2^2\gamma\omega_1^2 - 6\tau_2^2\gamma\omega_2^2 + \omega_1\omega_2^2 - \omega_1^3) = 0. \tag{5.89}$$

We immediately see that $\gamma = \frac{\omega_1}{4\tau_2^2}$, and therefore

$$\alpha = \frac{\omega_1}{4}, \qquad \beta = \frac{\omega_2}{4\tau_2}, \qquad \gamma = \frac{\omega_1}{4\tau_2^2}, \qquad \upsilon = \frac{\omega_2}{4\tau_2^3}, \tag{5.90}$$

gives a solution to (5.85). In fact, it describes a supersymmetric branch of the extremum equations (5.14). The cubic equation for γ in (5.89) has three non-susy solutions that can be described by the formula:

$$\gamma = \frac{2\mathrm{Re}(\omega) + |\omega|(|\omega|/\omega)^{1/3} + |\omega|(|\omega|/\omega)^{-1/3}}{8\tau_2^2}, \tag{5.91}$$

where one can choose any of three cubic root branches. It is obvious that all solutions (5.91) are real. Correspondingly, in this case

$$\begin{aligned}
\alpha &= \tfrac{1}{4}\mathrm{Re}(\omega) - \tfrac{3}{8}|\omega|(|\omega|/\omega)^{1/3} - \tfrac{3}{8}|\omega|(|\omega|/\omega)^{-1/3}, \\
\beta &= \frac{\mathrm{Im}(\omega)}{8\tau_2} \cdot \frac{2\mathrm{Re}(\omega)+|\omega|\big(|\omega|/\omega\big)^{1/3}+|\omega|\big(|\omega|/\omega\big)^{-1/3}}{\mathrm{Re}(\omega)+|\omega|\big(|\omega|/\omega\big)^{1/3}+|\omega|\big(|\omega|/\omega\big)^{-1/3}}, \\
\gamma &= \frac{2\mathrm{Re}(\omega)+|\omega|\big(|\omega|/\omega\big)^{1/3}+|\omega|\big(|\omega|/\omega\big)^{-1/3}}{8\tau_2^2}, \\
\upsilon &= \frac{\mathrm{Im}(\omega)}{8\tau_2^3} \cdot \frac{2\mathrm{Re}(\omega)+5|\omega|\big(|\omega|/\omega\big)^{1/3}+5|\omega|\big(|\omega|/\omega\big)^{-1/3}}{\mathrm{Re}(\omega)+|\omega|\big(|\omega|/\omega\big)^{1/3}+|\omega|\big(|\omega|/\omega\big)^{-1/3}}.
\end{aligned} \tag{5.92}$$

It is instructive to compute the values of the black hole potential (5.104) at the three non-supersymmetric extremal points (5.92). Using the second equation in (5.85), we obtain

$$V_{\rm BH} = \frac{64}{\tau_2}(\beta^2 + 2\gamma^2\tau_2^2 + v^2\tau_2^4). \tag{5.93}$$

If we apply (5.87), after some algebra we find

$$\begin{aligned}\beta^2 + 2\gamma^2\tau_2^2 + v^2\tau_2^4 &= \frac{128\tau_2^8\gamma^4 - 32\tau_2^6\gamma^3\omega_1 + 2\tau_2^4\gamma^2\omega_1^2 + 26\tau_2^4\gamma^2\omega_2^2 - 10\tau_2^2\gamma\omega_1\omega_2^2 + \omega_1^2\omega_2^2}{\tau_2^2(8\tau_2^2\gamma - \omega_1)^2} = \\ &= \frac{\omega_1^2 + \omega_2^2}{2\tau_2^2} + \frac{\tau_2^2\gamma + \omega_1/2}{\tau_2^2(8\tau_2^2\gamma - \omega_1)^2}(128\tau_2^6\gamma^3 - 96\tau_2^4\gamma^2\omega_1 + 18\tau_2^2\gamma\omega_1^2 - 6\tau_2^2\gamma\omega_2^2 + \omega_1\omega_2^2 - \omega_1^3).\end{aligned} \tag{5.94}$$

The last term in the second line vanishes at the non-supersymmetric extremum point due to (5.89), and we get a simple formula for the potential

$$V_{\rm BH}^{\rm n-susy} = 32\frac{|\omega|^2}{\tau_2^3}. \tag{5.95}$$

Notice that the value of the potential is the same for all three points (5.92). At the supersymmetric extremum point (5.90) we have

$$V_{\rm BH}^{\rm susy} = 8\frac{|\omega|^2}{\tau_2^3}, \tag{5.96}$$

so that, as in [19])

$$V_{\rm BH}^{\rm n-susy} = 4V_{\rm BH}^{\rm susy}. \tag{5.97}$$

Note that this relation is written in terms of Calabi-Yau moduli rather then in terms of the black hole charges.

As we will see in a moment, all three non-supersymmetric extremum points provide a minimum of the black hole potential. In order to show this, let us look at the Hessian

$$Hess(V_{\rm BH}) = \begin{pmatrix} \frac{\partial^2 V_{\rm BH}}{\partial\tau_1^2} & \frac{\partial^2 V_{\rm BH}}{\partial\tau_1\partial\tau_2} \\ \frac{\partial^2 V_{\rm BH}}{\partial\tau_2\partial\tau_1} & \frac{\partial^2 V_{\rm BH}}{\partial\tau_2^2} \end{pmatrix}. \tag{5.98}$$

Straightforward computation gives

$$Hess(V_{\rm BH}) = \frac{192}{\tau_2^3}\begin{pmatrix} 3\beta^2 - 2\alpha\gamma + (4\gamma^2 - 2\beta v)\tau_2^2 + v^2\tau_2^4 & 4\gamma\tau_2(-\beta + v\tau_2^2) \\ 4\gamma\tau_2(-\beta + v\tau_2^2) & -\beta^2 + 2\gamma^2\tau_2^2 + 3v^2\tau_2^4 \end{pmatrix}. \tag{5.99}$$

At the non-supersymmetic extremal point (5.92), using (5.87) and (5.89), we obtain the following expression

$$M = \frac{\tau_2^3}{962}Hess(V_{\rm BH}) = \\ = \begin{pmatrix} \frac{96\tau_2^4\gamma^2(2\omega_1^2+\omega_2^2)-8\tau_2^2\gamma\omega_1(6\omega_1^2+\omega_2^2)+3\omega_1^4-\omega_1^2\omega_2^2}{\tau_2^2(8\tau_2^2\gamma-\omega_1)^2} & \frac{8\gamma(4\tau_2^2\gamma-\omega_1)\omega_2}{8\tau_2^2\gamma-\omega_1} \\ \frac{8\gamma(4\tau_2^2\gamma-\omega_1)\omega_2}{8\tau_2^2\gamma-\omega_1} & \frac{32\tau_2^4\gamma^2(2\omega_1^2+5\omega_2^2)-8\tau_2^2\gamma\omega_1(2\omega_1^2+7\omega_2^2)+\omega_1^4+5\omega_1^2\omega_2^2}{\tau_2^2(8\tau_2^2\gamma-\omega_1)^2} \end{pmatrix} \tag{5.100}$$

The eigenvalues $h_{1,2}$ of the matrix (5.100) are solutions to the equation

$$0 = \det\left\| M - \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \right\| = h^2 - 4\frac{|\omega|^2}{\tau_2^2}h + 3\frac{|\omega|^4}{\tau_2^4} - \\ - \frac{8\omega_2^2(4\tau_2^2\gamma-\omega_1)(16\tau_2^4\gamma^2+4\tau_2^2\gamma\omega_1-\omega_1^2)}{\tau_2^4(8\tau_2^2\gamma-\omega_1)^4}(128\tau_2^6\gamma^3-96\tau_2^4\gamma^2\omega_1 + 6\tau_2^2\gamma(3\omega_1^2-\omega_2^2)+\omega_1\omega_2^2-\omega_1^3) \tag{5.101}$$

The last line vanishes because of the extremum equation (5.89), and we get

$$h^2 - 4\frac{|\omega|^2}{\tau_2^2}h + 3\frac{|\omega|^4}{\tau_2^4} = 0. \tag{5.102}$$

Therefore, the eigenvalues of the matrix (5.100)

$$\begin{aligned} h_1 &= \frac{|\omega|^2}{\tau_2^2} \geq 0 \\ h_2 &= 3\frac{|\omega|^2}{\tau_2^2} \geq 0 \end{aligned} \tag{5.103}$$

are always non-negative. Since $\tau_2 > 0$, this means that the eigenvalues of the Hessian (5.99) are also positive if $\omega \neq 0$, and thus the non-supersymmetric extremum points minimize the potential.

