Superstring Theory in AdS_3 and Plane Waves

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John Sang Won Son

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Shiraz Minwalla Superstring Theory John Sang Won Son
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Abstract

This thesis is devoted to the study of string theory in AdS_3 and its applications to recent developments in string theory. The difficulties associated with formulating a consistent string theory in AdS_3 and its underlying $SL(2,R)$ WZW model are explained. We describe how these difficulties can be overcome by assuming that the $SL(2,R)$ WZW model contains spectral flow symmetry. The existence of spectral flow symmetry in the fully quantum treatment is proved by a calculation of the one-loop string partition function. We consider Euclidean AdS_3 with the time direction periodically identified, and compute the torus partition function in this background. The string spectrum can be reproduced by viewing the one-loop calculation as the free energy of a gas of strings, thus providing a rigorous proof of the results based on spectral flow arguments.

Next, we turn to spacetimes that are quotients of AdS_3, which include the BTZ black hole and conical spaces. Strings propagating in the conical space are described by taking an orbifold of strings in AdS_3. We show that the twisted states of these orbifolds can be obtained by fractional spectral flow. We show that the shift in the ground state energy usually associated with orbifold twists is absent in this case, and offer a unified framework in which to view spectral flow.

Lastly, we consider the RNS superstrings in $AdS_3 \times S^3 \times \mathcal{M}$, where $\mathcal{M}$ may be $K3$ or T^4, based on supersymmetric extensions of $SL(2,R)$ and $SU(2)$ WZW models. We construct the physical states and calculate the spectrum. A subsector of this theory describes strings propagating in the six dimensional plane wave obtained by the Penrose limit of $AdS_3 \times S^3 \times \mathcal{M}$. We reproduce the plane wave spectrum by taking J and the radius to infinity. We show that the plane wave spectrum actually coincides with the large J spectrum at fixed radius, i.e. in $AdS_3 \times S^3$. Relation to some recent topics of interest such as the Frolov-Tseytlin string and strings with critical tension or in zero radius AdS are discussed.

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To my parents

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1. Introduction

The subject of this thesis is string theory on the three-dimensional Anti-de Sitter space, AdS_3, and also on spacetimes that are obtained as deformations of AdS_3. The reasons for studying string theory on AdS_3 are many, each of central importance. In this Introduction we will explain what those reasons are, and also put them in context of string theory in general.

The two biggest achievements of 20th century physics are quantum mechanics and general relativity. Quantum mechanics governs the behavior of atoms and elementary particles, while general relativity is the framework in which to describe massive objects such as planets and galaxies. Each theory in its own region of validity is fantastically successful in explaining the observed phenomena.

However, attempts to unite quantum mechanics and general relativity into a single theory have been met with very little success. The tools of quantum field theory–which deftly unified quantum mechanics and *special* relativity–proved to be inept at doing the same for quantum mechanics and general relativity. Extracting sensible, finite answers to results of scattering experiments involving gravitons, the quanta of gravitational field, proved to be out of reach. Since gravity couples to all forms of matter and energy, this state of affairs was clearly not satisfactory, even though in practice the effects of gravity are so weak that corrections due to gravitons are completely negligible in all scattering processes involving elementary particles.

Currently, the leading candidate for a unified theory of gravity and quantum mechanics is string theory. The elementary object in this theory is a string, which traces out a 2-dimensional *worldsheet* in spacetime[1]. Among the massless excitations of the string is a spin-2 multiplet, which describes the graviton. The graviton appears very naturally in string theory, and is described in a manner similar to the gauge field. So, as far as string theory is concerned, gauge theory of elementary particles and gravity are two sectors in its Hilbert space.

We have just described how string theory contains gauge theory and gravity. However, this is not sufficient to claim that string theory is the ultimate physical theory. String theory must also overcome the divergences associated with graviton interaction. The reason is that divergences in a physical theory signal "new physics", some degrees of freedom at a smaller length scale, that the theory is not equipped to describe. So, if we found that string theory did not give us finite amplitudes for graviton scattering we would have to conclude that there was some other theory that would supersede string theory. Happily, infinities that arise in interactions built out of a 1-dimensional world-line, described by conventional quantum field theory, were found to disappear due to the extra dimension of the string. So string amplitudes are finite, a requirement that must be satisfied by any theory claiming to be a theory that describes everything in our universe.

Since string theory represents a significant departure from conventional quantum field theory, we should be ready to encounter some

[1] What follows is a "traditional" understanding of string theory. Recent developments have indicated that in addition to the 2-dimensional string, there are higher dimensional "branes" present in string theory, and a string does not have a claim to be any more fundamental than the branes. However, we will in this Introduction consider the string to be truly fundamental, as many distinct features of string theory can be understood from this viewpoint.

peculiarities as we dive deeper into string theory. One of these peculiarities is that strings cannot propagate in arbitrary spacetime. In order for a spacetime to be a vacuum of string theory, it must satisfy the requirement of Weyl invariance[2]. The resulting theory on the string worldsheet is then a conformal field theory (CFT). In fact, even the dimension of spacetime is determined by the string itself, and we are not free to arbitrarily add or subtract dimensions without spoiling some consistency of string theory. For the special case of perturbative string theory in flat Minkowski space, the dimension must be ten[3].

If we wish to consider spacetimes that are more complicated than the flat Minkowski space, or for phenomological reasons we wish to consider a spacetime of the form

$$\mathcal{M}_4 \times X \tag{1.1}$$

where $\mathcal{M}_4$ is a four-dimensional Minkowski space, we can consider some of the spatial dimensions to curve into a closed manifold X. Again, string theory does not allow X to be arbitrary. Simple examples of allowed X are products of circles (toroidal compactification), and in more complicated situations X can be a Calabi-Yau manifold, which are important because they give rise to supersymmetry in $\mathcal{M}_4$.

Even though spacetimes such as (1.1) are extremely important because of their immediate application to the present day universe, they leave out an important class of spacetimes—namely, spacetimes in which the time direction is embedded non-trivially. This is our first reason for studying string theory on AdS_3.

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[2] One of the equations in demanding Weyl invariance turns out to be nothing other than Einstein equations. In this way string theory reproduces the field equations of general relativity.

[3] Again, this statement is made in the context of traditional string theory. Recent results from non-perturbative aspects of string theory suggest that in fact the most symmetric vacuum has eleven dimensions [1].

1.1. Time-dependent backgrounds

If string theory is the correct ultimate theory, it must be capable of describing cosmological models and the physics of early universe in particular. Such spacetimes are expected to be extremely curved in the time direction as well as the spatial ones[4]. Thus, we should learn how to do string physics in time-dependent backgrounds. Before we can tackle the extremely complicated cosmological scenarios, we should search for a relative simple example of a time-dependent spacetime.

But as we already mentioned, not any spacetime we can think of is a suitable vacuum of string theory. So we have the complicated task of finding a time-dependent spacetime that also satisfies the requirement of Weyl invariance. This immediately leads to AdS_3, which has a non-trivial coefficient of dt^2 in the metric, as the leading candidate. This is because, as we will explain in the next secion, the worldsheet theory of a string propagating in AdS_3 belongs to a class of theories known as a Wess-Zumino-Witten (WZW) model[5]. It is a fundamental result of WZW models that they are conformally invariant, i.e. they are CFT's. Hence Weyl invariance is satisifed and AdS_3 is an acceptable string vacuum.

Let us now turn to a discussion of some potential problems we might come across in trying to formulate string theory in time-dependent backgrounds. The immediate problem we are faced with is that of unitarity. To explain this, let us consider strings in ten dimensional Minkowski space. The string worldsheet, parametrized by

[4] It is also possible that the early universe underwent a discontinuous process, for example through tachyon condensation.

[5] Actually, AdS_3 without any flux is not described by a WZW model. We will explain this point in the next section.

(σ, τ), is mapped to spacetime by the fields $X^\mu(\sigma, \tau)$. Then we perform canonical quantization whereby each field is expanded in terms of Fourier modes

$$X^\mu = x^\mu - i\frac{\alpha'}{2}p^\mu \ln|z|^2 + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m\neq 0} \frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m} + \frac{\tilde{\alpha}_m^\mu}{\bar{z}^m}\right) , \quad (1.2)$$

where the complex coordinates $z, \bar{z}$ are defined by

$$\begin{aligned} z &= \sigma + i\tau \\ \bar{z} &= \sigma - i\tau \ , \end{aligned} \quad (1.3)$$

denoting the holomorphic (left-moving) and anti-holomorphic (right-moving) coordinates on the string, respectively. The canonical commutation relations give

$$\begin{aligned} [x^\mu, p^\nu] &= i\eta^{\mu\nu} \\ [\alpha_m^\mu, \alpha_n^\nu] &= m\eta^{\mu\nu}\delta_{m,-n} \ , \end{aligned} \quad (1.4)$$

where $\eta^{\mu\nu}$ is the Minkowski (mostly plus) metric. The first relation is familiar from quantization of point particles, while the second relation represents the higher modes on the string. The important point is that the commutator of oscillators along the time direction has a negative sign, and such oscillators will create states with negative norm. When we consider a more complicated background, we replace $\eta^{\mu\nu}$ with $g^{\mu\nu}$ but the argument proceeds in a similar manner and we again find that there are states with negative norm.

So it appears that the CFT spectrum is not unitary. How can we be certain that the resulting string spectrum is unitary? In flat space, it is well understood how the ghosts (negative-norm states) are eliminated. Among the many ways to understand this, the most intuitive one involves the use of lightcone coordinates

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^1) \; . \tag{1.5}$$

After gauge fixing, details of which can be found in [2], the independent degrees of freedom are identified with the eight transverse fields X^i, $i = 2, \ldots 9$, which have the correct sign in the canonical commutation relations (1.4) and do not create ghosts. Unitarity is then proved by showing that the Hilbert space obtained in the lightcone gauge is the same as obtained in other quantization schemes. This is the statement of the no-ghost theorem in flat space.

It is clear that the lightcone quantization will not work when the time direction is curved. Hence, we will have to resort to the so-called *covariant quantization* in AdS_3, where first we construct the Hilbert space as the Fock space of all oscillators, and then eliminate the ghosts via the Virasoro constraints

$$\begin{aligned} (L_0 - 1)|\text{physical}\rangle &= 0 \\ L_n|\text{physical}\rangle &= 0 \; , n \geq 1 \; , \end{aligned} \tag{1.6}$$

which is a consequence of worldsheet reparametrization invariance. It is an important test of string theory that (1.6) are sufficient to remove all ghosts from the physical spectrum.

Another important challenge we face in time-dependent backgrounds is that the CFT will be non-compact. Much of the powerful tools that are useful in understanding compact CFT's become difficult to handle for non-compact CFT's. This is why, despite being a WZW model, the CFT of strings in AdS_3 remained a difficult problem for a long time.

1.2. Relation to black hole physics

The next reason for studying string theory on AdS_3 is its intricate relation to black hole physics. As we shall see, AdS_3 shows up repeatedly in discussion of black holes.

The first connection between AdS_3 and black holes is that a black hole can be obtained by taking a quotient of AdS_3. This is the famous BTZ black hole [3]. In taking a quotient, there can be singular points corresponding to the fixed points of the identification. A string can propagate freely in regions where there are no fixed points, or it can be attached to such points, giving rise to the "twisted" states. This describes what is known as the orbifold. So, by taking an orbifold of string theory on AdS_3 we obtain string theory on the BTZ black hole—example of an exact description of strings propagating in a black hole background!

Another relation between string theory on AdS_3 and black holes is that by taking a coset of the worldsheet CFT on AdS_3, one finds a theory describing a two-dimensional black hole [4]. This black hole has a Euclidean metric and looks like a semi-infinite cigar, and was the first example of a black hole in string theory.

AdS_3 also appears in string theory computation of black hole entropy, an important topic that any theory claiming to be a quantum theory of gravity must address. As it turns out, every black hole whose entropy has been counted in string theory so far has in its near-horizon geometry an AdS_3 factor [5]. For example, in the famous five-dimensional Strominger-Vafa black hole [6], which was the setting for the first entropy computation, the near-horizon geometry is locally

$$AdS_3 \times S^3 \times \mathcal{M} , \tag{1.7}$$

where $\mathcal{M}$ is T^4 or $K3$. Brown-Henneaux showed that quantum gravity on AdS_3 is a conformal field theory [7], and the cetral charge of this conformal field theory determines the entropy through Cardy's formula. As shown in [8], this is sufficient to reproduce the entropy of the blackhole. This is a satisfying result since the black hole's entropy is coming from the degrees of freedom near the horizon, even though exactly what those degrees of freedom are remains to be understood.

1.3. AdS/CFT

Finally, we come to the important topic of AdS/CFT correspondence [9,10,11,12], which is a duality between string theory on AdS and a conformal field theory in one lower dimension[6]. For many applications of this duality the CFT can be thought of as living on the boundary of AdS. The most important case in terms of applications to everyday physics we observe at present is the AdS_5/CFT$_4$ correspondence, in which the CFT$_4$ is the Yang-Mills theory in four dimensions with four supersymmetries. This theory is expected to yield much insight into physics of four-dimensional gauge theories, which has QCD as an important example.

However, string theory on AdS_5 remains unsolved due to the presence of Ramond-Ramond (R-R) flux. Solving for the string spectrum in AdS_5 appears to be beyond our grasp at this point and the low energy supergravity (which only describes the massless string excitations) approximation has been used for most part.

[6] That gravity in a D dimensional spacetime can be described by a $D-1$ dimensional theory without gravity goes by the name "holographic principle" [13]. Although AdS/CFT correspondence so far is the only explicit example, it is believed that the holographic principle holds in general.

This is where AdS_3 comes in. As the only example of AdS that has been solved exactly, it can serve as a guide in understanding the more complicated cases. In particular, AdS_3/CFT_2 can be studied in a string theoretic setting without making approximations. Additionally, a distinguishing feature of AdS_3/CFT_2 is that two dimensional CFT's have an infinite dimensional conformal symmetry, allowing for more analytic control of the theory. This has led to a better understanding of the AdS_3/CFT_2 duality [14,15] than others[7].

1.4. Outline

This thesis will be focused on many of the issues address above. It is organized as follows. In Section 2, we explain the geometric features of AdS_3 needed for our discussion. We also explain why string propagation in AdS_3 is described by the $SL(2,R)$ WZW model, paying close attention to how current algebra gives rise to conformal symmetry. In Section 3 we explain the unitarity problem in AdS_3 and how it is resolved by the proposal of Maldacena and Ooguri [16]. The presence of long strings in the spectrum is also discussed. Sections 4 and 5 constitute a string theoretic proof of the spectrum. The proof consists of first computing the one-loop partition function on thermal AdS_3, and then checking that it agrees with the free energy of string states in Lorentzian AdS_3. Due to the non-compact nature of the underlying CFT, some features not seen in compact CFT's are present in the partition function. We give physical interpretations of these features and explain how they are appropriate for AdS_3.

[7] It is interesting to note that actually, the aforemonetioned work by Brown and Henneaux [7] was the first to propose a duality between quantum gravity in AdS_3 and a 2-dimensional CFT, an insight gained precisely because of the infinite dimensional conformal symmetry.

In Section 6, we describe the general solutions of three-dimensional gravity with negative cosmological constant, which are also solutions of string theory. Besides AdS_3, they are the BTZ black hole and conical spaces. String theory in the conical space is constructed as an orbifold of AdS_3.

Finally, the last topic is the relationship between string theory on $AdS_3 \times S^3$ and its plane wave limit. In Section 7 we consider the supersymmetric strings in $AdS_3 \times S^3$. Section 8 through 11 explain in detail how strings in the plane wave emerge in the double scaling limit. We conclude in section 12 and 13 with attempts to understand what the plane wave might teach us about $AdS \times S$ in other dimensions.

2. Geometry of AdS_3 and WZW models

In this Chapter we explain the geometry of AdS_3, and also explain why string theory on AdS_3 is described by a WZW model.