5.4.2 Solution of the direct problem

The black hole potential (5.78) is given by

$$\begin{aligned}V_{\rm BH} = \quad & \tfrac{4}{\tau_2^3}(u^2+6qu\tau_1+9q^2\tau_1^2-6pu\tau_1^2-18pq\tau_1^3-2uv\tau_1^3+9p^2\tau_1^4- \\ & -6qv\tau_1^4+6pv\tau_1^5+v^2\tau_1^6+3q^2\tau_2^2-12pq\tau_1\tau_2^2+12p^2\tau_1^2\tau_2^2- \\ & -6qv\tau_1^2\tau_2^2+12pv\tau_1^3\tau_2^2+3v^2\tau_1^4\tau_2^2+3p^2\tau_2^4+6pv\tau_1\tau_2^4+3v^2\tau_1^2\tau_2^4+v^2\tau_2^6).\end{aligned} \tag{5.104}$$

Straightforward calculation gives

$$\frac{\partial V_{\rm BH}}{\partial \tau_1} = \frac{24}{\tau_2^3}\Big((q-2p\tau_1-v\tau_1^2)(u+3q\tau_1-3p\tau_1^2-v\tau_1^3)-2(p+v\tau_1)(q-2p\tau_1-v\tau_1^2)\tau_2^2+(p+v\tau_1)v\tau_2^4\Big) \tag{5.105}$$

and

$$\frac{\partial V_{\rm BH}}{\partial \tau_2} = \frac{12}{\tau_2^4}\Big(-(u+3q\tau_1-3p\tau_1^2-v\tau_1^3)^2-(q-2p\tau_1-v\tau_1^2)^2\tau_2^2+(p+v\tau_1)^2\tau_2^4+v^2\tau_2^6\Big). \tag{5.106}$$

The extremal points are solutions to the equations $\frac{\partial V_{\rm BH}}{\partial \tau_1} = \frac{\partial V_{\rm BH}}{\partial \tau_2} = 0$. From (5.105) we find that for a generic set of charges (assuming $v\gamma \neq 0$)

$$\tau_2^2 = \frac{\beta\gamma \pm \sqrt{\beta\gamma(\beta\gamma - v\alpha)}}{v\gamma}, \tag{5.107}$$

where

$$\begin{aligned}\alpha &= u+3q\tau_1-3p\tau_1^2-v\tau_1^3, \\ \beta &= q-2p\tau_1-v\tau_1^2, \\ \gamma &= p+v\tau_1.\end{aligned} \tag{5.108}$$

If we plug (5.107) into (5.106), we obtain

$$\gamma\sqrt{\beta\gamma-v\alpha}(\beta\sqrt{\beta\gamma}(v^2\alpha-3v\beta\gamma-2\gamma^3)\mp\gamma\sqrt{\beta\gamma-v\alpha}(3v\beta^2+v\alpha\gamma+2\beta\gamma^2))=0. \tag{5.109}$$

Let us look at the solution $\beta\gamma - v\alpha = 0$ first. Due to (5.108) this is equivalent to

$$\tau_1 = \frac{pq - uv}{2(p^2 + qv)} \tag{5.110}$$

Then (5.107) gives, assuming $\tau_2 > 0$

$$\tau_2 = \frac{\sqrt{-\mathcal{D}}}{2(p^2 + qv)} \tag{5.111}$$

where

$$\mathcal{D} = -(3p^2q^2 + 4p^3u + 4q^3v + 6pquv - u^2v^2). \tag{5.112}$$

This is the supersymmetric solution obtained in [124]. Note that there is no such solution if the discriminant (5.112) is positive: $\mathcal{D} > 0$.

The non-supersymmetric solution will emerge from the second branch:

$$\beta\sqrt{\beta\gamma}(v^2\alpha - 3v\beta\gamma - 2\gamma^3) = \pm\gamma\sqrt{\beta\gamma - v\alpha}(3v\beta^2 + v\alpha\gamma + 2\beta\gamma^2) \tag{5.113}$$

Without loss of generality we can take the square of this equation. Then, after plugging in (5.108) we find massive cancellations, and obtain the following cubic equation

$$\begin{aligned}
&(2p^6 + 6p^4qv + 3p^2q^2v^2 - 4p^3uv^2 - 2q^3v^3 - 6pquv^3 + u^2v^4)\tau_1^3 - \\
&-3(p^5q + 5p^3q^2v + 3p^4uv + 5pq^3v^2 + 4p^2quv^2 - q^2uv^3 - pu^2v^3)\tau_1^2 - \\
&-3(p^4q^2 + 2p^5u + 2p^3quv - 2q^4v^2 - 2pq^2uv^2 - p^2u^2v^2)\tau_1 + \\
&+(2p^3q^3 + 3p^4qu + 3pq^4v + 6p^2q^2uv + p^3u^2v + q^3uv^2) \;=\; 0.
\end{aligned} \tag{5.114}$$

The discriminant of this equation is equal to

$$\Delta = 729\mathcal{D}^3(p^2 + qv)^6(2p^6 + 6p^4qv + 3p^2q^2v^2 - 4p^3uv^2 - 2q^3v^3 - 6pquv^3 + u^2v^4)^2. \tag{5.115}$$

Only one solution of this equation can be real, if $\mathcal{D} > 0$, which implies $\Delta > 0$, but this is exactly what we are looking for. It is given by

$$\begin{aligned}\tau_1 = \quad & \tfrac{1}{(2(p^2+qv)^3+v^2\mathcal{D})}\Big((p^2+qv)^2(pq-uv) - vp\mathcal{D} - \\ & -\tfrac{2^{1/3}(p^2+qv)^3\mathcal{D}}{\left(v(2p^3+3pqv-uv^2)\mathcal{D}^2+(2(p^2+qv)^3+v^2\mathcal{D})\mathcal{D}\sqrt{\mathcal{D}}\right)^{1/3}} + \\ & + \tfrac{p^2+qv}{2^{1/3}}(v(2p^3+3pqv-uv^2)\mathcal{D}^2 + (2(p^2+qv)^3+v^2\mathcal{D})\mathcal{D}\sqrt{\mathcal{D}})^{1/3}\Big). \end{aligned} \quad (5.116)$$

Corresponding expression for τ_2 is obtained by substituting (5.116) into (5.107).

5.4.3 Cubic equation

Consider a general cubic equation of the form

$$ax^3 + 3bx^2 - 3cx - d = 0. \quad (5.117)$$

The discriminant of this equation is

$$\Delta = -(3b^2c^2 + 4c^3a + 4b^3d + 6abcd - a^2d^2). \quad (5.118)$$

The solutions are given by

$$\begin{aligned} x_1 &= -\tfrac{b}{a} + \tfrac{2^{1/3}(b^2+ac)}{a\left(a^2d-3abc-2b^3+a\sqrt{\Delta}\right)^{1/3}} + \tfrac{\left(a^2d-3abc-2b^3+a\sqrt{\Delta}\right)^{1/3}}{2^{1/3}a}, \\ x_2 &= -\tfrac{b}{a} - \tfrac{2^{1/3}(1+i\sqrt{3})(b^2+ac)}{2a\left(a^2d-3abc-2b^3+a\sqrt{\Delta}\right)^{1/3}} - \tfrac{(1-i\sqrt{3})}{2^{1/3}2a}(a^2d-3abc-2b^3+a\sqrt{\Delta})^{1/3}, \\ x_3 &= -\tfrac{b}{a} - \tfrac{2^{1/3}(1-i\sqrt{3})(b^2-ac)}{2a\left(a^2d-3abc-2b^3+a\sqrt{\Delta}\right)^{1/3}} - \tfrac{(1+i\sqrt{3})}{2^{1/3}2a}(a^2d-3abc-2b^3+a\sqrt{\Delta})^{1/3} \end{aligned} \quad (5.119)$$

We are interested in the case $\Delta > 0$, when there is one real root and a pair of complex conjugate roots.

5.5 Semiclassical Entropy in the OSV Ensamble

In this section we develop a semiclassical version of OSV formalism which applies to both supersymmetric and non-supersymmetric black holes. We then illustrate it using $D0-D4$ system in the diagonal T^6 model as an example. This will serve as a preparation for the discussions in section 6 and the conjecture in section 7 taking into account perturbative corrections to the extremal black hole entropy.

We begin by recalling some basic ingredients of the OSV formalism. The formula [142]

$$Z_{\mathrm{BH}}(p^I,\phi^I) = \left| e^{F_{\mathrm{top}}(p^I + \frac{i}{\pi}\phi^I)} \right|^2. \tag{5.120}$$

describes a relation between the mixed partition function of the supersymmetric (BPS) black hole and topological string free energy. Here F_{top} denotes the topological string free energy. It is well known [22] that the higher genus contributions to F_{top} depend non-holomorphically on the background complex structure. This dependence, originally described in [22] as the holomorphic anomaly in the topological string amplitudes coming from the boundary of the moduli space, was interpreted in [169] as a dependence of the wave-function $\Psi_{\mathrm{top}} = e^{F_{\mathrm{top}}}$ on the choice of the polarization. This viewpoint on the topological string partition function as a wave-function was further studied in [163, 62, 143].

As noted in [142], the formula 5.1 can be inverted, and resulting expression

$$\Omega(p^I,q_I) = \int d\chi^I e^{-i\pi\chi^I q_I} \Psi^*_{\mathrm{top}}(p^I-\chi^I)\Psi_{\mathrm{top}}(p^I+\chi^I). \tag{5.121}$$

can be interpreted as the Wigner function[6] associated to the topological string wave

[6]Let us recall that in quantum mechanics the Wigner function defines the quasi-probablity mea-

function. Here $\Psi_{\rm top}(p^I) = \langle p^I|\Psi_{\rm top}\rangle$ represents the topological string wave function in real polarization (see [93] for a comprehensive review and references), and the chemical potentials are restored after deforming the integration contour as $\phi^I = -i\chi^I$.

5.5.1 Black hole potential and OSV transformation

Let us rewrite modified black hole potential (5.37) in the form

$$V_{\rm BH}^{(0)} = q_I {\rm Im} P^I + \Big(\tfrac{i}{4}(P^I - X^I)(P^J - X^J)\overline{F}_{IJ}^{(0)} + \tfrac{i}{2}(P^I - X^I)F_I^{(0)} + \tfrac{i}{2}F^{(0)} + c.c.\Big). \tag{5.122}$$

We put the superscript (0) to stress that the prepotential $F^{(0)}$ corresponds to a genus zero part of the topological string free energy. As in the OSV setup [142], we can parameterize the Lagrange multiplier P^I (which can also be viewed as a complexified magnetic charge) as

$$P^I = p^I + \frac{i}{\pi}\phi^I, \tag{5.123}$$

so that the first of the attractor equations (5.29) is automatically satisfied. At the next step, we rewrite the semiclassical entropy $S_{\rm BH}^{(0)} = \pi V_{\rm BH}^{(0)}$ as

$$S_{\rm BH}^{(0)} = q_I\phi^I - \pi {\rm Im}\mathcal{G}^{(0)}, \tag{5.124}$$

where we introduced a function $\mathcal{G}^{(0)}$ defined by

$$\mathcal{G}^{(0)} = \frac{1}{2}(P^I - X^I)(P^J - X^J)\overline{F}_{IJ}^{(0)} + (P^I - X^I)F_I^{(0)} + F^{(0)}. \tag{5.125}$$

In order to compute the entropy in (5.124) we should find the values of ϕ^I and X^I that extremize the black hole potential (5.122). Extremization with respect to the

sure $f({\rm x},{\rm p}) = \frac{1}{2\pi}\int dy e^{-iy{\rm p}}\psi^*({\rm x}-\frac{\hbar}{2}y)\psi({\rm x}+\frac{\hbar}{2}y)$ on the *phase* space, see e.g. [175]. Here the canonical commutation relation is $[\widehat{\rm p},\widehat{\rm x}] = -i\hbar$. In the topological string setup $\hbar = \frac{2}{\pi}$.

(extended) Calabi-Yau moduli $\partial_I V_{\rm BH}^{(0)} = 0$ gives the equations (5.49). Let us use the index s to label all solutions to these equations, $X_s^I = X_s^I(P)$. There are two types of these solutions, supersymmetric ($s = \rm susy$) and non-supersymmetric ($s = \rm n - susy$) ones. In particular, the supersymmetric solution is given by $X_{\rm susy}^I(P) = P^I$. By substituting these solutions in (5.125) we obtain the functions $\mathcal{G}_s^{(0)}(P^I) = \mathcal{G}_s^{(0)}(p^I, \phi^I)$. In the supersymmetric case $\mathcal{G}_{\rm susy}^{(0)}(P^I) = F^{(0)}(p^I + \frac{i}{\pi}\phi^I)$. Let us define a mixed partition functions corresponding to each of the solutions $X_s^I = X_s^I(P)$ by

$$Z_s^{(0)}(p^I, \phi^I) = e^{i\frac{\pi}{2}\mathcal{G}_s^{(0)}(p^I, \phi^I)}. \tag{5.126}$$

For example, the supersymmetric mixed partition function

$$Z_{\rm susy}^{(0)}(p^I, \phi^I) = e^{i\frac{\pi}{2}F^{(0)}(p^I + \frac{i}{\pi}\phi^I)} \tag{5.127}$$

describes the leading contribution to (5.120).

For a fixed charge vector (p^I, q_I) the extremal black hole degeneracy can be written symbolically as $\Omega_{\rm extrm} = \Omega_{\rm susy} + \Omega_{\rm n-susy}$. Therefore, the leading semiclassical contribution to $\Omega_{\rm extrm}$ is given by an OSV type integral

$$\Omega_{\rm extrm}^{(0)}(p^I, q^I) = \int d\phi^I e^{q_I \phi^I} \sum_s |Z_s^{(0)}(p^I, \phi^I)|^2, \tag{5.128}$$

where the sum is over all solutions to the extremum equations (5.49). We will discuss perturbative corrections to this formula later in section 5.7, but before that let us comment on the possible wave function interpretation of this expression.

Define

$$\Psi(X, P) = \exp\frac{i\pi}{2}\Big(\frac{1}{2}(P^I - X^I)(P^J - X^J)\overline{F}_{IJ}^{(0)} + (P^I - X^I)F_I^{(0)} + F^{(0)}\Big). \tag{5.129}$$

This is essentially the off-shell version of the partition function (5.126), since we have not substituted the extremum solution $X_s^I = X_s^I(P)$ into (5.129) yet. This can be achieved by integrating out the fields X^I in the semiclassical approximation

$$\sum_s |Z_s^{(0)}(p^I, \phi^I)|^2 \approx \int dX^I d\overline{X}^I \sqrt{\det\|\text{Im}F_{IJ}\|}\Psi(X,P)\Psi^*(X,P). \tag{5.130}$$

The function $\Psi(X,P)$ given in (5.129) is holomorphic in P^I and non-holomorphic in X^I. It turns out that (up to some numerical factors due to a difference in conventions) it coincides exactly with the DVV 'conformal block' [62] appearing in study of the five-brane partition function! In particular, as was shown in [62], it satisfies the holomorphic anomaly equation [22]. Using results of [93], it can be identified as the intertwining function $\Psi(X,P) ={}_{(X,\overline{X})}\langle X^I|P^I\rangle$ between the coherent state $|P^I\rangle$ in the real polarization and the coherent state $|X^I\rangle_{(X,\overline{X})}$ in the holomorphic polarization appearing in quantization of $H^3(M,\mathbb{C})$. The integral in (5.130) then can naturally be interpreted as averaging over the wave function polarizations, thus effectively removing the background dependence. We should stress, however, that only semiclassical approximation to this integral is needed for (5.128). This would be interesting to develop further, especially in connection with the topological M-theory [58, 133] interpretation of the black hole entropy counting.

We now turn to a simple example of the diagonal T^6 model, where semiclassical formula (5.128) for extremal black hole entropy can be illustrated.

5.5.2 Semiclassical entropy in the diagonal T^6 compactification

Consider Type IIB compactification on the diagonal T^6 threefold [124]. The prepotential is

$$F = \frac{(X^1)^3}{X^0}, \qquad f(\tau) = \tau^3, \tag{5.131}$$

where the complex structure parameter $\tau = \frac{X^1}{X^0}$. We compute:

$$F_{IJ} = \begin{pmatrix} 2\tau^3 & -3\tau^2 \\ -3\tau^2 & 6\tau \end{pmatrix}, \quad C_{IJ0} = -\tfrac{6\tau}{X^0}\begin{pmatrix} \tau^2 & -\tau \\ -\tau & 1 \end{pmatrix}, \quad C_{IJ1} = \tfrac{6}{X^0}\begin{pmatrix} \tau^2 & -\tau \\ -\tau & 1 \end{pmatrix}. \tag{5.132}$$

Let us denote $y^I = P^I - X^I$. The attractor equations (5.49) read

$$\begin{cases} C_{0IJ}\overline{y}^I\overline{y}^J = 4i\mathrm{Im}(F_{0I})y^I \\ C_{1IJ}\overline{y}^I\overline{y}^J = 4i\mathrm{Im}(F_{1I})y^I. \end{cases} \tag{5.133}$$

In order to compute the function $\mathcal{G}^{(0)}(p^I, \phi^I)$, we need to find from these equations a solution $X^I = X^I(P)$ of the direct problem. This can be done by inverting the solutions of the inverse problem (5.69)-(5.70). However, it turns out that it is easier to find $X^I = X^I(P)$ directly from (5.133).