AdS_3 is the hyperboloid

$$R^2 = X_{-1}^2 + X_0^2 - X_1^2 - X_2^2 \tag{2.1}$$

embedded in $R^{2,2}$, with metric

$$ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + dX_2^2 \ , \tag{2.2}$$

which makes manifest the $SO(2,2) \cong SL(2,R) \times SL(2,R)$ isometry. A convenient solution to (2.1) is

$$\begin{aligned} X_{-1} &= R\cos t\cosh\rho \\ X_0 &= R\sin t\cosh\rho \\ X_1 &= R\cos\phi\sinh\rho \\ X_2 &= R\sin\phi\sinh\rho \ , \end{aligned} \tag{2.3}$$

which gives for the metric

$$ds^2 = R^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\phi^2) \ . \tag{2.4}$$

This coordinate system is called the global coordinates in AdS_3, because by setting $\rho \geq 0$ and $2\pi > t > 0$ it covers the entire hyperboloid[8]. As the way it stands, there is a closed time like curve generated by

[8] In addition to the global coordinates, there is another popular choice known as the Poincare coordinates. However, this coordinate system only covers a patch of the AdS_3 described by (2.4), and in addition contains a coordinate horizon.

$t \to t + 2\pi$. We will always consider the universal covering of the hyperboloid by unwrapping the t coordinate so that $-\infty < t < \infty$.

Suppose we wish to consider string theory on some spacetime whose metric is $g_{\mu\nu}$. The Polyakov action is

$$S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{-\gamma}\gamma^{ab} g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \ , \tag{2.5}$$

where γ_{ab} is the worldsheet metric. We might at first take this to be our starting point for string theory on AdS_3, with the metric given in (2.4). However, as we have already emphasized in the Introduction, requirements for an acceptable string vacuum are rather stringent and as it turns out (2.5) will not work. The correct procedure is to turn on some NS-NS two-form field B. It is not obvious how this comes about, so let us explain this.

For the following discussion, it will be convenient to normalize the coordinates so that $R = 1$. Now consider the matrix

$$g = \begin{pmatrix} X_{-1} + X_1 & X_0 - X_2 \\ -X_0 - X_2 & X_{-1} - X_1 \end{pmatrix} \ , \tag{2.6}$$

which is an element of $SL(2,R)$. As a group manifold, $SL(2,R)$ is identified with AdS_3. The metric on $SL(2,R)$

$$dt^2 = \mathrm{tr}(dg^{-1}dg) \ , \tag{2.7}$$

coincides with (2.4). There is a natural action one can write down when the target space is a group manifold. It is the nonlinear sigma model action

$$S \sim \int d\tau d\sigma \ \mathrm{tr}(\partial^\mu g^{-1}\partial_\mu g) \ , \tag{2.8}$$

which is however not conformally invariant (the action (2.8) essentially reproduces (2.5)). Witten studied the beta function of (2.8) and found that upon the addition of the Wess-Zumino term

$$\Gamma_{WZ} = \frac{ik}{12\pi} \int \mathrm{tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg) \ , \tag{2.9}$$

where the integral is over a three-dimensional surface whose boundary is the string worldsheet, the resulting action posesses conformal symmetry [17].

The quantity k that appears in the above expression is known as the level or grade of the WZW model. Since the overall action has a factor of k in front we interpret it as being proportional to R^2. The level is quantized for compact groups, in order to ensure that under a large coordinate transformation (i.e. a transformation not connected to the identity) the action only changes by $S \to S + 2\pi i$. This condition is necessary for the path integral to be well defined. For non-compact groups such as $SL(2, R)$, k need not be quantized.

Locally, the Wess-Zumino term is a total derivative, so it can be written as a two-dimensional integral over the worldsheet coordinates. In terms of the target space variables the total action then can be written

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{-\gamma} \left(\gamma^{ab} g_{\mu\nu} + i\epsilon^{ab} B_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu \ , \tag{2.10}$$

revealing that in addition to the metric there is an antisymmetric field present. So in order to satisfy conformal invariance we must have some B field present. The reason why we know it is a NS-NS field is that fields from the Ramond sector do not couple to the string worldsheet.

The presence of NS-NS B field in this background will have a profound impact on the string spectrum. The most drastic effect is the possibility of having “long strings” [18,19]. We will discuss this in more detail when we consider the representations of $\widehat{SL}(2,R)$.

How does the WZW model automatically satisfy conformal invariance? The reason is the underlying *current algebra*. The WZW action is invariant under the action of independent left and right action by group elements[9]. This implies the existence of two sets of conserved currents

$$\begin{aligned} K^a &= \mathrm{tr}(t^a \partial g g^{-1}) \\ \bar{K}^a &= \mathrm{tr}(t^{*a} g^{-1} \bar{\partial} g) \end{aligned} \tag{2.11}$$

where we have switched to the complex coordinates introduced in (1.3), and t^a are the generators of $SL(2,R)$. This notation is a very useful reminder of the fact that the equation of motion simply forces K^a and $\bar{K}^b$ to be holomorphic and anti-holomorphic, respectively.

Let us focus our attention on the holomorphic sector. We can introduce the modes of the current by the Laurent expansion

$$K^a(z) = \sum_{n \in Z} \frac{K^a_n}{z^{n+1}} \ , \tag{2.12}$$

which satisfy the $\widehat{SL}(2,R)$ current algebra

$$\begin{aligned} \left[K^+_m, K^-_n\right] &= -2K^3_{m+n} + km\delta_{m+n} \\ \left[K^3_m, K^\pm_n\right] &= \pm K^\pm_{m+n} \\ \left[K^3_m, K^3_n\right] &= -\frac{k}{2} m \delta_{m+n} \ . \end{aligned} \tag{2.13}$$

The zero modes represent the integral of currents, i.e. they are conserved charges. It is convenient to choose a basis in which

$$\begin{aligned} K^3_0 &= \frac{1}{2}(E+L) \\ \bar{K}^3_0 &= \frac{1}{2}(E-L) \end{aligned} \tag{2.14}$$

[9] The addition of the Wess-Zumino term is what makes this possible.

where E and L represent the energy and angular momentum in AdS_3, respectively.

The current algebra is the key property of WZW models. We can construct the operators

$$L_n = \frac{1}{k-2} \sum_{m=-\infty}^{\infty} : \eta_{ab} K^a_m K^b_{n-m} : , \tag{2.15}$$

where η_{ab} is the metric on $SL(2,R)$ with signature $(+,+,-)$. The generators (2.15) obey the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \tag{2.16}$$

with central charge

$$c = \frac{3k}{k-2} , \tag{2.17}$$

and also

$$[L_n, K^a_m] = -mK^a_{n+m} . \tag{2.18}$$

The Virasoro algebra is the algebra of conformal symmetry. Hence the presence of current algebra implies, via (2.15), conformal symmetry.

3. Algebraic construction of bosonic strings on AdS_3

We now focus on the $SL(2,R)$ WZW model, which is the world-sheet CFT of string theory in AdS_3. Since the model possesses two copies of $\widehat{SL}(2,R)$ symmetry (one on the left and one on the right), the Hilbert space is a sum of products of $\widehat{SL}(2,R)$ representations. The question is which representations appear.

Representations of current algebra can be constructed by considering representations of the global algebra, generated by the zero modes of the currents K_0^a, to be the primary states annhiliated by $K_{m>0}^a$ (note that (2.18) implies that K_0^a generate a multiplet with the same L_0 eigenvalue). Then $K_{m<0}^a$ can be applied to these states, generating the representation of the current algebra. Hence, the first problem is to find the right representations of $SL(2,R)$. In [16], the following was proposed, based on an analysis of the point particle limit. The representations of $SL(2,R)$ that appear are $\mathcal{D}_\ell$ and $C_{\ell,\alpha}$, where $\mathcal{D}_\ell$ is the discrete lowest weight representation

$$\mathcal{D}_\ell = \{|\ell, n\rangle : n = \ell, \ell+1, \ell+2, \ldots\} , \qquad (3.1)$$

with $K_0^-|\ell,\ell\rangle = 0$. The representation is labeled by the value of the quadratic Casimir

$$\left(\frac{1}{2}(K_0^+K_0^- + K_0^-K_0^+) - (K_0^3)^2\right)|\ell,n\rangle = -\ell(\ell-1)|\ell,n\rangle , \qquad (3.2)$$

and n which is the eigenvalue of K_0^3, related to the spacetime energy by (2.14). The representation is unitary for real ℓ greater than zero [20]. However, we need that the wavefunctions are square integrable, and this requires $\ell > 1/2$.

$C_{\ell,\alpha}$ is the continuous representation

$$C_{\ell,\alpha} = \{|\ell, n, \alpha\rangle : n = \alpha, \alpha \pm 1, \alpha \pm 2, \ldots\} , \qquad (3.3)$$

where without loss of generality α may be restricted to $0 \leq \alpha < 1$. Unitarity requires $\ell = 1/2 + is$ with s real [20]. This gives for the quadratic Casimir

$$\left(\frac{1}{2}(K_0^+ K_0^- + K_0^- K_0^+) - (K_0^3)^2\right) |\ell, n, \alpha\rangle = \left(\frac{1}{4} + s^2\right) |\ell, n, \alpha\rangle \ . \quad (3.4)$$

Now starting with the above representations of $SL(2,R)$, representations of $\widehat{SL}(2,R)$ are generated by applying $K^a_{m<0}$. The resulting representations are denoted

$$\hat{\mathcal{D}}_\ell \ , \quad \hat{C}_{\ell,\alpha} \ . \quad (3.5)$$

However, we will now explain that based on these representations alone the resulting string theory would be fatally flawed.

The issue is unitarity, which we explained in the Introduction. The no ghost theorem for AdS_3 [21,22,23,24,25,26,27,28] restricts the value of ℓ in the discrete representations to be less than $k/2$. Let us look at the consequence of this restriction, by considering string theory on

$$AdS_3 \times \mathcal{X} \ . \quad (3.6)$$

We assume that the CFT on $\mathcal{X}$ is unitary, and that it has the right central charge to form, together with AdS_3, a critical string theory. The Virasoro operators are given by the sum of the Virasoro operators for each CFT, $L_m = L_m^{SL} + L_m^{\mathcal{X}}$. Consider a state in the discrete representation of $SL(2,R)$ WZW model, tensored with a state from $\mathcal{X}$ with conformal weight h. The combined state is labeled as

$$|\ell, n, N, h\rangle \ , \quad (3.7)$$

where N is the level[10] of the $\widehat{SL}(2,R)$ descendent, i.e. the conformal weight of (3.7) is

$$L_0 = -\frac{\ell(\ell-1)}{k-2} + N + h \ . \tag{3.8}$$

By the Virasoro condition (1.6) this must equal one, otherwise the state is not physical. But note that this means there is an upper bound to how large N can be, which follows directly from the upper bound on ℓ. Since N is related to the mass of the string state, we are forced to conclude that the tower of string excitations abruptly comes to an end. This sounds very unphysical—for example, it is hard to see how modular invariance, a key requirement of string theory, would be maintained.

There is an additional problem, which was only realized fairly recently. We mentioned that the background we are considering has the NS-NS B field turned on. The effect of this field is to expand the string, while the tension wants to contract the string. Since in AdS_3 the volume and area grow at the same rate asymptotically, these two effects almost completely cancel, and long strings can freely propagate far from the origin of AdS_3 [18,19]. So, we expect to find in the Hilbert space of string theory on AdS_3 states in the continuous representation of $\widehat{SL}(2,R)$. The problem is that all the states in $\hat{C}_{\ell,\alpha}$ are tachyonic, which can easily be seen using (1.6). When we consider the superstrings, such states get projected out and there would not be any long strings in the spectrum.

Maldacena and Ooguri proposed a solution [16] that solved both of these problems. Their suggestion was that (3.5) are not the only representations of $\widehat{SL}(2,R)$ that appear in the Hilbert space. There are

---

[10] Not to be confused with k that appears in the WZW action, which is also called a level.

additional representations that are generated by the action of spectral flow

$$\begin{aligned} K^3_m &\to \tilde{K}^3_m = K^3_m - \frac{k}{2} w \delta_{m,0} \\ K^+_m &\to \tilde{K}^+_m = K^+_{m+w} \\ K^-_m &\to \tilde{K}^-_m = K^-_{m-w} \ , \end{aligned} \tag{3.9}$$

and the resulting transformation on the Virasoro generators

$$\tilde{L}_m = L_m + w K^3_m - \frac{k}{4} w^2 \delta_{m,0} \ . \tag{3.10}$$

For each integer valued spectral flow, we generate the representations $\hat{\mathcal{D}}^w_\ell$ and $\hat{C}^w_{\ell,\alpha}$ from $\hat{\mathcal{D}}_\ell$ and $\hat{C}_{\ell,\alpha}$, respectively.

We return to (3.6) and see how this proposal overcomes the difficulties explained above. First, consider a state in the spectral flowed discrete representation $\hat{\mathcal{D}}^w_\ell$,

$$|\tilde{\ell}, \tilde{n}, \tilde{N}, w, h\rangle \tag{3.11}$$

Let us denote this state by $|\Omega\rangle$. Taking into account the spectral flow relations (3.9) and (3.10), the Virasoro constraints are

$$\begin{aligned} (L_0 - 1)|\Omega\rangle &= \left(-\frac{\tilde{\ell}(\tilde{\ell}-1)}{k-2} + \tilde{N} - w\tilde{n} - \frac{kw^2}{4} + h - 1 \right) |\Omega\rangle = 0 \\ L_m|\Omega\rangle &= (\tilde{L}^{SL}_m - w\tilde{K}^3_m + L^{\mathcal{X}}_m)|\Omega\rangle = 0 \ , \qquad m \geq 1 \ . \end{aligned} \tag{3.12}$$

For discrete representations, $\tilde{n} = \tilde{\ell} + q$, with q an integer. Using this relation with the first equation in (3.12), $\tilde{\ell}$ is determined to be

$$\tilde{\ell} = \frac{1}{2} - \frac{k-2}{2} w + \sqrt{\frac{1}{4} + (k-2)\left(N + h - \frac{1}{2} w(w+1) - 1 \right)} \ , \tag{3.13}$$

where N is the level measured by L_0, related to $\tilde{N}$ by $N = \tilde{N} - wq$. We impose the level matching condition $L_0 = \bar{L}_0$, and find the spacetime energy (2.14)

$$E = 1 + 2w + q + \bar{q} + \sqrt{1 + 4(k-2)\left(N_w + h - \frac{1}{2}w(w+1) - 1\right)} \ . \tag{3.14}$$

Note that the energy is discrete, even though ℓ took on continuous values in the $SL(2,R)$ WZW model.

Spectral flow by -1 gives the charge conjugated representations, $\hat{\mathcal{D}}_\ell^{+,w=-1} = \hat{\mathcal{D}}_{k/2-\ell}^{-}$, where the subscript minus (plus) indicates that it is a lowest (highest) weight representation. In all the discrete representations the $SL(2,R)$ spin must be in the range

$$\frac{1}{2} < \ell < \frac{k-1}{2} \ , \tag{3.15}$$

which is more restrictive than what is allowed by the no-ghost theorem. In the context of string theory on AdS_3, these representations correspond to the short strings that are trapped inside AdS_3.

For states coming from the continuous representations, we can proceed in a similar manner to obtain their spectrum. The difference in this case is that $\tilde{\ell}$ and $\tilde{n}$ are not related. The result is

$$E = \frac{kw}{2} + \frac{1}{w}\left(\frac{2s^2 + \frac{1}{2}}{k-2} + \tilde{N} + \tilde{\bar{N}} + h + \bar{h} - 2\right) \ . \tag{3.16}$$

Note that this time the level is measured by $\tilde{L}_0$. The spectrum is continous and s represents the momentum of the string in the radial direction of AdS_3. These are the long strings that can approach arbitrarily close to the boundary.

In conclusion, we may summarize the Hilbert space of $SL(2,R)$ WZW model by

$$\mathcal{H}_{SL} = \oplus_{w=-\infty}^{\infty} \left[\left(\int_{\frac{1}{2}}^{\frac{k-1}{2}} d\ell \, \hat{\mathcal{D}}_{\ell}^{w} \otimes \hat{\mathcal{D}}_{\ell}^{w} \right) \oplus \left(\int_{\frac{1}{2}+i\mathbf{R}} d\ell \int_{0}^{1} d\alpha \, \hat{C}_{\ell,\alpha}^{w} \otimes \hat{C}_{\ell,\alpha}^{w} \right) \right] , \tag{3.17}$$

and the string Hilbert space is obtained via the Virasoro constraints. With this spectrum the fictitious upper bound on the excitation level of the string is removed, as it can be shown from (3.14) and (3.16) that when a short string saturates the bound, it turns into a long string [16]. Also, there are now continuous representations that survive the projection.