According to (5.59) and (5.132), the third derivatives of the prepotential are related as $C_{0IJ} = -\tau C_{1IJ}$, and therefore (5.133) reduces to

$$2y^0\mathrm{Im}(\tau^3) - 3y^1\mathrm{Im}(\tau^2) = 3\tau y^0\mathrm{Im}(\tau^2) - 6\tau y^1\mathrm{Im}(\tau). \tag{5.134}$$

Apart from the supersymmetric solution $y^0 = y^1 = 0$, this gives

$$\frac{y^1}{y^0} = \mathrm{Re}\tau - \frac{i}{3}\mathrm{Im}\tau, \tag{5.135}$$

If we recall that $y^I = P^I - X^I$, we can solve (5.135) for X^1:

$$X^1 = X^0 \frac{4\mathrm{Re}(X^0\overline{P}^1) - 2\overline{P}^1 P^0 + P^1\overline{P}^0}{4\mathrm{Re}(X^0\overline{P}^0) - |P^0|^2}. \tag{5.136}$$

Then we plug this into the second equation of (5.133) and find[7]

$$(\overline{X}^0 - \overline{P}^0)^2 = 3X^0(X^0 - P^0). \tag{5.137}$$

This should be compared to (5.65). To solve the equation (5.137), is convenient to work with the real and imaginary parts of X^0 and P^0. Then (5.137) can be reduced to a quartic equation for $\mathrm{Re}X^0$. For a generic choice of $\mathrm{Re}P^0$ and $\mathrm{Im}P^0$, two of the roots of this quartic equation are complex, and two are real. These real roots lead to the two solutions of the attractor equations (5.133), supersymmetric

$$\begin{aligned} X^0 &= P^0, \\ X^1 &= P^1, \end{aligned} \tag{5.138}$$

and non-supersymmetric one. Explicit expression for the non-supersymmetric solution depends on the signs of $\mathrm{Re}P^0$ and $\mathrm{Im}P^0$. For example, when $\mathrm{Im}P^0 > |\mathrm{Re}P^0|$, it is given by[8]

$$\begin{aligned} \mathrm{Re}X^0 &= \tfrac{1}{4}\mathrm{Re}P^0 + \tfrac{3}{8}(\mathrm{Re}P^0 + \mathrm{Im}P^0)^{\frac{2}{3}}(\mathrm{Im}P^0 - \mathrm{Re}P^0)^{\frac{1}{3}} - \\ &\quad - \tfrac{3}{8}(\mathrm{Re}P^0 + \mathrm{Im}P^0)^{\frac{1}{3}}(\mathrm{Im}P^0 - \mathrm{Re}P^0)^{\frac{2}{3}}, \\ \mathrm{Im}X^0 &= \tfrac{1}{4}\mathrm{Im}P^0 - \tfrac{1}{4}\sqrt{9(\mathrm{Im}P^0)^2 - 8(\mathrm{Re}P^0)^2 - 8\mathrm{Re}(X^0)\mathrm{Re}(P^0) + 16(\mathrm{Re}X^0)^2}. \end{aligned} \tag{5.139}$$

We can use these solutions and study a system of $kD0$ and $ND4$ branes on the diagonal T^6. This corresponds to the charge vector of the form $(k, 0, N, 0)$. In this case

[7] assuming $\mathrm{Im}\big(P^0\overline{P}^1\big) \neq 0$.

[8] Corresponding solution for X^1 is obtained by plugging this expression into (5.136).

the discriminant $\mathcal{D} = -(3p^2q^2 + 4p^3u + 4q^3v + 6pquv - u^2v^2)$ reduces to $\mathcal{D} = -4kN^3$, so that the system is supersymmetric when $kN > 0$ and non-supersymmetric when $kN < 0$. Complexified magnetic charges are given by

$$P^0 = \frac{i}{\pi}\varphi, \qquad P^1 = N + \frac{i}{\pi}\phi, \tag{5.140}$$

and the black hole degeneracy (5.128) in this case reads

$$\Omega^{(0)}_{\text{extrm}}(k,N) = \int d\phi d\varphi e^{k\varphi}\Big(e^{-\pi \text{Im}\mathcal{G}^{(0)}_{\text{susy}}(\frac{i}{\pi}\varphi, N+\frac{i}{\pi}\phi)} + e^{-\pi \text{Im}\mathcal{G}^{(0)}_{\text{n-susy}}(\frac{i}{\pi}\varphi, N+\frac{i}{\pi}\phi)}\Big). \tag{5.141}$$

Let us now compute expressions for $\mathcal{G}^{(0)}$-functions entering into (5.141). Using (5.138), we find from (5.125)

$$-\pi \text{Im}\mathcal{G}^{(0)}_{\text{susy}}(\frac{i}{\pi}\varphi, N + \frac{i}{\pi}\phi) = \frac{N^3\pi^2 - 3N\phi^2}{\varphi}. \tag{5.142}$$

The non-supersymmetric solution (5.139) in the case (5.140) reads

$$\begin{aligned} X^0 &= \quad -\tfrac{i}{2\pi}\varphi \\ X^1 &= \tfrac{1}{2}(N - \tfrac{i}{2\pi}\phi). \end{aligned} \tag{5.143}$$

Therefore, from (5.125) we obtain the following expression

$$-\pi \text{Im}\mathcal{G}_{\text{n-susy}}(\frac{i}{\pi}\varphi, N + \frac{i}{\pi}\phi) = -\frac{N^3\pi^2 - 3N\phi^2}{\varphi}. \tag{5.144}$$

The integral over ϕ in (5.141) is quadratic, and (ignoring the convergence issue) in the semiclassical approximation $\phi = 0$. The critical points in the φ direction are given by

$$\partial_\varphi(k\varphi - \pi \text{Im}\mathcal{G}_{\text{susy}}) = 0 \quad \Rightarrow \quad \varphi_{\text{susy}} = \pi\sqrt{\frac{N^3}{k}} \tag{5.145}$$

for supersymmetric term, and

$$\partial_\varphi(k\varphi - \pi \mathrm{Im}\mathcal{G}_{\mathrm{n-susy}}) = 0 \quad \Rightarrow \quad \varphi_{\mathrm{n-susy}} = \pi\sqrt{-\frac{N^3}{k}} \tag{5.146}$$

for the non-supersymmetric term. Since we are integrating over the real axis, the leading contribution to (5.141) comes only from one of the two terms, depending on the sign of the ratio $\frac{N}{k}$. This gives:

$$\Omega^{(0)}_{\mathrm{extrm}}(k, N) \approx \exp\left(2\pi\sqrt{|N^3 k|}\right), \tag{5.147}$$

which is a correct expression for extremal black hole degeneracy, valid both in the supersymmetric and non-supersymmetric cases. Using the same method, it is also easy to obtain an expression $\Omega^{(0)}_{\mathrm{extrm}}(N_0, N_6) \approx \exp\left(\pi|N_0 N_6|\right)$ for the degeneracy of $D0 - D6$ system on diagonal T^6, which agrees with [69].

It is instructive to compare this prediction of (5.128) with the original OSV formula [142]

$$\Omega(p^I, q_I) = \int d\phi^I e^{q_I\phi^I + \mathcal{F}(p^I,\phi^I)}. \tag{5.148}$$

Because of our choice of the non-canonical $D3$-brane intersection matrix on T^6, we have $q_I\phi^I = -u\phi^0 - 3q\phi$. Also,

$$\mathcal{F}(p^I, \phi^I) = -\pi \mathrm{Im}\left(\frac{(p + \frac{i}{\pi}\phi)^3}{v + \frac{i}{\pi}\phi^0}\right). \tag{5.149}$$

In the semiclassical approximation, the leading contribution to $\ln\Omega(u, q, p, v)$ can be computed by extremizing the exponent in (5.148). This gives

$$\begin{aligned} 2q &= -\frac{(p+\frac{i}{\pi}\phi)^2}{v+\frac{i}{\pi}\phi^0} - \frac{(p-\frac{i}{\pi}\phi)^2}{v-\frac{i}{\pi}\phi^0}, \\ 2u &= \frac{(p+\frac{i}{\pi}\phi)^3}{(v+\frac{i}{\pi}\phi^0)^2} - \frac{(p-\frac{i}{\pi}\phi)^3}{(v-\frac{i}{\pi}\phi^0)^2}. \end{aligned} \tag{5.150}$$

which essentially are the supersymmetric attractor equations (5.36). The general solution to (5.150) is easy to write:

$$\begin{aligned} \phi^0 &= \pm\pi \tfrac{2p^3+2pqv-uv^2}{\sqrt{-\mathcal{D}}}, \\ \phi &= \mp\pi \tfrac{2p^2q+2q^2v+puv}{\sqrt{-\mathcal{D}}}, \end{aligned} \tag{5.151}$$

where the discriminant $\mathcal{D} = -(3p^2q^2+4p^3u+4q^3v+6pquv-u^2v^2)$. The sign ambiguity in (5.151) can be fixed by imposing physically natural condition

$$\mathrm{Im}\tau = \mathrm{Im}\frac{p+\frac{i}{\pi}\phi}{v+\frac{i}{\pi}\phi^0} > 0. \tag{5.152}$$

Notice that the potentials (5.151) become pure imaginary when $\mathcal{D} > 0$. Therefore, if one is allowed to do the analytical continuation when computing the integral (5.148), the answer for the microcanonical entropy reads

$$\ln \Omega(u,q,p,v) \approx \pi\sqrt{3p^2q^2+4p^3u+4q^3v+6pquv-u^2v^2}. \tag{5.153}$$

This expression, of course, becomes pure imaginary on the non-supersymmetric side $\mathcal{D} > 0$ of the discriminant hypersurface $\mathcal{D} = 0$, which is meaningless. This thus illustrates the shortcoming of OSV formalism in the context of non-BPS black holes.