4. Partition function on thermal AdS_3

In the previous section we described the current algebra approach to string theory on AdS_3. We have seen how the proposal of Maldacena and Ooguri [16] to include spectral flowed representations of $\widehat{SL}(2,R)$ produces a very sensible spectrum, which included states corresponding to long strings as well as the short strings. The seemingly arbitrary upper bound on the mass of the string was removed, thus recovering the infinite tower of masses on expects from string theory.

However, the existence of spectral flow as a symmetry of the $SL(2,R)$ WZW model was inferred on the basis of classical and semi-classical analysis. It is crucial to check in an independent manner the details that become important at finite values of k, where the intuition gained from semi-classical reasonings can break down. For example, the restriction on the $SL(2,R)$ spin (3.15) becomes trivial in the semi-classical limit, and we would like to derive this result from a fully quantum treatment.

In this and the next sections we verify the results of previous section by an explicit calculation of the one-loop string partition function. As shown in [15], the Euclidean black hole background is equivalent to the thermal AdS_3 background. So we will consider string theory on AdS_3 at a finite temperature, which is described by strings moving on a Euclidean AdS_3 background with the Euclidean time identified. The calculation of the partition function for this geometry is a minor variation on the calculation of Gawedzki in [29]. From this we can read off the spectrum of the theory in Lorentzian signature by interpreting the result as the free energy of a gas of strings.

This section is devoted to the calculation of the one-loop partition function on thermal AdS_3. First we explain the relation between various useful coordinate systems. Then we consider thermal

$AdS_3 = H_3/Z$ and show how the identification of Euclidean time in the global coordinates translates into particular boundary conditions for the target space fields. The partition function is then calculated by an explicit evaluation of the functional integral following [29].

4.1. Thermal AdS_3

The natural metric on H_3 is given by

$$ds^2 = \frac{k}{y^2}(dy^2 + dwd\bar{w}), \tag{4.1}$$

which is the Euclidean continuation of the Poincare metric on AdS_3. By the coordinate transformation

$$\begin{aligned} w &= \tanh\rho e^{t+i\theta} \\ \bar{w} &= \tanh\rho e^{t-i\theta} \\ y &= \frac{e^t}{\cosh\rho} \end{aligned} \tag{4.2}$$

we obtain the cylindrical coordinates on Euclidean AdS_3,

$$\frac{ds^2}{k} = \cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\theta^2. \tag{4.3}$$

For the purpose of calculating the partition function, however, it is convenient to use coordinates in which the metric reads [29]

$$\frac{ds^2}{k} = d\phi^2 + (dv + vd\phi)(d\bar{v} + \bar{v}d\phi), \tag{4.4}$$

which corresponds to the parametrization of H_3 as

$$g = \begin{pmatrix} e^{\phi}(1+|v|^2) & v \\ \bar{v} & e^{-\phi} \end{pmatrix} \tag{4.5}$$

The coordinate transformation from (4.3) to (4.4) is

$$
\begin{aligned}
v &= \sinh\rho e^{i\theta} \\
\bar{v} &= \sinh\rho e^{-i\theta} \\
\phi &= t - \log\cosh\rho \quad .
\end{aligned}
\tag{4.6}
$$

Thermal AdS_3 is given by the identification

$$
t + i\theta \sim t + i\theta + \hat{\beta} \ , \tag{4.7}
$$

where $\hat{\beta}$ represents the temperature T and the imaginary chemical potential $i\mu$ for the angular momentum,

$$
\hat{\beta} = \beta + i\mu\beta = \frac{1}{T} + i\frac{\mu}{T} \ . \tag{4.8}
$$

The corresponding identifications in the coordinates (4.4) are

$$
\begin{aligned}
v &\sim v e^{i\mu\beta} \\
\bar{v} &\sim \bar{v} e^{-i\mu\beta} \\
\phi &\sim \phi + \beta \ ,
\end{aligned}
\tag{4.9}
$$

which is a consistent symmetry of the WZW action,

$$
S = \frac{k}{\pi} \int d^2z \left(\partial\phi\bar{\partial}\phi + (\partial\bar{v} + \partial\phi\bar{v})(\bar{\partial}v + \bar{\partial}\phi v) \right) \ . \tag{4.10}
$$

4.2. Computation of the partition function on thermal AdS_3.

We consider a conformal field theory on a worldsheet torus with modular parameter τ ($z \sim z + 2\pi \sim z + 2\pi\tau$). The two-dimensional conformal field theory on the worldsheet is the sum of three parts: the conformal field theory on H_3, the internal conformal field theory on $\mathcal{M}$, and the (b, c) ghosts. First we start with the computation of the partition function for the conformal field theory describing the three dimensions of thermal AdS_3 and then we will multiply the result by the partition function of the ghosts and the internal conformal field theory.

Due to the identification (4.9), the string coordinates now satisfy the following boundary conditions

$$\begin{aligned} \phi(z+2\pi) &= \phi(z) + \beta n \\ \phi(z+2\pi\tau) &= \phi(z) + \beta m \\ v(z+2\pi) &= v(z) e^{in\mu\beta} \\ v(z+2\pi\tau) &= v(z) e^{im\mu\beta} \ . \end{aligned} \tag{4.11}$$

The thermal circle is non-contractible and therefore we get two integers (n, m) characterizing topologically nontrivial embeddings of the world-sheet in spacetime. In order to implement these boundary conditions it is convenient to define new fields $\hat{\phi}, \hat{v}$ such that they are periodic:

$$\begin{aligned} \phi &= \hat{\phi} + \beta f_{n,m}(z,\bar{z}) \\ v &= \hat{v} \exp(i\mu\beta f_{n,m}(z,\bar{z})) \ , \end{aligned} \tag{4.12}$$

with

$$f_{n,m}(z,\bar{z}) = \frac{i}{4\pi\tau_2} \left[z(n\bar{\tau} - m) - \bar{z}(n\tau - m) \right] \ . \tag{4.13}$$

When we substitute this into the action (4.10), we get

$$S = \frac{k\beta^2}{4\pi\tau_2} |n\tau - m|^2 + \frac{k}{\pi} \int d^2z \left(|\partial\hat{\phi}|^2 + \left| \left(\partial + \frac{1}{2\tau_2} U_{n,m} + \partial\hat{\phi} \right) \hat{\bar{v}} \right|^2 \right), \tag{4.14}$$

where

$$U_{n,m}(\tau) = \frac{i}{2\pi} (\beta - i\mu\beta)(n\bar{\tau} - m). \tag{4.15}$$

We are interested in the functional integral

$$\mathcal{Z}(\beta,\mu;\tau) = \int \mathcal{D}\phi \mathcal{D}v \mathcal{D}\bar{v} e^{-S} \ . \tag{4.16}$$

This integral is evaluated as explained in [29]. We can first do the integral over $\hat{v}, \hat{\bar{v}}$ which is quadratic, giving the determinant

$$\det \left| \partial + \frac{1}{2\tau_2} U_{n,m} + \partial \hat{\phi} \right|^{-2} . \tag{4.17}$$

We calculate the $\hat{\phi}$ dependence on the determinants by realizing that we can view (4.17) as an inverse of two fermion determinants. We can then remove $\hat{\phi}$ from the determinants by a chiral gauge transformation and using the formulas for chiral anomalies. The result is

$$\det \left| \partial + \frac{1}{2\tau_2} U_{n,m} + \partial \hat{\phi} \right|^{-2} = e^{\frac{2}{\pi} \int d^2 z \partial \hat{\phi} \bar{\partial} \hat{\phi}} \ \det \left| \partial + \frac{1}{2\tau_2} U_{n,m} \right|^{-2} . \tag{4.18}$$

The remaining integral over $\hat{\phi}$ gives the usual result for a free boson, except that $k \to k-2$ due to (4.18). The functional integral for the thermal AdS_3 partition function then gives

$$\begin{aligned} \mathcal{Z}(\beta,\mu;\tau) = & \frac{\beta(k-2)^{\frac{1}{2}}}{8\pi\sqrt{\tau_2}} \sum_{n,m} \frac{1}{|\sin(\pi U_{n,m})|^2} \\ & \times \frac{e^{-k\beta^2 |m-n\tau|^2/4\pi\tau_2 + 2\pi(\mathrm{Im} U_{n,m})^2/\tau_2}}{|\prod_{r=1}^{\infty}(1-e^{2\pi i r\tau})(1-e^{2\pi i r\tau + 2\pi i U_{n,m}})(1-e^{2\pi i r\tau - 2\pi i U_{n,m}})|^2} \\ & = \frac{\beta(k-2)^{\frac{1}{2}}}{2\pi\sqrt{\tau_2}} (q\bar{q})^{-\frac{3}{24}} \sum_{n,m} \frac{e^{-k\beta^2 |m-n\tau|^2/4\pi\tau_2 + 2\pi(\mathrm{Im} U_{n,m})^2/\tau_2}}{|\vartheta_1(\tau, U_{n,m})|^2}, \end{aligned} \tag{4.19}$$

where ϑ_1 is the elliptic theta function and $q = e^{2\pi i\tau}$. The factor $\beta(k-2)^{\frac{1}{2}}$ comes from the length of the circle in the ϕ direction. This partition function is explicitly modular invariant after summing over (n,m). In Appendix B of [16], there was a puzzle about the apparent lack of modular invariance of the $SL(2,R)$ partition functions with J^3 insertions. Here we have found that, if we introduce the twist by considering the physical set-up of thermal AdS_3, the result (4.19) turns out to be manifestly modular invariant. This resolves the puzzle raised in [16].

We also need to include the contribution of the (b, c) ghosts and the internal CFT. Partition function of the latter will be of the form

$$\mathcal{Z}_{\mathcal{M}} = (q\bar{q})^{-\frac{c_{int}}{24}} \sum_{h,\bar{h}} D(h, \bar{h}) q^h \bar{q}^{\bar{h}}, \tag{4.20}$$

where $D(h, \bar{h})$ is the degeneracy at left-moving weight h and right-moving weight $\bar{h}$, and c_{int} the central charge of the internal CFT. Modular invariance requires that $h - \bar{h} \in Z$, a fact which will be useful in the next section. Vanishing of the total conformal anomaly gives

$$c_{SL(2,R)} + c_{int} = 26 \ . \tag{4.21}$$

We can calculate now the total contribution to the ground state energy. We found a ground state energy of $-3/24$ in (4.19), the ghosts contribute with $2/24$ and the internal CFT with $-c_{int}/24 = (c_{SL(2,R)} - 26)/24$. Using $c_{SL(2,R)} = 3 + \frac{6}{k-2}$, we find the overall factor

$$(q\bar{q})^{-(1+c_{int})/24} = e^{4\pi\tau_2(1-\frac{1}{4(k-2)})} \ . \tag{4.22}$$

Note that $c_{int} \geq 0$, $k > 2$, and (4.21) imply that there will always be a tachyon in the bosonic theory.

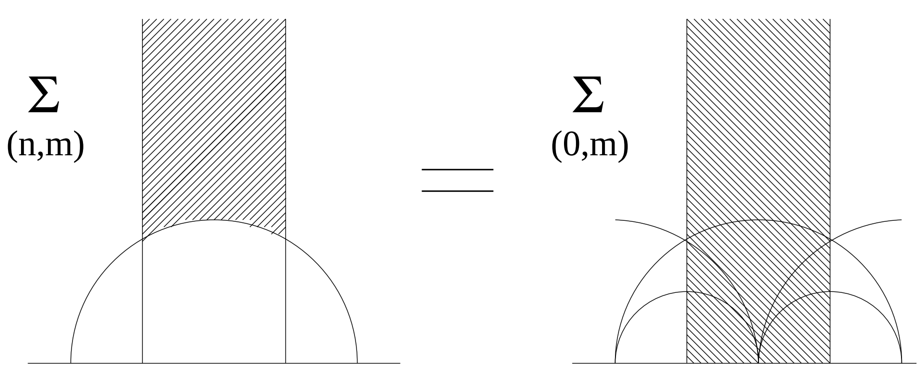

Fig. 1 : The sum over n is traded for the sum over copies of the fundamental domain.

After multiplying (4.19) by the (b,c) ghosts and the internal CFT partition functions, we should integrate the resulting expression over the fundamental domain F_0 of the modular parameter τ. The computation is much facilitated by the trick invented in [30,31] to trade the sum over n in (4.19) for the sum over copies of the fundamental domain. See Figure 1. This is possible since (n,m) transforms as a doublet under the modular group $SL(2,Z)$. If $(n,m)\neq(0,0)$, it can be mapped by an $SL(2,Z)$ transformation to $(0,m)$, $m>0$. The $SL(2,Z)$ transformation also maps the fundamental domain into the strip $\mathrm{Im}\,\tau\geq 0$, $|\mathrm{Re}\,\tau|\leq 1/2$. On the other hand, $(n,m)=(0,0)$ is invariant under the $SL(2,Z)$ transformation, and the corresponding term still has to be integrated over the fundamental domain F_0. This term represents the zero temperature contribution to the cosmological constant, or the zero temperature vacuum energy. In addition to the usual tachyon divergence of bosonic string theory at large τ_2, it is also divergent due to the $\sin^{-1}$ factor in (4.19); this divergence can be interpreted as coming from the infinite volume of AdS_3. Since the temperature dependence of this term is trivial we will ignore it from now on. The final result then is that we fix $n=0$ in (4.19) and we integrate over the entire strip shown in Figure 1. The one-loop partition function of bosonic string theory on $H_3/Z\times\mathcal{M}$ is then

$$Z(\beta,\mu)=\frac{\beta(k-2)^{\frac{1}{2}}}{8\pi}\int_0^\infty\frac{d\tau_2}{\tau_2^{3/2}}\int_{-1/2}^{1/2}d\tau_1 e^{4\pi\tau_2\left(1-\frac{1}{4(k-2)}\right)}$$
$$\times\sum_{h,\bar h}D(h,\bar h)q^h\bar q^{\bar h}\sum_{m=1}^\infty\frac{e^{-(k-2)m^2\beta^2/4\pi\tau_2}}{|\sinh(m\hat\beta/2)|^2}\tag{4.23}$$
$$\times\left|\prod_{n=1}^\infty\frac{1-e^{2\pi in\tau}}{(1-e^{m\hat\beta+2\pi in\tau})(1-e^{-m\hat\beta+2\pi in\tau})}\right|^2 .$$

5. Derivation of the spectrum from the partition function

In this section we show how to extract the spectrum of Lorentzian string theory on AdS_3 from the one-loop partition function. First we present a qualitative analysis, which is then followed by a precise calculation. During the course of this investigation we will find a rather novel phenomenon–singularities in the interior of the one-loop moduli space. We explain how this is due to the presence of long strings. We regulate the divergences and find a physical interpretation for how the different parts of the spectrum arise from this calculation. Furthermore, we show how the one-loop result contains information about the $SL(2,R)$ and Liouville reflection amplitudes.

5.1. The free energy

The one-loop partition function (4.23) can be interpreted as the single particle contribution to the thermal free energy, $Z(\beta,\mu) = -\beta F$. To this each string state makes a contribution $\beta^{-1}\log(1-e^{-\beta(E+i\mu L)})$, where E and L are the energy and the angular momentum of the state. The total free energy is the sum over all such factors:

$$\begin{aligned} F(\beta,\mu) &= \frac{1}{\beta}\sum_{string\in\mathcal{H}} \log\left(1-e^{-\beta(E_{string}+i\mu L_{string})}\right) \\ &= \sum_{m=1}^{\infty} f(m\beta, m\mu)\ , \end{aligned} \tag{5.1}$$

where

$$f(\beta,\mu) = \frac{1}{\beta}\sum_{string\in\mathcal{H}} e^{-\beta(E_{string}+i\mu L_{string})}\ . \tag{5.2}$$

Here $\mathcal{H}$ is the physical Hilbert space of single string states. In both (4.23) and (5.1), we have the sums over functions of $(m\beta, m\mu)$. It is therefore sufficient to compare the $m=1$ terms in these expressions.

In other words, we want to verify that E_{string} and L_{string} extracted from the identification,

$$
\begin{aligned}
f(\beta,\mu) &= \sum_{string\in\mathcal{H}} \frac{1}{\beta} e^{-\beta(E_{string}+i\mu L_{string})} \\
&= \frac{(k-2)^{\frac{1}{2}}}{8\pi} \int_0^\infty \frac{d\tau_2}{\tau_2^{3/2}} \int_{-1/2}^{1/2} d\tau_1 e^{4\pi\tau_2\left(1-\frac{1}{4(k-2)}\right)} \sum_{h,\bar{h}} D(h,\bar{h}) q^h \bar{q}^{\bar{h}} \\
&\quad \times \frac{e^{-(k-2)\beta^2/4\pi\tau_2}}{|\sinh(\hat{\beta}/2)|^2} \left| \prod_{n=1}^\infty \frac{1-e^{2\pi i n\tau}}{(1-e^{\hat{\beta}+2\pi i n\tau})(1-e^{-\hat{\beta}+2\pi i n\tau})} \right|^2 .
\end{aligned}
\tag{5.3}
$$

agree with the string spectrum reviewed in Section 3. We will see that the sum over the Hilbert space breaks up into a sum over the discrete states and an integral over the continuous states, with the correct expressions for the energies. Since the one-loop calculation presented here is independent of the assumptions made in [16] about strings in Lorentzian AdS_3, we can regard this as a derivation of the spectrum starting from the well-defined Euclidean path integral.

5.2. Qualitative analysis

In this subsection we will analyze (5.3) in a qualitative way and explain where the different contributions to the spectrum come from. To keep the notation simple, we set $\mu = 0$ in this subsection, leaving the exact computation for the next subsection.

As expected, in (5.3) there is an exponential divergence as $\tau_2 \to \infty$, coming from the tachyon. This is just as in the flat space case, where $(\text{mass})^2 < 0$ of the tachyon causes its contribution to be weighted with a positive exponential. We will disregard this divergence[11].

[11] A skeptical reader could think that we are really doing the superstring partition function (the fermions included in the internal CFT, etc.). Then

However, rather unexpectedly, the expression above has additional divergences at finite values of τ. In string theory one might naively expect that divergences come only from the corners of the fundamental domain in the τ-plane, but in this case the divergence is coming from points in the interior of the fundamental domain. Overcoming the initial panic, one realizes that these divergences are related to the presence of long strings. In fact, as with any other string divergence, it can be interpreted as an IR effect. This divergence is due to the fact that long strings feel a flat potential as they go to infinity and become free. This causes their contribution to the free energy to be weighed by an infinite volume factor[12]. To see this, note that near the pole (see Figure 2)

$$\tau = \tau_{pole} + \epsilon \ , \tag{5.4}$$

where

$$\tau_{pole} = \frac{r}{w} + i\frac{\beta}{2\pi w} \ , \tag{5.5}$$

we can expand the partition function and replace τ in all terms by its value at the pole, except in the one term that has the pole.

---

the tachyon divergence will disappear but one would still find the divergences that we discuss below. Of course, the one-loop partition function is non-vanishing even in the supersymmetric case since the thermal boundary conditions break supersymmetry.

[12] One can avoid the appearance of these infinities by considering the free energy density. However, then the short strings would not be visible.

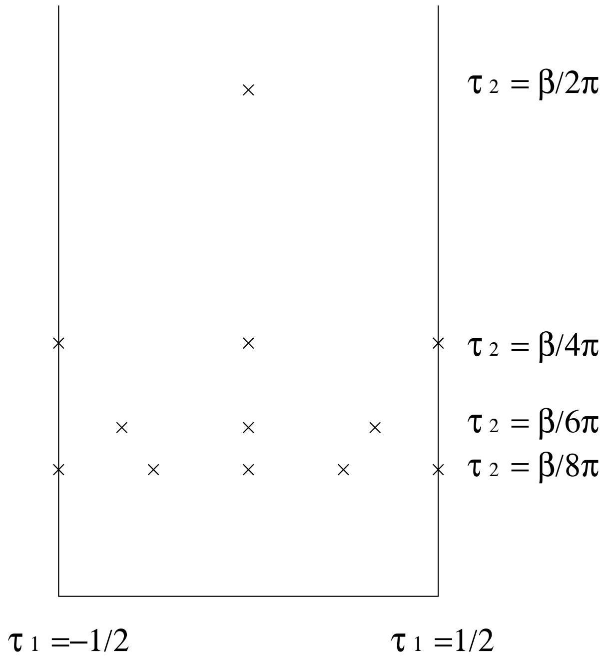

Fig. 2: Poles in the τ plane, shown for $w = 1$ to 4.

If we integrate (5.3) near the pole, i.e. in the region

$$\epsilon < |\tau - \tau_{pole}| \ll 1 \ , \tag{5.6}$$

we find that it diverges as $\log \epsilon$ with coefficient

$$\begin{aligned} \frac{1}{\sqrt{w\beta^3}} \exp \Bigg[-\beta \left(\frac{k}{2} w + \frac{1}{w} (\tilde{N} + h + \bar{\tilde{N}} + \bar{h} - 2 + \frac{1}{2(k-2)}) \right) \\ + \frac{2\pi i r}{w} (\tilde{N} + h - \bar{\tilde{N}} - \bar{h}) \Bigg] \ . \end{aligned} \tag{5.7}$$

We now sum over r, with $|r/w| \leq 1/2$, since these are the ones corresponding to the poles in the strip[13]. This sum constrains $\tilde{N} + h - \bar{\tilde{N}} - \bar{h}$ to be an integer multiple of w, and it introduces an additional factor of

[13] If some poles are on the boundaries of the strip, $\tau_1 = \pm 1/2$, then we only count them once, since the right and left boundaries of the strip are identified.

w in (5.7). The log divergence in τ-integral can therefore be expressed as

$$f(\beta,\mu) \sim \frac{1}{\beta}\log\epsilon \int_0^\infty ds e^{-\beta E(s)} + \cdots\ , \tag{5.8}$$

where $E(s)$ is the energy spectrum given by (3.16). Note that the s-integral and the sum over r we mentioned above give the factor $\sqrt{w/\beta}$ needed to match the prefactor in (5.7) to that in (5.8). This reproduces the expected contribution from the long strings in the left hand side of (5.3). The logarithmic divergence should be interpreted as a volume factor due to the fact that the long string can be at any radial position. In the next subsections, we will see more precisely that it is indeed associated to the infinite volume in spacetime by relating ϵ to a long distance cutoff.

Now we would like to calculate the short string spectrum. Since the long string spectrum gives a divergent result, while the short string spectrum gives a finite one, it might appear at first that extracting the contributions due to the short strings from a divergent expression such as (5.3) will be problematic. Fortunately we can get around this difficulty since the temperature dependence of the long string free energy is different from that of the short string free energy. In the next subsection we will explain how to do this precisely and reproduce the short string spectrum which agrees with [16]. One of the more puzzling aspects of the short string spectrum found there is the cutoff $1/2 < \tilde{\ell} < (k-1)/2$ in the value of the $SL(2,R)$ spin $\tilde{\ell}$. In the remainder of this section we will explain in a qualitative way how this cutoff arises by doing the calculation for large k.

If we were to evaluate the right hand side of (5.3) naively (and incorrectly), we would expand the integrand in powers of $q = e^{2\pi i\tau}$ and then perform the τ integral. If we did this, we would obtain the short string spectrum with $w = 0$ and no upper bound on the value of $\tilde{\ell}$. However this expansion is not correct. How we can expand the integrand in (5.3) depends on the value of τ_2. When we cross the poles at $\tau_2 = \frac{\beta}{2\pi w}$, a different expansion should be used for the denominator:

$$\begin{aligned} \frac{1}{1-e^{\beta+2\pi i w\tau}} &= \sum_{q=0}^{\infty} e^{q(\beta+2\pi i w\tau)} \qquad \left(\tau_2 > \frac{\beta}{2\pi w}\right), \\ &= -\sum_{q=0}^{\infty} e^{-(q+1)(\beta+2\pi i w\tau)}, \quad \left(\tau_2 < \frac{\beta}{2\pi w}\right). \end{aligned} \tag{5.9}$$

When τ_2 is in the range

$$\frac{\beta}{2\pi(w+1)} < \tau_2 < \frac{\beta}{2\pi w}, \tag{5.10}$$

the product over n in the first term in the denominator in (5.3) is broken up into two factors, a product in $1 \leq n \leq w$ and a product in $w+1 \leq n$. The first factor is expanded in powers of $e^{-(\beta+2\pi i n\tau)}$ and the second factor is expanded in powers of $e^{\beta+2\pi i n\tau}$. Combining them together with the terms coming from the expansion of the remaining products in (5.3), we get an exponent of the form

$$-\left(\frac{1}{2}+q+w\right)\beta + 2\pi i\tau\left(N_w - \frac{1}{2}w(w+1)\right), \tag{5.11}$$

for some integers q and N_w (the first term $-\beta/2$ comes from expanding $1/\sinh(\beta/2)$ in (5.3)). There is a similar term for $\tau \to \bar{\tau}$. We are then to do the τ-integral of the form,

$$\int \frac{d^2\tau}{\tau_2^{3/2}} \exp\left[4\pi\tau_2\left(1-\frac{1}{4(k-2)}\right) - \frac{\beta^2(k-2)}{4\pi\tau_2} - \beta(1+q+\bar{q}+2w) \right. \\ \left. + 2\pi i\tau\left(N_w + h - \frac{1}{2}w(w+1)\right) - 2\pi i\bar{\tau}\left(\bar{N}_w + \bar{h} - \frac{1}{2}w(w+1)\right)\right], \tag{5.12}$$

over the region (5.10). The integral over τ_1 produces the level matching condition . Now we evaluate the integral over τ_2 using the saddle point approximation. We find that the saddle point is at

$$\tau_{saddle} = \frac{(k-2)\beta}{2\pi\sqrt{1+4(k-2)(N_w+h-1-\frac{1}{2}w(w+1))}} \tag{5.13}$$

and the integral gives

$$\frac{1}{\beta}e^{-\beta X} \ , \tag{5.14}$$

where the exponent X is equal to

$$1+q+\bar{q}+2w+\sqrt{1+4(k-2)\left(N_w+h-1-\frac{1}{2}w(w+1)\right)} \tag{5.15}$$

This is the correct form of the contributions due to the short strings in the left hand side of (5.3). Moreover we obtain the bound on $\tilde{\ell}$ exactly, because τ_{saddle} has to be in the range (5.10) in order for the saddle point approximation to give a non-zero result. By (5.13), this condition is equivalent to the bound $1/2 < \tilde{\ell} < (k-1)/2$ using the physical state condition. (It is a bit surprising that we get all factors precisely right from the saddle point approximation.) Notice then that the cutoff in $\tilde{\ell}$ is associated to the fact that we expand the integrand in (5.3) in different ways depending on the value of τ. The value of τ making the biggest contribution to the integral depends on the values of N and h of the string state.

5.3. A precise evaluation of the τ-integral

Now let us study the partition function (5.3) more systematically. In this subsection, we go back to the general case with $\mu \neq 0$. From what we saw in the previous subsection, we expect to find the discrete states from the integral over the range (5.10), and the continuous states from the poles after a suitable regularization.

In order to evaluate the τ-integral exactly, it is useful to introduce a new variable c by

$$e^{-(k-2)\frac{\beta^2}{4\pi\tau_2}} = -\frac{8\pi i}{\beta}\left(\frac{\tau_2}{k-2}\right)^{\frac{3}{2}} \int_{-\infty}^{\infty} dc\ c\ e^{-\frac{4\pi\tau_2}{k-2}c^2+2i\beta c} . \tag{5.16}$$

Now suppose τ_2 is in the range,

$$\frac{\beta}{2\pi(w+1)} < \tau_2 < \frac{\beta}{2\pi w}, \tag{5.17}$$

and expand the integrand in (5.3) as explained in the previous subsection. The right hand side of (5.3) becomes a sum of terms like

$$\begin{aligned}
&\frac{4}{\beta(k-2)i}\int_{-\infty}^{\infty} dc\ c \int_{\frac{\beta}{2\pi(w+1)}}^{\frac{\beta}{2\pi w}} d\tau_2 \int_{-1/2}^{1/2} d\tau_1 \\
&\quad\times \exp\left[-\hat{\beta}\left(q+w+\frac{1}{2}\right) - \bar{\hat{\beta}}\left(\bar{q}+w+\frac{1}{2}\right)\right. \\
&\qquad\qquad + 2\pi i\tau_1(N_w+h-\bar{N}_w-\bar{h}) + 2ic\beta \\
&\qquad\qquad \left. -2\pi\tau_2\left(h+\bar{h}+N_w+\bar{N}_w+\frac{2c^2+\frac{1}{2}}{k-2}-w(w+1)-2\right)\right] .
\end{aligned} \tag{5.18}$$

The integral over τ_1 gives a delta function enforcing $N_w+h=\bar{N}_w+\bar{h}$, which is the level matching condition . Integrating over τ_2 in the range (5.17) gives

$$\begin{aligned}
&\frac{1}{\beta\pi i}\int_{-\infty}^{\infty} dc\ c\ \frac{\exp\left[2ic\beta-\hat{\beta}\left(q+w+\frac{1}{2}\right)-\bar{\hat{\beta}}\left(\bar{q}+w+\frac{1}{2}\right)\right]}{c^2+\frac{1}{4}+(k-2)\left(N_w+h-1-\frac{1}{2}w(w+1)\right)} \\
&\quad\times\left\{-\exp\left[-\frac{\beta}{w}\left(2N_w+2h-2+\frac{2c^2+\frac{1}{2}}{k-2}-w(w+1)\right)\right]\right. \\
&\quad\left. +\exp\left[-\frac{\beta}{w+1}\left(2N_w+2h-2+\frac{2c^2+\frac{1}{2}}{k-2}-w(w+1)\right)\right]\right\} ,
\end{aligned} \tag{5.19}$$

where we used the level matching condition.

Let us first look at the first term (the second line) in (5.19). We note that the exponent can be expressed in the form of a complete square if we set $c = s + \frac{i}{2}(k-2)w$. As it will become clear shortly, it is natural to shift the contour of the c-integral from Im $c = 0$ to Im $c = \frac{1}{2}(k-2)w$ so that s becomes real. During this process the contour crosses some poles in the integrand, picking up the residues of the poles in the range $0 <$ Im $c < \frac{1}{2}(k-2)w$. See Figure 3. The poles are located at

$$-\frac{c^2}{(k-2)} = N_w + h - \frac{1}{2}w(w+1) - 1 + \frac{1}{4(k-2)} < \frac{k-2}{4}w^2. \quad (5.20)$$

Similarly, for the second exponential term (the third line) in (5.19) we shift the contour to $c = s + \frac{i}{2}(k-2)(w+1)$ with s real. This picks up the poles at

$$-\frac{c^2}{(k-2)} = N_w + h - \frac{1}{2}w(w+1) - 1 + \frac{1}{4(k-2)} < \frac{k-2}{4}(w+1)^2. \quad (5.21)$$

It is important to note that the residues of these poles have a sign opposite to that of the residues of the poles obeying (5.20). So the result is that we are left with only those poles in the range

$$\frac{k-2}{2}w \; < \; \text{Im}\, c \; < \; \frac{k-2}{2}(w+1), \quad (5.22)$$

with residues

$$\frac{1}{\beta}e^{-Y}, \quad (5.23)$$

where the exponent Y is

$$\hat{\beta}q + \bar{\hat{\beta}}\bar{q} + \beta\left(1 + 2w + \sqrt{1 + 4(k-2)\left(N_w + h - 1 - \frac{1}{2}w(w+1)\right)}\right). \quad (5.24)$$

This is the expected contribution of the short strings to the right hand side of (5.3), and we see also that (5.22) translates into the correct bound on $\tilde{\ell}$.