5.6 Including Higher Derivative Corrections: The Entropy Function Approach

The Wald's formula provides a convenient tool for computing the macroscopic black hole entropy in the presence of higher derivative terms. It can be written as

$$S_{\mathrm{BH}} = 2\pi \int_H d^2x\sqrt{h}\,\epsilon_{\mu\nu}\epsilon_{\lambda\rho}\frac{\delta\mathcal{L}}{\delta\mathcal{R}_{\mu\nu\lambda\rho}}, \tag{5.154}$$

where $\mathcal{L}$ is the Lagrangian density and the integral is computed over the black hole horizon. Sen [151, 152] showed that in the case of a spherically symmetric extremal black holes with $AdS^2 \times S_2$ near horizon geometry Wald's formula simplifies drastically. This gives an effective method for computing a macroscopic entropy of a spherically symmetric extremal black holes in a theory of gravity coupled to gauge and scalar fields, called the entropy function formalism.

In this section we briefly describe, following [123], a formulation of $\mathcal{N} = 2$ supergravity coupled to n_V abelian gauge fields, in the presence of higher-derivative corrections. Then we review recent computations of the extremal black hole entropy in this setup [148, 7, 31], performed in the framework of the entropy function formalism.

5.6.1 $d = 4,\ \mathcal{N} = 2$ Supergravity with F-term $\mathcal{R}^2$ corrections

The Lagrangian density of $\mathcal{N} = 2$ Poincare supergravity coupled to n_V vector multiplets can be conveniently formulated using the off-shell description [51]. The idea is to start with an $\mathcal{N} = 2$ conformal supergravity and then reduce it to Poincare supergravity by gauge fixing and adding appropriate compensating fields. The advantage of working with $\mathcal{N} = 2$ superconformal approach is that it provides many powerful tools, such as superconformal tensor calculus and a general *density formula* for the Lagrangian.

One introduces the Weil and matter chiral superfields

$$\begin{aligned} W_{\mu\nu}(x,\theta) &= T^-_{\mu\nu} - \tfrac{1}{2}\mathcal{R}^-_{\mu\nu\lambda\rho}\epsilon_{\alpha\beta}\theta^\alpha\sigma^{\lambda\rho}\theta^\beta + \ldots \\ \Phi^I(x,\theta) &= X^I + \tfrac{1}{2}\mathcal{F}^{-I}_{\mu\nu}\epsilon_{\alpha\beta}\theta^\alpha\sigma^{\mu\nu}\theta^\beta + \ldots \end{aligned} \tag{5.155}$$

where $T^-_{\mu\nu}$ is an auxiliary antiself-dual tensor field[9], and $\mathcal{F}^{-I}_{\mu\nu}$ and $\mathcal{R}^-_{\mu\nu\lambda\rho}$ denote the anti-selfdual parts the field-strength and curvature tensors correspondingly. The conventions are $*T_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}T^{\rho\sigma}$ and $T^\pm_{\mu\nu} = \frac{1}{2}(T_{\mu\nu} \pm i * T_{\mu\nu})$, so that $T^-_{\mu\nu} = \overline{T}^+_{\mu\nu}$ for Minkovski signature. The superconformally covariant field strength

$$\mathbf{F}^I_{\mu\nu} = \mathcal{F}^I_{\mu\nu} - (\frac{1}{4}\overline{X}^I T^-_{\mu\nu} + \epsilon_{ij}\overline{\psi}^i{}_{[\mu}\gamma_{\nu]}\Omega^{jI} + \epsilon_{ij}\overline{X}^I\overline{\psi}^i{}_\mu\psi^j_\nu + h.c.) \tag{5.156}$$

enters into the bosonic part of the Lagrangian through the combination $\mathcal{F}^{+I}_{\mu\nu} - \frac{1}{4}X^I T^+_{\mu\nu}$. The F-terms can be reproduced from the generalized prepotential

$$F(X^I, W) = \sum_g F^{(g)}(X^I)W^{2g}, \tag{5.157}$$

where $F^{(g)}$ can be computed from the topological string amplitudes [22, 10]. In particular, the topological string free energy is given by

$$F_{\rm top}(X^I, g_{\rm top}) = \sum_g (g_{\rm top})^{2g-2}F^{(g)}(X^I). \tag{5.158}$$

The function $F^{(g)}$ is homogeneous of degree $2-2g$, so that

$$F(\lambda X^I, \lambda W) = \lambda^2 F(X^I, W). \tag{5.159}$$

This homogeneity relation for the generalized prepotential (5.157) can also be written as

$$X^I\partial_I F + W\partial_W F = 2F. \tag{5.160}$$

Notice that another notation

$$\widehat{A} \equiv W^2, \qquad F(X^I, \widehat{A}) \equiv F(X^I, W) \tag{5.161}$$

[9] At tree-level this field is identified with the graviphoton by the equations of motion.

is sometimes used in the supergravity literature.

The coupling of the vector fields to the gravity is governed by the generalized prepotential (5.157) as follows

$$\begin{aligned} 8\pi S_{\rm vect} &= 8\pi S^{\rm tree}_{\rm vect} + \int d^4x d^4\theta \sum_{g=1}^{\infty} F_g(\Phi^I)(W_{\mu\nu}W^{\mu\nu})^g + h.c. = \\ &= 8\pi S^{\rm tree}_{\rm vect} + \int d^4x \sum_{g=1}^{\infty} F_g(X^I)(\mathcal{R}^2_- T^{2g-2}_- + \dots) + h.c. \end{aligned} \quad (5.162)$$

The terms in the Lagrangian density, relevant for the computation of the entropy are [123]

$$\begin{aligned} 8\pi\mathcal{L} = -\tfrac{i}{2}\Big[&\tfrac{1}{2}(\mathcal{F}^{+I}_{\mu\nu} - \tfrac{1}{4}X^I T^+_{\mu\nu})(\mathcal{F}^{+J\mu\nu} - \tfrac{1}{4}X^J T^{+\mu\nu})\overline{F}_{IJ} + \tfrac{T^{+\mu\nu}}{4}(\mathcal{F}^{+I}_{\mu\nu} - \tfrac{1}{4}X^I T^+_{\mu\nu})F_I + \tfrac{\widehat{\overline{A}}}{16}F - \\ &-\overline{X}^I F_I \mathcal{R} - F_{\widehat{A}}\widehat{C} - h.c.\Big] + \dots \end{aligned} \quad (5.163)$$

Here

$$\begin{aligned} \widehat{C} &= 64\mathcal{R}^-_{\nu\mu\rho\sigma}\mathcal{R}^{-\nu\mu\rho\sigma} + 16T^{-\mu\nu}f^\rho_\mu T^+_{\rho\nu} + \dots \\ f^\nu_\mu &= -\tfrac{1}{2}\mathcal{R}^\nu_\mu + \tfrac{1}{32}T^-_{\mu\rho}T^{+\nu\rho} + \dots \\ F &= F(X^I, \widehat{A}), \quad F_{\widehat{A}} \equiv \partial_{\widehat{A}}F, \end{aligned} \quad (5.164)$$

and ... in (5.163)-(5.164) denotes the terms (auxiliary fields, fermions, etc.) that will vanish or cancel out on the black hole ansatz.