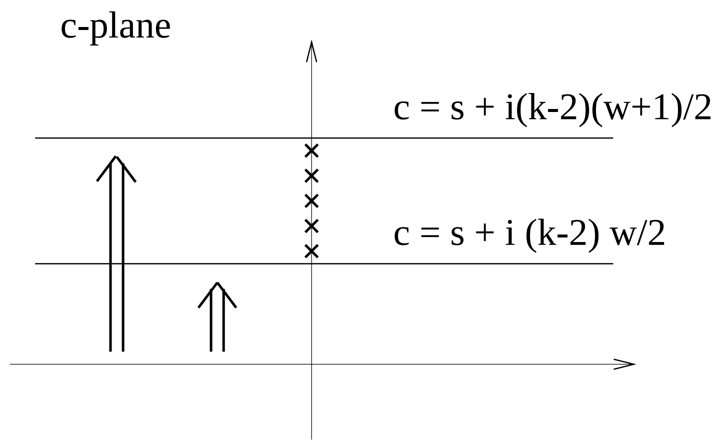

Figure 3: Shifting the contour of integration picks up the pole residues corresponding to the short string spectrum.

It remains to examine the resulting integral over s. Notice that the term coming from just above the pole at $\tau = \hat{\beta}/2\pi w$ has a very similar w dependence in the exponent as that coming from just below the pole. In other words, we combine the first term of (5.19) with the second term of an expression similar to (5.19) but with $w \to w-1$ and we find, after shifting the countours as above,

$$\frac{1}{2\pi i\beta}\int_{-\infty}^{\infty} ds \left(\frac{2s}{w(k-2)}+i\right) \times \left(\frac{\exp\left[-\hat{\beta}q-\bar{\hat{\beta}}\bar{q}-\beta\left(\frac{k}{2}w+\frac{2}{w}\left(\frac{s^2+1/4}{k-2}+N_{w-1}+h-1\right)\right)\right]}{\frac{1}{2}+is-\frac{k}{4}w+\frac{1}{w}\left(N_{w-1}+h-1+\frac{s^2+1/4}{k-2}\right)} - \frac{\exp\left[-\hat{\beta}q-\bar{\hat{\beta}}\bar{q}-\beta\left(\frac{k}{2}w+\frac{2}{w}\left(\frac{s^2+1/4}{k-2}+N_{w}+h-1\right)\right)\right]}{-\frac{1}{2}+is-\frac{k}{4}w+\frac{1}{w}\left(N_{w}+h-1+\frac{s^2+1/4}{k-2}\right)}\right). \tag{5.25}$$

Let us concentrate for now on the third line of this expression. We first note that the sum of such terms over all states gives rise to the log divergence. To see this, it is useful to notice that the combinations

$$\tilde{N} = qw + N_w \ , \qquad \tilde{\bar{N}} = \bar{q}w + \bar{N}_w \tag{5.26}$$

that appear in the exponent of the third line of equation (5.25) are the levels before spectral flow. Thus, for a given state $|\psi\rangle$, states of the form $(\tilde{K}_0^+ \tilde{\bar{K}}_0^+)^n |\psi\rangle$ all have the same value of $\tilde{N}$ and $\tilde{\bar{N}}$. Acting with $\tilde{K}_0^+ \tilde{\bar{K}}_0^+$ on $|\psi\rangle$ does not change the exponent in (5.25), but it does change the denominator by one. This implies that when we sum over all the states of this type, we will find a divergent sum of the form

$$\sum_{n=0}^{\infty} \frac{1}{A-n} \ . \tag{5.27}$$

This divergence has the same origin as the divergence of the right hand side of (5.3) at the pole $\tau_{pole} = \hat{\beta}/2\pi w$. In fact, if we regularize the τ-integral by removing a small strip near the pole as $|\tau - \tau_{pole}| > \epsilon$, we find an additional factor $e^{-n\epsilon}$ in the sum. In the next subsection, we will give the spacetime interpretation of this procedure. With this regularization, the sum behaves as $\log \epsilon$. More precisely we have

$$-\sum_{n=0}^{\infty} \frac{1}{A-n} e^{-n\epsilon} = \log \epsilon + \frac{d}{dA} \log \Gamma(-A) + \mathcal{O}(\epsilon) \tag{5.28}$$

where

$$A = -\frac{1}{2} + is - \frac{k}{4} w + \frac{1}{w} \left(\frac{s^2 + \frac{1}{4}}{k-2} + \tilde{N} + h - 1 \right) . \tag{5.29}$$

Here we have assumed that

$$\tilde{\bar{N}} + \bar{h} \leq \tilde{N} + h, \tag{5.30}$$

but it can be seen that the other case gives the same result.

Now we turn our attention to the second line of (5.25). In those terms we have one less unit of spectral flow, as compared to the third line in (5.25) that we analyzed above. In other words, now we will have that $(w-1)q + N_{w-1} = \tilde{N}'$. These states are in the spectral flow image of $\mathcal{D}_\ell^+$. Since we want to combine these states with the states coming from the third line in (5.25) it is convenient to do spectral flow one more time and think of these states as in the spectral flow image of $\mathcal{D}_\ell^-$ under w units of spectral flow. In this case we find that $q + \tilde{N}' = \tilde{N}$ where now $\tilde{N}$ is the level of the $\mathcal{D}_\ell^-$ representation before spectral flow. From now on the discussion is very similar to what we had above. The states with $(\tilde{K}_0^- \bar{\tilde{K}}_0^-)^n|\psi\rangle$ all have the same energies but they will contribute to the denominator of the second line in (5.25) with

$$\sum_{n=0}^{\infty} \frac{1}{B+n} e^{-n\epsilon} = \log \epsilon - \frac{d}{dB} \log \Gamma(B) + \mathcal{O}(\epsilon) \qquad (5.31)$$

where

$$B = \frac{1}{2} + is - \frac{k}{4}w + \frac{1}{w}\left(\frac{s^2+\frac{1}{4}}{k-2} + \bar{\tilde{N}} + \bar{h} - 1\right), \qquad (5.32)$$

again assuming (5.30).

After we perform these two sums, we find that (5.25) can be written in the form

$$\frac{2}{\beta}\int_0^\infty ds \rho(s) \exp\left[-\beta\left(E(s) + i\frac{\mu}{w}(\tilde{N} + h - \bar{\tilde{N}} - \bar{h})\right)\right] \qquad (5.33)$$

with $E(s)$ the energy of long strings (3.16) and $\rho(s)$ the density of states. We will later see that the physical momentum p of a long string in the ρ direction is equal to $p = 2s$. The angular momentum $L = (\tilde{N} + h - \bar{\tilde{N}} - \bar{h})/w$ is an integer since the states in (5.25) were obeying the level matching condition and the definition (5.26) ensures that

$$\tilde{N} + h = \bar{\tilde{N}} + \bar{h} + w \times (\text{integer}) \; . \tag{5.34}$$

The density of states $\rho(s)$ derived from this analysis is

$$\rho(s) = \frac{1}{2\pi} 2 \log \epsilon + \frac{1}{2\pi i} \frac{d}{2ds} \log \left(\frac{\Gamma(\frac{1}{2} - is + \bar{\tilde{m}})\Gamma(\frac{1}{2} - is - \tilde{m})}{\Gamma(\frac{1}{2} + is + \bar{\tilde{m}})\Gamma(\frac{1}{2} + is - \tilde{m})} \right) \; , \tag{5.35}$$

where

$$\begin{aligned} \tilde{m} = & \; -\frac{k}{4} w + \frac{1}{w} \left(\frac{s^2 + \frac{1}{4}}{k-2} + \tilde{N} + h - 1 \right) \; , \\ \bar{\tilde{m}} = & \; -\frac{k}{4} w + \frac{1}{w} \left(\frac{s^2 + \frac{1}{4}}{k-2} + \bar{\tilde{N}} + \bar{h} - 1 \right) \; . \end{aligned} \tag{5.36}$$

Note that, despite appearances to the contrary, (5.35) is actually symmetric under $\tilde{m} \leftrightarrow \bar{\tilde{m}}$ since $\tilde{m} - \bar{\tilde{m}} = L$ is an integer. In the next subsection we will show that this density of states (5.35) is what is expected from the spacetime meaning of the cutoff ϵ. In going from (5.25) to (5.33) we have states which could be interpreted as coming from the spectral flow of the discrete representations $\mathcal{D}_\ell^+$ and $\mathcal{D}_\ell^-$, with the zero modes essentially stripped off since they were explicitly summed over in (5.28) and (5.31). This implies that the states we have in the end belong to the continuous representation. Note also that the integral over s in (5.33) has only half the range in (5.25). We rewrote it in this way using the fact that the exponent is invariant under $s \to -s$, and that is the reason why we have four Gamma functions in (5.35). In going from (5.25) to (5.33) we have also used that $\frac{d}{dA} = \frac{1}{\frac{dA(s)}{ds}} \frac{d}{ds}$ in (5.29) and similarly in (5.32).

Combining eqns. (5.23) and (5.33), we have finally

$$\begin{aligned} f(\beta, \mu) = & \frac{1}{\beta} \sum D(h, \bar{h}, \tilde{N}, \bar{\tilde{N}}, w) \\ & \times \left[\sum_{q, \bar{q}} e^{-\beta(E + i\mu L)} + \int_0^\infty ds \rho(s) e^{-\beta(E(s) + i\mu L)} \right] \end{aligned} \tag{5.37}$$

which is the free energy due to the short strings and the long strings, respectively.

5.4. The density of long string states

What remains to be shown is the interpretation of $\rho(s)$ given by (5.35) as the density of long string states. Whenever we have a continuous spectrum the density of states may be calculated by first introducing a long distance cutoff which will make the spectrum discrete, and then removing the cutoff. If the cutoff is related to the volume of the system then the density of states will have a divergent part, proportional to the volume and dependent only on the bulk physics, and a finite part which encodes information about the scattering phase shift and also has some dependence on the precise cutoff procedure. To see this, let us consider a one-dimensional quantum mechanical model on the half line, $\rho > 0$, with a potential $V(\rho)$. We assume that $V(\rho)$ vanishes sufficiently fast for large ρ, and that there is continuous spectrum above a certain energy level. To define the density of states, it is convenient to introduce a long distance cutoff at large ρ so that the spectrum becomes discrete. Let us first consider a cutoff by an infinite wall at $\rho = L$. If L is sufficiently large, an energy eigenfunction $\psi(\rho)$ near the wall can be approximated by the plane wave

$$\psi(\rho) \sim e^{-ip\rho} + e^{ip\rho + i\delta(p)}, \tag{5.38}$$

where $\delta(p)$ is the phase shift due to the original potential $V(\rho)$. Imposing Dirichlet boundary condition $\psi(L) = 0$ at the wall, we have

$$2pL + \delta(p) = 2\pi\left(n + \frac{1}{2}\right) \tag{5.39}$$

for some integer n. If L is sufficiently large, there is a unique solution $p = p(n)$ to this equation for a given n. As we remove the cutoff by sending $L \to \infty$, the spectrum of p becomes continuous. We then define the density of states $\rho(p)$ by

$$dn = \rho(p)dp \ . \tag{5.40}$$

From (5.39), we obtain

$$\rho(p) = \frac{1}{2\pi}\left(2L + \frac{d\delta}{dp}\right) \ . \tag{5.41}$$

Thus the finite part of the density of states is given by the derivative of the phase shift.

Instead of the infinite wall at $\rho = L$, we may consider a more general potential $V_{wall}(\rho - L)$ which vanishes for $\rho < L$ but rises steeply for $L < \rho$ to confine the particle. Let us denote by $\delta_{wall}(p)$ the phase shift due to scattering from $V_{wall}(\rho)$. We then obtain the condition on the allowed values of momenta by matching these two wavefunctions and their derivatives at $\rho = L$ as

$$\begin{aligned} \psi(\rho) &\sim e^{-ip\rho} + e^{ip\rho + i\delta(p)} \\ &\sim A\left[e^{-ip(\rho-L)} + e^{ip(\rho-L)+i\delta_{wall}(p)}\right] , \qquad (\rho \sim L) \ . \end{aligned} \tag{5.42}$$

It follows that

$$pL + \delta(p) = -pL + \delta_{wall}(p) + 2\pi n \ . \tag{5.43}$$

In the limit $L \to \infty$, the density of states given by $dn = \rho(p)dp$ is then

$$\rho(p) = \frac{1}{2\pi}\left(2L + \frac{d\delta}{dp} - \frac{d\delta_{wall}}{dp}\right) \ . \tag{5.44}$$

When we have the infinite wall, the phase shift due to the wall is independent of p ($\delta_{wall} = \pi$), and (5.44) reduces to (5.41).

In order to apply this observation to our problem, it is useful to first identify the origin of the logarithmic divergence in the one-loop amplitude $Z(\beta,\mu)$ by examining the functional integral (4.16) near the boundary of AdS_3. In the cylindrical coordinates (4.3), the string worldsheet action (4.10)for large ρ takes the form

$$S \sim \frac{k}{\pi}\int d^2z\left(\partial\rho\bar{\partial}\rho+\frac{1}{4}e^{2\rho}|\bar{\partial}(\theta-it)|^2+\cdots\right) . \tag{5.45}$$

Because of the factor $e^{2\rho}$, the functional integral for large ρ restricts (t,θ) to be a harmonic map from the worldsheet to the target space. Since (t,θ) are coordinates on the torus,

$$\theta-it \sim \theta-it+2\pi n+i\hat{\beta}m, \qquad (n,m \text{ integers}) , \tag{5.46}$$

the harmonic map from the torus to the torus is

$$\begin{aligned}\theta-it &= (2\pi w+i\hat{\beta}m)\sigma^1+(2\pi r+i\hat{\beta}n)\sigma^2 \\ &= \left[(2\pi w+i\hat{\beta}m)\tau-(2\pi r+i\hat{\beta}n)\right]\frac{\bar{z}}{2i\tau_2} \\ &\quad -\left[(2\pi w+i\hat{\beta}m)\bar{\tau}-(2\pi r+i\hat{\beta}n)\right]\frac{z}{2i\tau_2} ,\end{aligned} \tag{5.47}$$

where $z=\sigma^1+\tau\sigma^2$ is the worldsheet coordinate and (r,w,n,m) are integers. In particular, the map $(\theta-it)$ with $(n,m)=(1,0)$ becomes w-to-1 and *holomorphic* when τ takes the special value

$$\tau_{pole}=\frac{r}{w}+i\frac{\hat{\beta}}{2\pi w} . \tag{5.48}$$

On the other hand, if τ is not at one of these points, $\bar{\partial}(\theta-it)$ cannot be set to zero[14]. This gives rise to an effective potential $e^{2\rho}$ for ρ, which

[14] For any τ, we also have a trivial holomorphic map $(t,\theta)=$ const. The functional integral around such a map gives a result independent of β and we can neglect it in the following discussion.

keeps the worldsheet from growing towards the boundary. If τ is near τ_{pole}

$$\tau = \tau_{pole} + \epsilon \ , \tag{5.49}$$

the harmonic map (5.47) with $(n, m) = (1, 0)$ gives

$$|\bar{\partial}(\theta - it)|^2 \sim \left(\frac{2\pi^2 w^2}{\beta} \right)^2 \epsilon^2 \ . \tag{5.50}$$

Thus the action (5.45) generates the Liouville potential $\epsilon^2 e^{2\rho}$. When we computed the one-loop amplitude in sections 4.1 and 4.2, we regularized the τ-integral by removing a small disk $|\tau - \tau_{pole}| < \epsilon$ around each of these special points. Near $\tau = \tau_{pole}$, this is equivalent to adding the infinitesimal Liouville potential $\epsilon^2 e^{2\rho}$ to the worldsheet action. For $|\tau - \tau_{pole}| \gg \epsilon$, the worldsheet can never grow large enough and the effect of the Liouville term is negligible. To be precise, the Gaussian functional integral of (t, θ) shifts $k \to (k-2)$ as in (4.18) and the effective action for ρ near $\tau = \tau_{pole}$ is

$$S_{Liouville} = \frac{k-2}{\pi} \int d^2 z \left(\partial\rho\bar{\partial}\rho + \epsilon^2 e^{2\rho} \right) \ . \tag{5.51}$$

Therefore, we find that our choice of regularization in (5.28) and (5.31) amounts to introducing the Liouville wall which prevents the longs strings from going to very large values of ρ. By looking at the potential in (5.51), we see that the effective length of the interval is $L \sim \log \epsilon$. The central charge of this Liouville theory is such that the $e^{2\rho}$ term has conformal weight one,

$$c_{Liouville} = 1 + 6 \left(b + \frac{1}{b} \right)^2 \ , \qquad b \equiv \frac{1}{\sqrt{k-2}} \ . \tag{5.52}$$

The finite part of the density of states will be given through (5.44) by $\delta(s)$, the phase shift in the $SL(2,R)$ model, and $\delta_{wall}(s)$, the corresponding quantity in Liouville theory. The first one was calculated in [32,33],

$$i\delta(s) = \log\left(\frac{\Gamma(\frac{1}{2}+is-\tilde{m})\Gamma(\frac{1}{2}+is+\tilde{\bar{m}})\Gamma(-2is)\Gamma(\frac{2is}{k-2})}{\Gamma(\frac{1}{2}-is-\tilde{m})\Gamma(\frac{1}{2}-is+\tilde{\bar{m}})\Gamma(2is)\Gamma(\frac{-2is}{k-2})}\right) , \quad (5.53)$$

while the second one was obtained in [34,35]

$$i\delta_{wall}(s) = \log\left(\frac{\Gamma(-2is)\Gamma(\frac{2is}{k-2})}{\Gamma(2is)\Gamma(\frac{-2is}{k-2})}\right) . \quad (5.54)$$

(In order to compare with the expressions in [34,35] we use the value of b given in (5.52) and note that the relevant values of α are $\alpha = Q/2 + isb$.) Using these two formulas we can check that indeed the density of states (5.35) is given by (5.44). We can view this as an independent calculation of (5.53) or as an overall consistency check. Notice that the physical momentum p of a long string along the ρ direction is $p = 2s$. This can be seen by comparing the energy of a long string (3.16) with the energy expected from (5.51) with spacetime momentum p along the radial direction, $p = (k-2)w\dot{\rho}$. We have chosen the variable s since it is conventional to denote by $\ell = 1/2 + is$ the $SL(2,R)$ spin of a continuous representation.

6. Orbifolds of AdS_3

6.1. Introduction

AdS_3 is a solution of General Relativity in three dimensions with a negative cosmological constant, described by the action

$$S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left(R + \frac{2}{l^2} \right) + \text{surface terms} . \tag{6.1}$$

This theory actually has a family of solutions labelled by two parameters M and J [36,3]

$$\begin{aligned} ds^2 &= -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2 , \\ N^2 &= -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} , \\ N^\phi &= -\frac{J}{2r^2} . \end{aligned} \tag{6.2}$$

What the resulting spacetimes look like depend on the values of the two parameters. When $M > 0$ and $Ml > |J|$, these spacetimes correspond to black holes. The second condition ensures that a horizon exists. The constants are then identified with the mass and angular momentum of the black hole, respectively. These spaces may be thought of as excitations of the $M = 0$ case.

However, $M = 0$ is not the lowest energy state possible. It turns out that by setting $M = -1$, the result is nothing but the familiar AdS_3.

For the spacetimes with $-1 < M < 0$ (and $J = 0$), a rescaling of the coordinates brings the metric into the form

$$ds^2 = -\left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\phi^2 , \tag{6.3}$$

which is the same as AdS_3 (related to the coordinates we have been using by the simple transformation $r = l \sinh \rho$), but with a deficit angle $\delta = 2\pi(1 - \sqrt{|M|})$ for ϕ. Thus, these spaces correspond to AdS_3 with conical singularities.

In fact, even the black holes corresponding to (6.2) with $M > 0$ are locally AdS_3, and can be obtained from AdS_3 by a quotient. This is consistent with the equations of motion resulting from (6.1), which implies that the curvature is constant. The black hole solutions do not have a curvature singularity, and differ from AdS_3 only by some global identifications.

The solutions that are being discussed here are easily lifted to solutions of string theory. By including a three form H (the field strength of B), which must be proportional to the volume form in three dimensions, these spaces provide a background in which it is possible to describe string propagation via the $SL(2,R)$ WZW model. At the level of low energy effective action, (6.1) arises when one takes the action for the massless fields of string theory $g_{\mu\nu}$, H, ϕ and sets $H_{\mu\nu\sigma} = \frac{2}{l}\epsilon_{\mu\nu\sigma}$, and $\phi = 0$ [37].

In this section we study string propagation on the conical spaces. For the special values of the opening angle $2\pi/N$, where N is an integer, the spaces may be obtained as a Z_N orbifold of AdS_3.[15] The singularity present is then just an orbifold singularity, and it is possible to formulate a consistent string theory on this background given the knowledge of string theory on AdS_3.

It is interesting to note that the conical spaces we are considering can be formed by adding mass to empty AdS_3 [38]. Relative to the AdS_3 vacuum, an object of mass less than 1 would create a conical singularity. One can imagine a process where a collision taking place

[15] ϕ corresponds to rotation in $X_1 - X_2$ plane in the covering space $ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + dX_2^2$, and is always a space-like killing vector, ensuring causality in the resulting quotient space.

inside of AdS_3 leaves a lump of stable matter, not enough to produce a black hole but distorting the geometry to what we are studying here. This provides a controlled setting to study black hole formation, as in [39].

Another reason for studying this theory is that one would like to gain further insight into the spectral flow symmetry of the $SL(2,R)$ WZW model. From this study we will learn that on the conical spaces spectral flow acts as a twist, in the orbifold sense.

6.2. Z_N quotient

Taking string theory on AdS_3 as the starting point, the conical spaces with opening angles $2\pi/N$ are obtained by taking a Z_N orbifold. Let us first note how spectral flow acts on this quotient space. The effect of spectral flow is to take a solution of the WZW equation of motion

$$g = g_+(x^+)g_-(x^-) \tag{6.4}$$

and generate a new solution [16]

$$g_+(x^+) \to e^{\frac{i}{2}wx^+\sigma_2} g_+(x^+) \ , \qquad g_-(x^-) \to g_-(x^-) e^{\frac{i}{2}wx^-\sigma_2} \ . \tag{6.5}$$

Under this operation, $t \to t + w\tau$ and $\phi \to \phi + w\sigma$. In regular AdS_3 closure of the string worldsheet required that w be an integer, but now we see that w only needs to be a multiple of $1/N$.

When we spectral flow by a fractional amount the $\widehat{SL}(2,R)$ currents obey twisted boundary conditions. Consider the nth twisted sector:

$$K^+(x^+ + 2\pi) = K^+(x^+)\, e^{-2\pi i n/N} \ , \quad K^-(x^+ + 2\pi) = K^-(x^+)\, e^{2\pi i n/N} \ . \tag{6.6}$$

Then the mode expansion is

$$K^+(z) = \sum_{r\in Z+n/N} K_r^+ z^{-r-1} \ , \qquad K^-(z) = \sum_{s\in Z-n/N} K_s^- z^{-s-1} \ , \tag{6.7}$$

where $z = e^{ix^+}$. The commutation relations are

$$\begin{aligned} \left[K_r^+, K_s^-\right] &= -2J^3_{r+s} + kr\delta_{r+s} \\ \left[K_m^3, K_r^\pm\right] &= \pm J^\pm_{m+r} \\ \left[K_m^3, K_l^3\right] &= -\frac{k}{2}m\delta_{m+l} \ . \end{aligned} \tag{6.8}$$

Note that K^3_m are integrally moded, a condition preserved by the algebra. There is a total of N sectors to consider, and in each sector $K^\pm$ are quantized with different periodicity. We now turn to the first step in taking an orbifold, which is to construct the twisted states. As we will see, there will be a close connection to spectral flow.

6.3. Twisted states and spectral flow

Consider a state obtained by repeated applications of the raising operators on a lowest weight state,

$$\prod_{m_i} K^3_{m_i} \prod_{r_j} K^+_{r_j} \prod_{s_k} K^-_{s_k} |\ell, \ell\rangle \ . \tag{6.9}$$

If necessary, commutation relations may be used to change the order in which the generators appear. However, in what follows the ordering will be immaterial. The conformal weight of (6.9) lies $-(\sum m_i + \sum r_j + \sum s_k)$ above the ground state and $K^3_0 = \ell + N^+ - N^-$ where N^+ (N^-) is the number of times K^+ (K^-) appears in the above expression. Also note that the fractional part of the level is given by $(N^+ - N^-)n/N$.

If we take this state and spectral flow by $w = -n/N$, we find that the new generators acting on it are integrally moded. Thus, one can think of this state as belonging to $\hat{\mathcal{D}}_{\tilde{j}}^{+,w=n/N}$. To obtain a string state in spacetime (including $\mathcal{X}$), we impose the Virasoro constraints (1.6) and obtain the same expression for the energy that was found in $AdS_3 \times \mathcal{X}$, (3.14). The discussion for the continuous states is similar and once again we conclude that the energy is given by (3.16).

Normally, twisting the currents as in (6.6) gets rid of the zero mode and the corresponding total charge $Q^{\pm} = \frac{1}{2\pi}\int K^{\pm} d\sigma$ vanishes. This results in breaking of the gauge symmetry [40,41]. What we have found here, in the case of AdS_3, is that such twists are nothing but fractional spectral flows. One might worry that there is still a distinction between those states built with integrally moded $K^{\pm}$ and those states built with fractionally moded $K^{\pm}$, in that the latter are expected to have a different ground state energy. However, in the next section we will show from the partition function calculation that this does not happen. As such, by taking $\hat{\mathcal{D}}_{\tilde{\ell}}^{+}$, $\hat{\mathcal{C}}_{1/2+is}^{\alpha}$ and their images under fractional spectral flows, we automatically include the twisted states. Of course, the integer-valued spectral flows are still allowed and all the flowed sectors are treated in equal footing. In particular, the form of the Virasoro constraints remains the same and so does the expression for the energy and angular momentum. It is tempting to think that even in the case of AdS_3, spectral flow arises as a kind of twisting of some underlying theory, possibly with ϕ noncompact. But one probably needs a better understanding of the $SL(2,R)/U(1)$ parafermion theory [42] in order to pursue this idea.

6.4. Invariant subspace

Having constructed the twisted sectors, only the states that are invariant under the identification $\phi \sim \phi + 2\pi/N$ are to be retained in the spectrum. There is a simple way to see what one should expect.

If one considers the wave equation for a scalar field in the background (6.3), the solution may be expressed as $\Psi = \sum R(r,\omega,m)e^{-i\omega t+im\phi}$. Then single-valuedness of the wave function implies $m = N \times$ integer. The effect of the projection, then, is to restrict the angular momentum to be a multiple of N.

It is straightforward to see how this condition comes about. For all the sectors that we have, we are to project on to the states invariant under the operator

$$e^{-2\pi i(K_0^3-\bar{K}_0^3)/N} \ . \tag{6.10}$$

Therefore, the states that remain carry angular momentum that is a multiple of N, $K_0^3 - \bar{K}_0^3 = N \times$ integer, for both the discrete and continuous representations.

6.5. Thermal partition function

As in the case of AdS_3, we can check that the spectrum derived above agrees with what one gets by evaluating the finite temperature partition function. The calculation was explained in detail in Chapter 4, so our focus will only be on the effects due to the conical singularity.

As before, we first transform to the coordinates that are well suited for carrying out the path integral. We reproduce the transformation here to make the identifications transparent:

$$\begin{aligned} v &= \sinh\rho\; e^{i\phi} \\ \bar{v} &= \sinh\rho\; e^{-i\phi} \\ \theta &= t - \log\cosh\rho \ . \end{aligned} \tag{6.11}$$

Under the identification $\phi \sim \phi + 2\pi/N$, the fields are identified as $v \sim ve^{2\pi i/N}$ and $\bar{v} \sim \bar{v}e^{-2\pi i/N}$. We take the worldsheet to be a torus with modular parameter τ. Then the boundary conditions are

$$v(z+2\pi) = v(z)e^{2\pi ia/N} \ , \qquad v(z+2\pi\tau) = v(z)e^{2\pi ib/N} \ . \tag{6.12}$$

We will denote by $\mathcal{Z}_{ab}$ the path integral $\int e^{-S}\mathcal{D}\theta\mathcal{D}v\mathcal{D}\bar{v}$ with the above boundary conditions. We remind the reader that these boundary conditions are in addition to those introduced by identifying the Euclidean time $t \sim t + \beta$.

Let us first calculate $\mathcal{Z}_{a0}$. We can implement the right boundary condition by setting

$$v(z) = \tilde{v} \exp\left(-\frac{a}{2N\tau_2}(z\bar{\tau} - \bar{z}\tau)\right) , \tag{6.13}$$

with $\tilde{v}$ periodic. Then $\bar{U}_{n,m}$, defined in eqn (4.15), picks up an additional term, $\bar{U}_{n,m} \to \bar{U}_{n,m} + a\bar{\tau}/N$. With this change, we can repeat the calculation that was done in Chapter 4, and obtain the partition function as (4.19). Similarly, for $\mathcal{Z}_{0b}$ all we need to do is twist along the other direction of the torus, to obtain $\bar{U}_{n,m} \to \bar{U}_{n,m} + b/N$ and once gain the partition function takes the same functional form. In this way we obtain for the partition function of thermal AdS_3/Z_N,

$$\mathcal{Z} = \frac{1}{N}\sum_{a,b} \mathcal{Z}_{ab} . \tag{6.14}$$

To obtain the free energy of strings on $AdS_3/Z_N \times \mathcal{M}$, we multiply (6.14) by the partition function of the CFT on $\mathcal{M}$ and the reparametrization ghosts, and integrate τ over the fundmental domain:

$$\int_{F_0} \mathcal{Z}_{AdS_3/Z_N}\mathcal{Z}_{\mathcal{M}}\mathcal{Z}_{bc} = -\beta F = -\sum_{physical} \log(1 - e^{-\beta E}) . \tag{6.15}$$

From this point on one can follow exactly the same steps as before to reproduce the spectrum. We will explain some of the new features that arise in the course of this computation.

As usual the sum over a represents the twisted sectors and the sum over b serves as a projection down to the invariant states. Consider

$\mathcal{Z}_{a0}$ and its expansion. From $\exp\{2\pi(\mathrm{Im}U_{0,1})^2/\tau_2\}$ and $|\sin(\pi U_{0,1})|^{-2}$ we obtain the additional factor

$$\exp\left[2\pi\tau_2\left(\frac{a^2}{N^2}-\frac{a}{N}\right)\right] . \tag{6.16}$$

A conformal field theory of 2 bosons with periodicity θ has ground state energy

$$(q\bar{q})^{-\frac{1}{2}(\theta^2-\theta)-\frac{1}{12}} , \tag{6.17}$$

so we have reproduced what might have been the expected shift in the ground state energy. However, this is not the end of story. The oscillator terms are changed to

$$\left|\prod_{n=1}^{\infty}(1-e^{\beta+2\pi i\tau(n-a/N)})(1-e^{-\beta+2\pi i\tau(n+a/N)})\right|^{-2} , \tag{6.18}$$

which has poles when $\tau_2=\frac{\beta}{2\pi(n-a/N)}$. Earlier it was shown that the location of the poles correspond to spectral flow parameters. So we see that w is given by $w=n-a/N$ with n being positive integers. It will be explained shortly that $w=-a/N$ arises from τ_2 above the first pole at $\frac{\beta}{2\pi(1-a/N)}$. The shift in the location of the poles also causes the expansion of (6.18) to be slightly different from the AdS_3 case. One finds the terms (compare to eqn. (5.18))

$$\ldots\exp\left[2\pi\tau_2\left(w(w+1)-\frac{a^2}{N^2}+\frac{a}{N}\right)\right]\ldots \tag{6.19}$$

The extra terms on the right serve to cancel the shift in ground state energy, (6.16), and we are left with the correct expression for the energy. Note that this cancellation is in agreement with what we found in the previous section. What appears to be twisting is actually a fractional spectral flow.

To see that summing over b corresponds to a projection down to the invariant states, take $\mathcal{Z}_{ab}$ and its expansion. The only additional change is the appearance of a new term

$$\exp\left[-\frac{2\pi i b(q-\bar{q})}{N}\right] \tag{6.20}$$

in every state. Hence, $\frac{1}{N}\sum_b \mathcal{Z}_{ab}$ only includes the states with the correct condition on angular momentum. This shows that from (6.14) we obtain the spectrum that agrees with what was found in the algebraic analysis.