5.6.2 Review of the entropy function computation

We are interested in a spherically symmetric extremal black hole solutions arising in the supergravity theory defined by the Lagrangian (5.163). Consider the most general $SO(2,1) \times SO(3)$ ansatz [148] for a field configurations consistent with the $AdS_2 \times S^2$ near horizon geometry of the black hole

$$
\begin{aligned}
ds^2 = v_1\Big(- r^2 dt^2 + \frac{dr^2}{r^2}\Big) + v_2(d\theta^2 + \sin^2\theta d\varphi^2),\\
X^I = x^I, \quad \mathcal{F}^I_{rt} = -\frac{\phi^I}{\pi}, \quad \mathcal{F}^I_{\theta\varphi} = p^I \sin\theta, \quad T^-_{rt} = v_1 w,
\end{aligned}
\tag{5.165}
$$

and all other fields presents in (5.163) are set to zero[10]. The entropy function [151] is defined as

$$
\mathcal{E} = q_I \phi^I - 2\pi \int_H d\theta d\varphi \sqrt{-\text{det} g}\mathcal{L}\Big). \tag{5.166}
$$

This function depends on free parameters $(x^I, v_1, v_2, w, \phi^I)$ of the $SO(2,1)\times SO(3)$ ansatz (5.165). The entropy of an extremal black hole is obtained as an entropy of a non-extremal black hole in the extremal limit, when the function (5.166) is extremized with respect to a free parameters

$$
\frac{\partial \mathcal{E}}{\partial x^I} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_1} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_2} = 0, \quad \frac{\partial \mathcal{E}}{\partial w} = 0, \quad \frac{\partial \mathcal{E}}{\partial \phi^I} = 0. \tag{5.167}
$$

The black hole entropy (5.154) is given by the value of $\mathcal{E}$ at the extremum

$$
S_{\text{BH}} = \mathcal{E}|_{\partial\mathcal{E}=0}. \tag{5.168}
$$

The result of computation [148] reads

$$
\begin{aligned}
\mathcal{E} = q_I\phi^I - i\pi v_1 v_2\Big[\tfrac{1}{4}\Big(- \tfrac{\phi^I}{\pi v_1} + i\tfrac{p^I}{v_2} - \tfrac{1}{2}x^I\overline{w}\Big)\Big(- \tfrac{\phi^J}{\pi v_1} + i\tfrac{p^J}{v_2} - \tfrac{1}{2}x^J\overline{w}\Big)\overline{F}_{IJ} +\\
+\tfrac{\overline{w}}{4}\Big(- \tfrac{\phi^I}{\pi v_1} + i\tfrac{p^I}{v_2} - \tfrac{1}{2}x^J\overline{w}\Big)F_I + \tfrac{\overline{w}^2}{8}F -\\
-(\tfrac{1}{v_1} - \tfrac{1}{v_2})\overline{x}^I F_I + \Big(|w|^4 - 8|w|^2(\tfrac{1}{v_1} + \tfrac{1}{v_2}) + 64(\tfrac{1}{v_1} - \tfrac{1}{v_2})^2\Big)F_{\widehat{A}} - c.c.\Big],
\end{aligned}
\tag{5.169}
$$

where

$$
\widehat{A} = -4w^2. \tag{5.170}
$$

[10]The dilaton is set to $1/3\mathcal{R}$, so that the combination $D - 1/3\mathcal{R}$ vanishes.

Note that the entropy function (5.169) is invariant under the following rescaling

$$x^I \to \lambda x^I, \quad w \to \lambda w, \quad v_{1,2} \to \frac{1}{\lambda\bar{\lambda}} v_{1,2}, \quad \phi^I \to \phi^I, \quad q_I \to q_I, \quad p^I \to p^I, \tag{5.171}$$

since the Lagrangian (5.163) was derived from a superconformally invariant expression. This means that there is a linear relation between the extremum equations (5.167). One can switch to inhomogeneous variables to fix this symmetry.

The above form of the entropy function does not take into account all the relevant higher derivative corrections needed for the non-supersymmetric black hole, as has been observed in [148]. For example at least an $\mathcal{R}^2$ term is needed in certain cases. We will come back to this point in the next section when we propose our conjecture.

To further motivate our conjecture, let us analyze the structure of the entropy function (5.169). First of all, compared to the topological string partition function, it depends on one more parameter. Indeed, using the scaling invariance of the entropy function (inherited from the formulation in terms of the superconformal action) we can gauge away w, and identify $(X^I, W^2) \sim (x^I, \widehat{A})$. However, after that the entropy function still depends on the relative magnitude of the variables v_1 and v_2, describing correspondingly the radii squared of AdS^2 and S_2 factors in the black hole near horizon geometry, and there is no such parameters in (5.122). Therefore, in order to match with the macroscopic computations on the supergravity side we need a modification of the topological string depending on an additional parameter. Moreover because of the observations of [148, 149] this extension of topological string should be computing additional higher derivative corrections, including extra $\mathcal{R}^2$ terms. These observations naturally lead to our conjecture in the next section.

5.7 A Conjecture

In the last section we saw that we need a one parameter extension of topological string which captures non-antiself-dual $4d$ geometries, for higher derivative corrections for non-supersymmetric black holes. In fact on the topological string side there is a natural candidate that can be used for this purpose: a one parameter extension of the topological string that appeared in the works of Nekrasov [139, 119, 135, 137, 138, 140] on instanton counting in Seiberg-Witten theory. There, a function $F(X^I, \epsilon_1, \epsilon_2)$ was introduced. In the special limit $-\epsilon_2 = \epsilon_1 = g_{\text{top}}$ this function reduces to the ordinary topological string free energy (5.158) according to

$$F(X^I, \epsilon_1, \epsilon_2)\Big|_{\epsilon_1+\epsilon_2=0} = F_{\text{top}}(X^I, g_{\text{top}}), \qquad g_{\text{top}}^2 = -\epsilon_1\epsilon_2, \tag{5.172}$$

In order to make a connection with the supergravity ansatz (5.165) we will need to identify the parameters as

$$\epsilon_1 = \frac{16}{|w|^2 v_1}, \quad \epsilon_2 = -\frac{16}{|w|^2 v_2}. \tag{5.173}$$

This is consistent with the fact that the field theory limit $\epsilon_{1,2} \to 0$ in the Nekrasov's approach corresponds to the flat space approximation in the ansatz (5.165).

Since the Nekrasov's extension of the topological string may not be familiar, we will first review the necessary background from [139, 138, 131]. Then we will be able to make a proposal about the corresponding generalization of the OSV formula.

5.7.1 Review of the Nekrasov's extension of the topological string

The instanton corrections to the prepotential of $\mathcal{N} = 2$ gauge theory can be computed by a powerful application of localization technique introduced by Nekrasov [139]. This localization, in the physical context gets interpreted as turning on non-antiself-dual graviphoton background,

$$T = \epsilon_1 dx^1 \wedge dx^2 + \epsilon_2 dx^3 \wedge dx^4. \tag{5.174}$$

This reproduces the $\mathcal{N} = 2$ prepotential by considering the most singular term as $\epsilon_i \to 0$, which scales as $F^{(0)}/\epsilon_1\epsilon_2$. However there is more information in the localization computation of Nekrasov: One can also look at the subleading terms and identify their physical significance. For the case of $\epsilon_1 = -\epsilon_2$ there is a natural answer, as this gets mapped to the $\mathcal{N} = 2$ F-terms which capture (anti)-selfdual graviphoton corrections, of the type studied in [22, 10]. In fact the two can get identified using geometric engineering of $\mathcal{N} = 2$ gauge theories [111, 110] by considering, in the type IIA setup, a local Calabi-Yau given by ALE fibrations over some base space (e.g. $\mathbb{P}^1$). Thus Nekrasov's gauge theory computation leads, *indirectly*, to a computation of topological string amplitudes, upon the specialization $\epsilon_1 = -\epsilon_2 = g_{\text{top}}$:

$$\lim_{\epsilon_2 \to -\epsilon_1} F(X^I, \epsilon_1, \epsilon_2) = \sum_{g=0}^{\infty} (g_{\text{top}})^{2g-2} F^{(g)}(X^I), \quad g_{\text{top}} = \epsilon_1. \tag{5.175}$$

It has been checked [104, 103, 100, 102] using the topological vertex formalism [2, 101] that this indeed agrees with the direct computation of topological string amplitudes in such backgrounds, see also [67, 156].