6.6. Bound on $\tilde{\ell}$

In expanding the partition function, the presence of poles in the oscillator terms meant that the range of τ_2 was broken up into

$$\frac{\beta}{2\pi(w+1)} < \tau_2 < \frac{\beta}{2\pi w} \ , \tag{6.21}$$

and a different expansion was used in each interval. This gave rise to the states with spectral flow by amount w. In the case of AdS_3, this included the sector with $w=0$. But now that w is no longer limited to be an integer, we need to re-examine the special case

$$\frac{\beta}{2\pi(1-a/N)} < \tau_2 < \infty \ . \tag{6.22}$$

In this range, the energy is found to be

$$E = 1+q+\bar{q}-\frac{2a}{N}+\sqrt{1+4(k-2)\left(N_w+h-1-\frac{1}{2}\left(\frac{a^2}{N^2}-\frac{a}{N}\right)\right)} \ . \tag{6.23}$$

So we see that these states are in the sector flowed by $w=-a/N$. Thus, the allowed values of spectral flow are $w=n-a/N$, including

$n = 0$. We expect that these states will have a different range of $\tilde{\ell}$, because the integral over τ_2 is broken up in a different way from all the other states.[16] Repeating the saddle point calculation as was done earlier, $\tilde{\ell}$ is seen to satisfy

$$\frac{(k-2)\frac{a}{N}+1}{2} < \tilde{\ell} < \frac{k-1}{2} \; . \tag{6.24}$$

On the algebraic side, this change in the lower bound can be seen from solving the physical state condition

$$\tilde{\ell} = \frac{1}{2} - \frac{k-2}{2} w + \sqrt{\ldots} \, , \tag{6.25}$$

with $w = -a/N$. The semi-classical limit (large k, h) of this bound translates into

$$0 < \sqrt{\frac{4h}{k}} < 1 - \frac{a}{N} \; , \tag{6.26}$$

which is consistent with the analysis of [43], extended to negative values of w. In AdS_3, states with negative w automatically had negative energy, but now we find that in the quotient space it is possible for states with negative fractional spectral flow to have positive energy.

6.7. *Discussion*

We have formulated a description of strings moving on AdS_3 but with an opening angle of $2\pi/N$ for ϕ. The twisted states arising from the orbifold construction found a natural description as states with fractional spectral flow. Specifically, we have shown that the nth twisted sector is obtained by taking spectral flow with $w = n/N$. Rather than thinking of the original states with integral w as being

[16] That is to say, in the variable $1/\tau_2$ these states occupy a strip of length less than $2\pi/\beta$.

"untwisted" and fractional w as being "twisted", we have proposed that there is only one untwisted sector, namely those with $w = 0$, and all the spectral flowed sectors should be thought of as being twisted.

We have also computed the thermal partition function on this background and extracted the spectrum that agrees with the results of the algebraic description. Despite the fact that there are states constructed by acting with fractionally-moded generators, it was shown that this does not cause a change in the ground state energy.

The fact that twisted states may be obtained by spectral flow means that we are also able to write down the corresponding vertex operators, by bosonizing the K^3 current [44,16]. Thus, unlike what usually happens in orbifolds we have explicit formulas for the twisted state vertex operators. Using these vertex operators we can compute the long string scattering amplitude on AdS_3/Z_N.

One might wonder whether we can extend our analysis to the case with rational values of the opening angle. Indeed, it is fairly simple to generalize the algebraic construction given here, by first going to the covering space in which ϕ has period $2\pi P$ and taking a Z_Q orbifold. The resulting space would have an opening angle $2\pi P/Q$. However, it is not clear whether one can calculate the partition function with this geometry, and that prevents us from concluding at present that such descriptions are possible.

As already mentioned, an important application of AdS_3 orbifold is the BTZ black hole. The idea of generating twisted states by fractional spectral flow was used in [45,46] to determine the string spectrum in the BTZ background. The quotient involved in that calculation is more complicated than what we considered here, and the orbifold is an asymmetric one, meaning that a different identification is made for the left and right. It is worth noting, however, that the asymmetry only manifests itself in having different fractional spectral flow numbers on each side.

Lastly, it would be interesting to apply the conicals discussed here to the study of closed string tachyon condensation. In many respects AdS_3/Z_N and C/Z_N are similar, but they differ in one important aspect: time does not decouple in AdS_3/Z_N. Extending the recent results in tachyon condensation in C/Z_N [47,48,49] to AdS_3/Z_N would represent a significant progress.

7. Strings in Plane Wave and $AdS \times S$

7.1. Introduction

As mentioned in the Introduction to this thesis, it has not been possible to explore AdS/CFT correspondence to the extent one would like because string backgrounds with R-R fields are difficult to solve. Recently, however, Berenstein, Maldacena, and Nastase [50] showed how to take AdS/CFT beyond the supergravity approximation. By taking a limit of $AdS_5 \times S^5$ in which the geometry becomes that of a plane wave, one obtains a background that allows for exact string quantization, in Green-Schwarz formalism [51]. The limiting procedure involves taking the radius of $AdS_5 \times S^5$ to infinity and is an example of Penrose's limit [52]. At the same time, on the CFT side one focuses on those states with large conformal weight and R-charge: $\Delta, J \to \infty$ as R^2, but with finite $\Delta - J$. In this way each AdS/CFT duality gives rise to a plane wave/CFT duality, in which one may go beyond the supergravity approximation. Specifically, BMN was able to reproduce, from the CFT point of view, some of the *stringy* excitations in the plane wave. This represents remarkable progress towards establishing the correspondence between a fully string theoretic description of gravity on AdS and the CFT on the boundary.

Furthermore, it has been shown that some physical quantities of interest may be computed perturbatively on both sides of the BMN correspondence [53,54,55,56]. This differs from AdS/CFT, in which the duality relates the weak coupling physics on one side to the strong coupling physics on the other. This development has led to an intense level of activity[17] which has resulted in significant understanding of both gauge theory and string field theory.

[17] See, for example, [57,58,59,60,61,63,64,65], and [66,67,68,69] for reviews.

For these reasons string theory on plane waves that arise as Penrose limits of $AdS \times S$ has emerged as a topic of great importance. However, the GS formulation of superstrings is technically cumbersome and much insight would be gained from an example of an exact CFT description of string propagation on a plane wave. Happily, such an example exists: the plane wave obtained via the Penrose limit of $AdS_3 \times S^3$ with a purely NS-NS field strength[18].

Actually, the $AdS_3 \times S^3$ plane wave with NS background is special for another reason–string theory is solvable even before the Penrose limit is taken! The CFT on the string worldsheet is given by the $SL(2, R)$ and $SU(2)$ WZW models with the level of the current algebras determined by the radius of $AdS_3 \times S^3$. The solvability of string theory on $AdS_3 \times S^3$ allows us to view string theory on the plane wave as one of its subsectors. This is similar in spirit to how the $\mathcal{N} = 4$ SYM theory is studied in the ten dimensional BMN duality, in that one does not in anyway change the theory while trying to study the correspondence. Rather, one restricts focus onto a particular subclass of operators, such as the (nearly) chiral operators, for which it is possible to say something about the dual objects in the string side.

Our goal is to understand string theory on plane wave from the viewpoint of the underlying supersymmetric $SL(2, R) \times SU(2)$ WZW model. We begin with the superstring spectrum on $AdS_3 \times S^3 \times \mathcal{M}$ at arbitrary values of the level k and angular momentum J on S^3. As we take $k, J \to \infty$, we can "see" how the Hilbert space breaks apart, and a subspace arising in this limit corresponds to the plane wave Hilbert

[18] For earlier work on the $AdS_3 \times S^3$ plane wave, see [70,71,72,73,74,75].

space. The spectral flow symmetry of the $SL(2,R)$ WZW model once again will play a key role in this discussion[19].

Moreover, since our treatment is fully string theoretic from the start, or in other words valid for arbitrary values of the radius, we can attempt to address the following important question: What can string theory on plane waves tell us about string theory on $AdS \times S$? Even though it is believed that the former represents a great simplification of the latter (the plane wave is, after all, just the first term in an R^{-2} expansion of $AdS \times S$), we find some strong evidence that in fact some aspects of string theory in the plane wave could be trusted away from the strict $R^2 \to \infty$ limit. Specifically, we will show that the large J spectrum of strings on $AdS_3 \times S^3$ with NS background at finite R^2 coincides with the plane wave spectrum, found in [50,70,71,72]. This is rather surprising since the spacetime geometry in each case is drastically different. Our result provides an explicit and compelling evidence in support of some of the recent ideas [76,77,78,79,80] about extrapolating the semiclassical relationship between energy and spins in $AdS_5 \times S^5$ down to the stringy regime.

The plan is as follows. We begin by reviewing in this section the $SU(2)$ WZW model, which is needed to describe the S^3 part of the target space. We will briefly describe the Hilbert space of the $SU(2)$ WZW model, in order to introduce notation and also because as we will see, the analog of spectral flow (3.9), (3.10) in the $SU(2)$ WZW model will prove to be an useful tool in studying superstrings in the

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[19] Previous work on the plane wave limit of $AdS_3 \times S^3$ either did not address the issue of spectral flow, or discussed it as a symmetry of the WZW model based on the extended Heisenberg group, i.e. after the Penrose limit was taken.

plane wave. Then we formulate superstrings in $AdS_3 \times S^3 \times \mathcal{M}$, where $\mathcal{M}$ may be T^4 or $K3$.

Section 8 explains the Penrose limit which takes $AdS_3 \times S^3$ to the plane wave, and in section 9 we study the semi-classical limit of strings that will be relevant in the plane wave limit. We do this by computing the Nambu action of a string near the origin of AdS_3 and moving with high angular momentum on a great circle of S^3. This is the six dimensional analog of the particle trajectory used by BMN to obtain the ten dimensional plane wave from $AdS_5 \times S^5$ [50]. The resulting Nambu action displays the same behavior as what was shown in [16]. Namely, new representations that do not obey the usual highest weight conditions appear. These representations are obtained from the usual representations by spectral flow, and it is shown that the amount of spectral flow depends on the ratio of the angular momentum to R^2. Armed with this knowledge, in section 10 we obtain the exact string spectrum on $AdS_3 \times S^3$, valid for arbitrary values of R^2 and J. The plane wave spectrum is reproduced by taking $R^2, J \to \infty$ and expanding to leading order. In section 11 we discuss the decoupling of the Hilbert space in the Penrose limit. In section 12 we discuss what happens when the radius of $AdS_3 \times S^3$ is finite. Conclusions are presented in section 13. In Appendix A we show how the spectral flow number violation rule found in [81] can be understood in terms of angular momentum conservation in the plane wave.

7.2. $SU(2)$ WZW model

String theory on S^3 is described by the $SU(2)$ WZW model, and its Hilbert space can be constructed in a manner similar to what we described for $SL(2,R)$. Again, we will restrict our attention to the holomorphic sector. Every statement we make regarding the holomorphic sector has an analogous statement for the anti-holomorphic sector.

The action takes the same form as the $SL(2,R)$ WZW model, but now with g labelling an element of $SU(2)$. The parametrization of the $SU(2)$ group manifold is very similar to what was used for $SL(2,R)$. The metric on S^3 reads

$$ds^2 = \cos^2\theta d\psi^2 + d\theta^2 + \sin^2\theta d\varphi^2 \ . \tag{7.1}$$

The symmetry of $SU(2)$ WZW model is generated by two copies of the $\widehat{SU}(2)$ current algebra at level k'

$$\begin{aligned}
\left[J^+_m, J^-_n\right] &= 2J^3_{m+n} + k' m\delta_{m+n} \\
\left[J^3_m, J^\pm_n\right] &= \pm J^\pm_{m+n} \\
\left[J^3_m, J^3_n\right] &= \frac{k'}{2} m\delta_{m+n} \ ,
\end{aligned} \tag{7.2}$$

and the Virasoro algebra given by the Sugawara form

$$L_n = \frac{1}{k'+2}\sum_{m=-\infty}^{\infty} : \delta_{ab} J^a_m J^b_{n-m} : \ . \tag{7.3}$$

The representations of the $SU(2)$ WZW model are built from the familiar $SU(2)$ angular momentum representations D_j. A state is labeled as $|j,m,M\rangle$, with

$$\begin{aligned}
L_0|j,m,M\rangle &= \left(\frac{j(j+1)}{k'+2} + M\right)|j,m,M\rangle \\
J^3_0|j,m,M\rangle &= m|j,m,M\rangle \ .
\end{aligned} \tag{7.4}$$

It will be convenient to choose our basis so that the zero modes of J^3 and $\bar{J}^3$ are related to translation along ψ direction in (7.1):

$$-i\frac{\partial}{\partial\psi} = J^3_0 + \bar{J}^3_0 \ . \tag{7.5}$$

The possible values of j that may appear are restricted to $0 \leq j \leq k'/2$, in half-integer steps [82]. The complete Hilbert space of $SU(2)$ WZW model is therefore

$$\mathcal{H}_{SU} = \oplus_{j=0,\frac{1}{2},\ldots\frac{k'}{2}} \hat{D}_j \otimes \hat{D}_j \ . \tag{7.6}$$

7.3. Superstrings on $AdS_3 \times S^3 \times \mathcal{M}$

So far we have discussed the bosonic string theory. Our main interest is in the supersymmetric case, and in this subsection we will describe the supersymmetric extension of WZW models. For simplicity, we will limit our discussion to the $SL(2,R)$ model; the corresponding modifications for the $SU(2)$ model is straightforward. Further details on superstrings on group manifolds can be found in [83]. Superstrings on $AdS_3 \times S^3$ was also studied in [84], and the no-ghost theorem was proved in [85,86].

To extend the above results to the case of superstrings in RNS formalism, we need to introduce free worldsheet fermions χ^a which together with the total current K^a comprise the WZW supercurrent:

$$C^a = \chi^a + \theta K^a \ , \tag{7.7}$$

with θ a holomorphic Grassmann variable. The OPE's of K^a and χ^a are

$$\begin{aligned} K^a(z)K^b(w) &\sim \frac{k}{2}\frac{\eta^{ab}}{(z-w)^2} + \frac{i\epsilon^{ab}{}_c K^c(w)}{z-w} \\ K^a(z)\chi^b(w) &\sim \frac{i\epsilon^{ab}{}_c \chi^c(w)}{z-w} \\ \chi^a(z)\chi^b(w) &\sim \frac{k}{2}\frac{\eta^{ab}}{z-w} \ . \end{aligned} \tag{7.8}$$

This shows that K^a and χ^a do not form independent algebras. By subtracting the fermionic contribution to the total current, we obtain the bosonic current

$$k^a = K^a + \frac{i}{k}\epsilon^a{}_{bc}\chi^b\chi^c \ , \tag{7.9}$$

which have the OPE's

$$\begin{aligned} k^a(z)k^b(w) &\sim \frac{k+2}{2}\frac{\eta^{ab}}{(z-w)^2} + \frac{i\epsilon^{ab}{}_c k^c(w)}{z-w} \\ k^a(z)\chi^b(w) &\sim 0 \ . \end{aligned} \tag{7.10}$$

Hence the level of the bosonic WZW model is shifted from k to $k+2$. Similarly, for the supersymmetric $SU(2)$ WZW model one introduces three fermions ζ^a which together with J^a form the supercurrent. The purely bosonic current j^a is defined analogous to (7.9), and the level of the bosonic part is shifted from k' to $k'-2$. The stress tensor and the Virasoro supercurrent are given by

$$\begin{aligned} T &= \frac{1}{k}(\eta_{ab}k^ak^b - \eta_{ab}\chi^a\partial\chi^b) + \frac{1}{k'}(\delta_{ab}j^aj^b - \delta_{ab}\zeta^a\partial\zeta^b) \\ G &= \frac{2}{k}\left(\eta_{ab}\chi^ak^b - \frac{i}{3k}\epsilon_{abc}\chi^a\chi^b\chi^c\right) + \frac{2}{k'}\left(\delta_{ab}\zeta^aj^b - \frac{i}{3k'}\epsilon_{abc}\zeta^a\zeta^b\zeta^c\right) \ . \end{aligned} \tag{7.11}$$

Criticality of superstring theory on $AdS_3 \times S^3 \times \mathcal{M}$, where $\mathcal{M}$ is $K3$ or T^4, requires the central charge to satisfy

$$\frac{3(k+2)}{k} + \frac{3}{2} + \frac{3(k'-2)}{k'} + \frac{3}{2} = 9 \ , \tag{7.12}$$

which relates the levels of the current algebras

$$k = k' \ . \tag{7.13}$$

It is worthwhile to use variables commonly used when discussing AdS/CFT duality. In deriving the AdS_3/CFT_2 correspondence from

D-branes, S-duality can be used to transform the D1-D5 system into an NS1-NS5 system. Taking the near horizon limit, the level of $SL(2,R)$ WZW model is identified with Q_5, the number of 5 branes (for details see [11], [87]). Hence the bosonic levels of the $SL(2,R)$ and $SU(2)$ WZW models are Q_5+2 and Q_5-2, respectively.