However, it is clear that there is still more to the story: Nekrasov's computation has more information than the topological string in such backgrounds as it depends on an extra parameter, which is visible when $\epsilon_1+\epsilon_2 \neq 0$. In fact Nekrasov's extension $F(X^I,\epsilon_1,\epsilon_2)$ satisfies the homogeneity condition

$$\Big[\epsilon_1\frac{\partial}{\partial\epsilon_1}+\epsilon_2\frac{\partial}{\partial\epsilon_2}+X^I\frac{\partial}{\partial X^I}\Big]F(X^I,\epsilon_1,\epsilon_2)=0. \tag{5.176}$$

which means that it does depend on one extra parameter compared to the topological strings. Below we will use a shorthand notation

$$F(X,\epsilon)\equiv F(X^I,\epsilon_1,\epsilon_2). \tag{5.177}$$

Even though the exact effective field theory terms that $F(X,\epsilon)$ computes has not been worked out, it is clear from the derivation that it has to do with constant, non-antiself-dual configurations of graviphoton and Riemann curvature. The origin of first such correction has been identified in [131] which we will now review. In general one can expand $F(X,\epsilon)$ as follows [138, 140, 131]

$$F=\frac{1}{\epsilon_1\epsilon_2}F^{(0)}+\frac{\epsilon_1+\epsilon_2}{\epsilon_1\epsilon_2}H_{\frac{1}{2}}+\frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}G_1+F^{(1)}+\mathcal{O}(\epsilon_1,\epsilon_2). \tag{5.178}$$

Let us discuss a geometrical meaning of the genus one terms in (5.178). Recall a general relations

$$\frac{1}{32\pi^2}\int_{\mathcal{X}}\mathrm{Tr}\mathcal{R}\wedge *\mathcal{R}=\chi, \qquad \frac{i}{32\pi^2}\int_{\mathcal{X}}\mathrm{Tr}\mathcal{R}\wedge\mathcal{R}=\frac{3}{2}\sigma, \tag{5.179}$$

where χ is the Euler characteristic of a Euclidean 4-manifold $\mathcal{X}$ and σ is the signature. The curvature tensor $\mathcal{R}$ in (5.179) is viewed as a 2-form $\mathcal{R}^a{}_b=\mathcal{R}^a{}_{b\mu\nu}dx^\mu\wedge dx^\nu$ with values in Lie algebra of $SO(4)$. As is clear from (5.162), the ordinary topological strings compute contributions to the effective action of the form[11]

$$\frac{1}{16\pi^2}\int_{\mathcal{X}} F^{(1)}(X)\mathcal{R}_- \wedge \mathcal{R}_- + \text{higher genus} = \frac{1}{2}F^{(1)}(X)(\chi - \frac{3}{2}\sigma) + \text{higher genus}. \tag{5.180}$$

On the other hand, more general couplings to χ and σ can be seen in the Donaldson theory. As was explained by Witten [171], the low energy effective action of twisted $\mathcal{N} = 2$ supersymmetric Yang-Mills theory on an arbitrary four-manifold $\mathcal{X}$ contains terms proportional to χ and σ. The Donaldson invariant D_ξ in general has three contributions

$$D_\xi = Z_u + Z_+ + Z_-, \tag{5.181}$$

where $Z_\pm$ are Seiberg-Witten invariants defined via the moduli space of monopoles, and Z_u is non-zero when $b_+(\mathcal{X}) = 1$ and is given by the u-plane integral [127]

$$Z_u = \int_{u-\text{plane}} da d\bar{a} A(u)^\chi B(u)^\sigma e^{pu+S^2T}\Psi. \tag{5.182}$$

As shown in [131], the functions A and B are related to genus one terms in (5.178) as

$$F^{(1)} = \ln A - \frac{2}{3}\ln B, \qquad G_1 = \frac{1}{3}\ln B \tag{5.183}$$

Note that the equivariant integral of the superfield $\Phi = \Phi^{(0)} + \Phi^{(1)}\theta + \ldots + \Phi^{(4)}\theta^4$ in the case $\mathcal{X} = \mathbb{C}^2$ is given by

$$\int_{\mathcal{X}} d^4x \int d^4\theta \Phi = \frac{\Phi^{(0)}(0)}{\epsilon_1\epsilon_2}. \tag{5.184}$$

[11] there is of course a similar antiholomorphic contribution starting with $\overline{F}^{(1)}(\chi - \frac{3}{2}\sigma)$.

It is also instructive to write down [131] the equivariant Euler number and signature for $\mathbb{C}^2$:

$$\chi(\mathbb{C}^2) = \epsilon_1\epsilon_2, \qquad \sigma(\mathbb{C}^2) = \frac{\epsilon_1^2 + \epsilon_2^2}{3}. \tag{5.185}$$

Let us introduce another notation:

$$\widetilde{F}^{(1)} = 4G_1 + F^{(1)}, \qquad G_1 = \frac{1}{4}(\widetilde{F}^{(1)} - F^{(1)}). \tag{5.186}$$

Then (5.178) can be rewritten as

$$\epsilon_1\epsilon_2 F = F^{(0)} + (\epsilon_1 + \epsilon_2)H_{\frac{1}{2}} + \frac{1}{2}(\chi - \frac{3}{2}\sigma)F^{(1)} + \frac{1}{2}(\chi + \frac{3}{2}\sigma)\widetilde{F}^{(1)} + \epsilon_1\epsilon_2\mathcal{O}(\epsilon_1, \epsilon_2). \tag{5.187}$$

The term $\widetilde{F}^{(1)} = 4G_1 + F^{(1)}$ is not captured by the ordinary topological string!

Extra terms are needed to obtain a correct macroscopic entropy for non-supersymmetric black holes in addition to the standard terms computed by the topological strings [148, 149]. In fact the particular term needed, which is discussed in [149] reduces, upon compactification to $4d$, to the term of the form $t \cdot \mathrm{Tr}\mathcal{R} \wedge \mathcal{R}$ for large t, where t is the overall Kähler moduli of the CY. Such a correction is indeed captured by the leading behavior of $G_1(t)$ for large t, as follows from (5.183). This gives us further confidence about the relevance of Nekrasov's extension of topological strings for a correct accounting of the non-supersymmetric black hole entropy.

In general, as pointed out in [100] one would expect that implementation of Nekrasov's partition function for general Calabi-Yau will mix hypermultiplet and vector multiplets. The case studied in [139] involved the case where there were no hypermultiplets so the question of mixing does not arise. In the context of the conjecture in the next section, this would suggest that higher derivative corrections may also fix the vevs for the hypermultiplet moduli in the context of non-supersymmetric black holes.

We now turn to a minimal conjecture for extremal black hole entropy which uses Nekrasov's extension of topological strings.

5.7.2 Minimal ϵ-deformation

Let us start with a semiclassical expression (5.125 for the $\mathcal{G}^{(0)}$-function

$$\mathcal{G}^{(0)} = \frac{1}{2}(P^I - X^I)(P^J - X^J)\overline{F}^{(0)}_{IJ} + (P^I - X^I)F^{(0)}_I + F^{(0)}, \tag{5.188}$$

where $F^{(0)} = F^{(0)}(X)$ is the Calabi-Yau prepotential, identified with genus zero topological string free energy, and $P^I = p^I + \frac{i}{\pi}\phi^I$. Our goal is to find an ϵ-deformation $\mathcal{G}^{(0)} \to \mathcal{G}$ of (5.188), such that corresponding extremum equations

$$\frac{\partial \mathrm{Im}\mathcal{G}}{\partial \epsilon_1} = \frac{\partial \mathrm{Im}\mathcal{G}}{\partial \epsilon_2} = \frac{\partial \mathrm{Im}\mathcal{G}}{\partial X^I} = 0 \tag{5.189}$$

still admit a supersymmetric attractor solution

$$\epsilon_1 = 1, \quad \epsilon_1 + \epsilon_2 = 0, \quad X^I = P^I_\epsilon = p^I + \frac{i}{\pi}\phi^I, \tag{5.190}$$

and the extremum value of $\mathrm{Im}\mathcal{G}$ computed using this solution is such that it describes correctly corresponding contribution [142] to the supersymmetric black hole entropy

$$-\mathrm{Im}\mathcal{G}_{\mathrm{susy}}(p^I, \phi^I) = -\mathrm{Im}F(p^I + \frac{i}{\pi}\phi^I, 256) = 2\mathrm{Re}F_{\mathrm{top}}(p^I + \frac{i}{\pi}\phi^I). \tag{5.191}$$

We will obtain this deformation of $\mathcal{G}$-function in two steps. First, we will use Nekrasov's refinement of the topological string to deform the prepotential as

$$F^{(0)}(X) \;\to\; F(X^I, \epsilon_1, \epsilon_2), \tag{5.192}$$

and at the same time, motivated from [148], deform the complexified magnetic charge as[12]

$$P^I \;\to\; P^I_\epsilon = -\epsilon_2 p^I + \frac{i}{\pi}\epsilon_1 \phi^I. \tag{5.193}$$

Second, in order to satisfy conditions (5.189)-(5.191) after the deformation (5.192)-(5.193), we will need to add some compensating terms to $\mathcal{G}$. As we will see, there is some freedom in choosing these terms, but there is a minimal choice that does the job.

At the first step, after substituting (5.192)-(5.193) directly into (5.188), we obtain

$$\widetilde{\mathcal{G}} = \frac{1}{2}(P^I_\epsilon - X^I)(P^J_\epsilon - X^J)\overline{F}_{IJ}(\overline{X},\bar\epsilon) + (P^I_\epsilon - X^I)F_I(X,\epsilon) + F(X,\epsilon). \tag{5.194}$$

This, however, is not the full answer, since the derivatives of $\mathrm{Im}\widetilde{\mathcal{G}}$ with respect to ϵ-parameters are not zero on the supersymmetric solution (5.190). This can be corrected at the second step, by adding to $\widetilde{\mathcal{G}}$ two terms, proportional to $\epsilon_1+\epsilon_2$, so that the value (5.191) of the potential is not affected when $\epsilon_1+\epsilon_2=0$. This leads to the following minimal ϵ-deformation

$$\boxed{\begin{aligned}\mathcal{G} = {}& \tfrac{1}{2}(P^I_\epsilon - X^I)(P^J_\epsilon - X^J)\overline{F}_{IJ}(\overline{X},\bar\epsilon) + (P^I_\epsilon - X^I)F_I + F(X,\epsilon) + \\ &+ \tfrac{1}{2}(\epsilon_1+\epsilon_2)\overline{X}^I F_I - \tfrac{1}{2}(\epsilon_1+\epsilon_2)(\epsilon_1\partial_{\epsilon_1} - \epsilon_2\partial_{\epsilon_2})F(X,\epsilon)\end{aligned}} \tag{5.195}$$

We call (5.195) a minimal ϵ-deformation because we can also add to (5.195) any terms proportional to $(\epsilon_1+\epsilon_2)^2$ without affecting conditions (5.189)-(5.191):

$$\mathcal{G} \to \mathcal{G} + \mathcal{O}(\epsilon_1+\epsilon_2)^2. \tag{5.196}$$

[12] When $\epsilon_2 = -\epsilon_1$, this is just a rescaling of P^I, while general deformation with $\epsilon_2 \neq -\epsilon_1$ involves a change of the complex structure in $H^3(M,\mathbb{C})$.

It is straightforward to check, using the homogeneity condition (5.176) and the relations

$$p^I = -\frac{1}{2\epsilon_2}(P^I_\epsilon + \overline{P}^I_\epsilon), \qquad \phi^I = -\frac{i\pi}{2\epsilon_1}(P^I_\epsilon - \overline{P}^I_\epsilon), \tag{5.197}$$

which follow from the definition

$$P^I_\epsilon = -\epsilon_2 p^I + \frac{i}{\pi}\epsilon_1 \phi^I, \tag{5.198}$$

that the extremum equations (5.189) for (5.195) indeed admit a solution (5.190), which corresponds to a supersymmetric BPS black hole. Moreover, in this case (5.191) is also satisfied.

Expression $q_I\phi^I - \pi \mathrm{Im}\mathcal{G}$ should be compared to the entropy function (5.169). Then our notations are related to those of [148] as follows. We identify

$$\epsilon_1 = \frac{16}{|w|^2 v_1}, \quad \epsilon_2 = -\frac{16}{|w|^2 v_2}. \tag{5.199}$$

The supersymmetric attractor equations of [148] read $p^I = -\frac{i}{4}v_2(\overline{w}x^I - w\overline{x}^I)$, while in our conventions the supersummetric case is $p^I = \mathrm{Re}X^I$. Therefore,

$$X^I = -\frac{i}{2}\overline{w}x^I, \qquad x^I = \frac{2i}{\overline{w}}X^I. \tag{5.200}$$

We also set in this case

$$w\overline{w} = 16, \quad v_1 = v_2 = 1. \tag{5.201}$$

5.7.3 Putting it all together

Now we are ready to make a proposal about the extremal black holes entropy. We want to write down a generalization of the semiclassical expression for the extremal black hole degeneracy from section 5, that would reduce to the OSV formula

(5.2) for the supersymmetric charge vector (p^I, q_I). The expression (5.195) for the deformed black hole potential provides a natural way to do this, and allows to treat supersymmetric and non-supersymmetric cases simultaneously.

We introduce a function $\mathcal{G} = \mathcal{G}(p, \phi; X, \epsilon)$ defined by

$$\begin{aligned}\mathcal{G} = &\frac{1}{2}(P^I_\epsilon - X^I)(P^J_\epsilon - X^J)\overline{F}_{IJ}(\overline{X}, \overline{\epsilon}) + (P^I_\epsilon - X^I)F_I(X, \epsilon) + F(X, \epsilon) + \\ &+\frac{1}{2}(\epsilon_1 + \epsilon_2)\overline{X}^I F_I(X, \epsilon) - \frac{1}{2}(\epsilon_1 + \epsilon_2)(\epsilon_1\partial_{\epsilon_1} - \epsilon_2\partial_{\epsilon_2})F(X, \epsilon) + \mathcal{O}(\epsilon_1 + \epsilon_2)^2,\end{aligned} \tag{5.202}$$

where $\mathcal{O}(\epsilon_1 + \epsilon_2)^2$ denotes an ambiguity that cannot be fixed just by requiring that Im$\mathcal{G}$ gives correct description of the supersymmetric black holes. In the minimal deformation case we set $\mathcal{O}(\epsilon_1 + \epsilon_2)^2 = 0$. In general, there are two types of solutions to the extremum equations

$$\frac{\partial}{\partial X^I}\mathrm{Im}\mathcal{G} = \frac{\partial}{\partial \epsilon_i}\mathrm{Im}\mathcal{G} = 0, \tag{5.203}$$

the supersymmetric one (5.190) $X^I = p^I + \frac{i}{\pi}\phi^I$, and non-supersymmetric ones (all other). Let us denote the functions obtained by substituting these non-supersymmetric solutions $X^I = X^I(p, \phi),\ \epsilon_{1,2} = \epsilon_{1,2}(p, \phi)$ into (5.202) as $\mathcal{G}(p^I, \phi^I)$. For supersymmetric solution $\mathcal{G}_{\mathrm{susy}}(p^I, \phi^I) = F(p^I + \frac{i}{\pi}\phi^I)$. We conjecture the following relation for the extremal black hole degeneracy

$$\boxed{\Omega_{\mathrm{extrm}}(p^I, q_I) = \int d\phi^I e^{q_I\phi^I}\left(\left|e^{\frac{i\pi}{2}F(p^I+\frac{i}{\pi}\phi^I)}\right|^2 + \sum_{\mathrm{n-susy}}\left|e^{\frac{i\pi}{2}\mathcal{G}(p^I,\phi^I)}\right|^2\right)} \tag{5.204}$$

which is expected to be valid asymptotically in the limit of large charges. The sum in (5.204) runs over all non-supersymmetric solutions to the extremum equations

(5.203). However, it is expected that for a given set of charges (p^I, q_I) only one solution (supersymmetric or non-supersymmetric, depending on the value of the discriminant) dominates, and contributions from all other solutions, including the ones with non-positive Hessian, are exponentially suppressed.

As noted before, it is expected that for general non-toric Calabi-Yau compactifications, which lead to hypermultiplets, the analog of Nekrasov's partition function would mix hypermultiplets with vector multiplets and therefore will fix their values at the horizon. This would be interesting to develop further.

5.8 Conclusions and Further Issues

We studied the black hole potential describing extremal black hole solutions in $\mathcal{N} = 2$ supergravity and found a new formulation of the semi-classical attractor equations, utilizing homogeneous coordinates on the Calabi-Yau moduli space. This allowed us to solve the inverse problem (that is, express the black hole charges in terms of the attractor Calabi-Yau moduli) completely in the one-modulus Calabi-Yau case. We found three non-supersymmetric solutions in addition to the supersymmetric one. In the higher dimensional case we found a bound $\#_{\mathrm{n-susy}} \leq 2^{n_V+1} - 1$ on the possible number of non-supersymmetric solutions to the inverse problem.

We then investigated a generalization of the attractor equations and OSV formula in the case when other corrections are turned on. We conjectured that corresponding corrected extremal black hole entropy needs an additional ingredient: the Nekrasov's extension of the topological string free energy $F(X^I, \epsilon_1, \epsilon_2)$. We related this to the black hole entropy using a minimal deformation conjecture given in (5.195),(5.204), that reduces to $F_{\text{top}}(X^I + \frac{i}{\pi}\phi^I)$ for the choice of the black hole charges that support a supersymmetric solution. We were unable to fix the $\mathcal{O}(\epsilon_1 + \epsilon_2)^2$ ambiguity in (5.202), though it could be that the minimal conjecture is correct.

One important open question is how to test our conjecture. One possible test may be using the local Calabi-Yau geometry for which Nekrasov's partition function is known. Another important question is to find out what is exactly computed by Nekrasov's partition function[13] and how to extend it to the case where there are both hypermultiplets and vector multiplets. Clearly there is a long road ahead. We hope to have provided strong evidence that Nekrasov's extension of topological string is a key ingredient in a deeper understanding of non-supersymmetric black holes.

[13]for example, in the $AdS_2 \times S^2$ setup of [148]), the ϵ-parameters corresponding to the radii of AdS_2 and S^2 factors were real, but from the topological string viewpoints it is natural to consider a complexification of $\epsilon_{1,2}$. This suggests that there should exist corresponding deformation of the $AdS_2 \times S^2$ near horizon geometry.

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