The supersymmetric generalization of spectral flow in $SL(2,R)$ WZW model was given in [88]. The spectral flow operation, given by the action of what was referred to as the "twist field" in that work, not only induces transformation on the $\widehat{SL}(2,R)$ quantum numbers but also on the CFT describing the internal space. Physically, this coupling between the $SL(2,R)$ part and the internal CFT has its roots in the fact that in order for the spacetime theory to admit supersymmetry, one needs to pair χ^3 with a fermion from the internal CFT and then bosonize [14,89,90]. In the case of $AdS_3\times S^3\times\mathcal{M}$ the internal fermion is identified with ζ^3 and in the language of [88] every time the twist in $\widehat{SL}(2,R)$ is taken there is a corresponding twist in $\widehat{SU}(2)$.

Thinking of spectral flow as a twist is equivalent to the parafermion decomposition $SL(2,R)\simeq SL(2,R)/U(1)\times U(1)$ and $SU(2)\simeq SU(2)/U(1)\times U(1)$, in the following way. Introduce free bosons ϕ and ψ, normalized such that

$$\langle\phi(z)\phi(z')\rangle=\log(z-z')\ ,\qquad \langle\psi(z)\psi(z')\rangle=-\log(z-z')\ . \tag{7.14}$$

In terms of which k_0^3 and j_0^3 can be expressed as

$$k^3(z)=-i\sqrt{\frac{k}{2}}\partial\phi\ ,\qquad j^3(z)=-i\sqrt{\frac{k'}{2}}\partial\psi\ . \tag{7.15}$$

Throughout this discussion k and k' stand for the bosonic $SL(2,R)$ and $SU(2)$ levels, respectively. Then the bosonic $SL(2,R)$ primary field $\Phi_{ln\bar{n}}$ is decomposed into a field of $SL(2,R)/U(1)$ times a field in $U(1)$, where the $U(1)$ is generated by ϕ:

$$\Phi_{ln\bar{n}} = e^{in\sqrt{\frac{2}{k}}\phi + i\bar{n}\sqrt{\frac{2}{k}}\phi}\Phi^{SL/U(1)}_{ln\bar{n}} \ . \tag{7.16}$$

Similarly, a bosonic $SU(2)$ primary $\Psi_{jm\bar{m}}$ is written as

$$\Psi_{jm\bar{m}} = e^{im\sqrt{\frac{2}{k'}}\phi + i\bar{m}\sqrt{\frac{2}{k'}}\phi}\Psi^{SU/U(1)}_{jm\bar{m}} \ . \tag{7.17}$$

The fields $\Phi^{SL/U(1)}_{ln\bar{n}}$ are $\Psi^{SU/U(1)}_{jm\bar{m}}$ parafermions, with weight

$$\begin{aligned} h(\Phi^{SL/U(1)}_{ln\bar{n}}) &= -\frac{l(l-1)}{k-2} + \frac{n^2}{k} \ , \\ h(\Psi^{SU/U(1)}_{jm\bar{m}}) &= \frac{j(j+1)}{k'+2} - \frac{m^2}{k'} \ , \end{aligned} \tag{7.18}$$

so that (7.16) and (7.17) have the expected weights. Note that under the shift $n \to n + wk/2$ and $m \to m + wk'/2$, the weights of the primary fields change to

$$\begin{aligned} h(\Phi_{ln\bar{n}}) &\to -\frac{l(l-1)}{k-2} - nw - \frac{kw^2}{4} \ , \\ h(\Psi_{jm\bar{m}}) &\to \frac{j(j+1)}{k'+2} + mw + \frac{k'w^2}{4} \ . \end{aligned} \tag{7.19}$$

Spectral flow in the supersymmetric theory consists of the above shift in n, m, plus an additional contribution from the fermions [88], which gives

$$\begin{aligned} h(\Phi^{w}_{ln\bar{n}}) &= -\frac{l(l-1)}{Q_5} - nw - \frac{Q_5 w^2}{4} \ , \\ h(\Psi^{w}_{jm\bar{m}}) &= \frac{j(j+1)}{Q_5} + mw + \frac{Q_5 w^2}{4} \ . \end{aligned} \tag{7.20}$$

There is a similar relation on the anti-holomorphic side as well, with the same w. Note that the parafermion formalism also provides a convenient way of defining the vertex operators for states belonging to the spectral flowed representations [91,16,81]. The physical state condition is $(L_n - a\delta_{n,0})|\Omega\rangle = 0$ for $n \geq 0$, where $a = \frac{1}{2}$ in the NS sector and $a = 0$ in the R sector, as well as $G_r|\Omega\rangle = 0$ for $r \geq 0$. In addition, the analogue of GSO projection is the requirement of mutual locality with the supercharges that are constructed by bosonizing the worldsheet fermions [88].

8. Penrose limit of $AdS_3 \times S^3$ with NS background

In this section we explain the Penrose limit [52] of $AdS_3 \times S^3$ that results in the plane wave geometry [50,92].

The six dimensional plane wave is obtained from $AdS_3 \times S^3$ by expanding around a particular class of geodesics. These geodesics correspond to a particle near the center of AdS_3 and moving with very high angular momentum around a great circle of S^3. For this purpose, we begin with the spacetime metric

$$ds^2 = R^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\phi^2 + \cos^2\theta d\psi^2 + d\theta^2 + \sin^2\theta d\varphi^2) \tag{8.1}$$

and introduce the coordinates

$$\begin{aligned} t &= \mu x^+ \\ \psi &= \mu x^+ - \frac{x^-}{\mu R^2} \ . \end{aligned} \tag{8.2}$$

Rescaling $\rho = r/R$, $\theta = y/R$, the metric is expanded around $\rho = \theta = 0$ by taking the limit $R \to \infty$. This results in the six dimensional plane wave

$$ds^2 = -2dx^+dx^- - \mu^2(r^2+y^2)dx^+dx^+ + dr^2 + r^2d\phi^2 + dy^2 + y^2d\varphi^2 \ . \tag{8.3}$$

String spectrum in this background with NS three form field strength was found by quantizing the light cone action in [50,70,72]. For our purposes we will find it convenient to take the light cone Hamiltonian as given in [71], adapted to the conventions of this paper and supersymmetrized,

$$H_{lc} = p^- = \mu(2 + q + \bar{q}) + \frac{N + \bar{N} + h^{\mathcal{M}} + \bar{h}^{\mathcal{M}} - 1}{p^+\alpha'} \ . \tag{8.4}$$

This applies to the NS-NS sector, and the last term needs to be appropriately changed for the R sector. The quantities appearing in this expression have the following physical interpretation. N is the total level of excitations along the pp-wave. $h^{\mathcal{M}}$ is the weight of the state coming from the CFT on $\mathcal{M}$. Finally, q is the net number of times the spacetime light cone energy raising and lowering operators have been applied to the ground state. The ground state in question may or may not be physical, i.e. we are referring to the ground state before the GSO projection. We have chosen the letter q to denote this number because as we shall see the physical meaning of this quantity is the same as the q we used in labelling the current algebra representations, see the remark below (3.12). There are corresponding contributions from the anti-holomorphic side to (8.4), subject to the constraint that the net momentum along the worldsheet vanishes,

$$N + h = \bar{N} + \bar{h} \ . \tag{8.5}$$

The lightcone variables p^- and p^+ are related to observables measured in the global coordinates (8.1) by

$$\begin{aligned} p^- &= i\partial_{x^+} = \mu(E - J) \\ p^+ &= i\partial_{x^-} = \frac{J}{\mu R^2} \ . \end{aligned} \tag{8.6}$$

E is the spacetime energy and J is the angular momentum around the ψ direction in S^3. Our choice of basis in labeling the $SU(2)$ representations (7.4) corresponds to diagonalizing the action of rotation in ψ, hence J is given by $J_0^3 + \bar{J}_0^3$.

The radius of AdS_3 and S^3 is related to Q_5 by $R^2 = \alpha' Q_5$, so the second equation in (8.6) is equivalent to

$$\mu p^+ \alpha' = \frac{J}{Q_5}. \tag{8.7}$$

Hence the string spectrum in the NS-NS sector is

$$E - J = 2 + q + \bar{q} + \frac{Q_5}{J}(N + \bar{N} - 1) + \frac{Q_5}{J}(h^{\mathcal{M}} + \bar{h}^{\mathcal{M}}) \,, \tag{8.8}$$

with the condition (8.5).

We make a few comments about the brane charges. Note that Q_1, the number of 1branes, actually never appears in any of the formulas[20]. But it should be kept in mind that Q_1 is being taken to infinity as well. As explained in [92], the plane wave limit can be described in terms of the brane charges by taking Q_1, $Q_5 \to \infty$, with fixed Q_1/Q_5. The scaling used to obtain the plane wave requires that finite energy excitations of the resulting geometry have Δ, $J \to \infty$ as $\sqrt{Q_1 Q_5}$, with finite $\Delta - J$. Since $Q_1 \propto Q_5$, this actually implies that Δ, $J \to \infty$ as $Q_5 \sim k$, the level of the current algebra. We could have seen this directly from the fact that J/R^2 is held fixed as the limit $R^2 \to \infty$ is taken, but then it would not be clear that Q_1 is scaled to infinity as well. Also note that in the case of $Q_5 = 1$, due to the aforementioned shift in the level of the bosonic WZW model the bosonic $SU(2)$ part has a negative level. This is in conflict with the well-known result that the $SU(2)$ level must be a non-negative integer. We will return to the issue of $Q_5 = 1$ later.

[20] This is a feature of the NS1-NS5 description [87].

9. Nambu action near the origin of $AdS_3 \times S^3$

One of the things we want to understand is how the string spectrum on $AdS_3 \times S^3 \times \mathcal{M}$ reduces to (8.8) in the limit $Q_5, J \to \infty$. In order to answer this question we must first understand how (8.8) takes into account the spectral flow parameter w. In this section we explain the physical significance of spectral flow in the plane wave.

The plane wave limit described above is essentially a semi-classical expansion about $AdS_3 \times S^3$, combined with the unusual procedure of boosting to infinite (angular) momentum. Indeed, the large k limit in WZW models corresponds to the semi-classical limit, since the WZW action is proportional to k. Motivated by these concerns we will consider the Nambu action, upto quadratic order in the fields, of a string moving moving near $\rho \sim \theta \sim 0$ of $AdS_3 \times S^3$. When J is taken to be large, of order Q_5, the resulting action displays spectral asymmetry which is then related to spectral flow [16].

The Nambu action is given by

$$S = \frac{1}{2\pi\alpha'} \int d\tau d\sigma (\sqrt{|g|} - \epsilon_{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu) \qquad (9.1)$$

with g the induced metric and $B_{\mu\nu}$ the NS-NS two form. The non-zero components of the B field are

$$B_{t\phi} = \frac{1}{4}\alpha' Q_5 \cosh 2\rho \ , \quad B_{\psi\varphi} = \frac{1}{4}\alpha' Q_5 \cos 2\theta \ . \qquad (9.2)$$

We will consider a string located at small values of ρ and θ, and moving along the ψ direction. Since we will be interested in states with fixed angular momentum around ψ, we take as our classical solution $\psi(\tau, \sigma) = \psi(\tau)$. This corresponds to a string collapsed to a point and rotating around a great circle[21]. The components of the induced metric $g_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$ are, in the gauge $t = \tau$,

$$\begin{aligned} g_{00} &= \alpha' Q_5(-(1+(X^a)^2) + \partial_0 X^a \partial_0 X^a \\ &\qquad + (1-(Y^a)^2)(\partial_0\psi)^2 + \partial_0 Y^a \partial_0 Y^a) \\ g_{01} &= \alpha' Q_5(\partial_0 X^a \partial_1 X^a + \partial_0 Y^a \partial_1 Y^a) \\ g_{11} &= \alpha' Q_5(\partial_1 X^a \partial_1 X^a + \partial_1 Y^a \partial_1 Y^a) \ , \end{aligned} \tag{9.3}$$

where $X^1 + iX^2 = \rho e^{i\phi}$, and $Y^1 + iY^2 = \theta e^{i\varphi}$. The coupling to B field simplifies in this gauge to

$$-\frac{Q_5}{2\pi}\int d\tau d\sigma(\rho^2\partial_1\phi - \theta^2\partial_0\psi\partial_1\varphi) \ , \tag{9.4}$$

where we have used the fact that ψ has no dependence on σ,

$$\int d\tau d\sigma \partial_0\psi\partial_1\varphi = \int d\tau d\sigma \partial_1(\partial_0\psi\varphi) = 0 \ . \tag{9.5}$$

The resulting action (9.1) shows that ψ is a cyclic coordinate. Hence, the conjugate momentum $J_0 = \frac{\partial L}{\partial(\partial_0\psi)}$ is constant and it is advantageous to perform a Legendre transformation for ψ. The resulting Routhian,

$$R(X^a, Y^a; J_0) = L - J_0\partial_0\psi \ , \tag{9.6}$$

is then the Lagrangian that describes the dynamics of X^a and Y^a, while treating J_0 as a constant of motion. The subscript 0 is added to J here to indicate that it is the angular momentum of the ground state, because we are discussing the point particle limit. Taking J_0 to

[21] The importance of studying such solutions were pointed out in [93,94].

be large, of order Q_5, the action for X^a and Y^a upto quadratic order in the fields is found to be

$$S = \frac{J_0}{2\pi}\int d^2\sigma \left[1 - \frac{1}{2}|\partial_0\Theta|^2 + \frac{1}{2}\frac{1}{A^2}|(\partial_1 - iA)\Theta|^2 \right. \\ \left. -\frac{1}{2}|\partial_0\Phi|^2 + \frac{1}{2}\frac{1}{A^2}|(\partial_1 - iA)\Phi|^2\right] , \tag{9.7}$$

where $A = J_0/Q_5$, and $X^1 + iX^2 = \Phi$, $Y^1 + iY^2 = \Theta$. We see that Φ and Θ are two massless charged scalar fields on $R \times S^1$, coupled to a constant gauge field $A_a = A\delta_{a,1}$. As shown in [16], this implies that if A is not an integer, the states of Φ and Θ belong to the discrete representations with spectral flow number w equal to the integer part of A. Let us explain how this arises. The solution to the equation of motion that follows from (9.7) is

$$\begin{aligned} \Phi &= \sum_n \left(a_n^\dagger e^{i(n-A)(\tau/A+\sigma)} + b_n e^{-i(n-A)(\tau/A-\sigma)}\right)\frac{e^{iA\sigma}}{n-A} \\ \Theta &= \sum_n \left(c_n^\dagger e^{i(n-A)(\tau/A+\sigma)} + d_n e^{-i(n-A)(\tau/A-\sigma)}\right)\frac{e^{iA\sigma}}{n-A} . \end{aligned} \tag{9.8}$$

Canonical quantization gives for the commutation relations

$$\begin{aligned} [a_n, a_m^\dagger] &\sim (n-A)\delta_{n,m} , \quad [b_n, b_m^\dagger] \sim (n-A)\delta_{n,m} \\ [c_n, c_m^\dagger] &\sim (n-A)\delta_{n,m} , \quad [d_n, d_m^\dagger] \sim (n-A)\delta_{n,m} . \end{aligned} \tag{9.9}$$

Hence, for $n > A$, $a_n^\dagger$ is the creation operator while for $n < A$, a_n should be thought of as the creation operator. Similar comments apply to the other sets of operators. The holomorphic currents constructed from Φ and Θ are

$$\begin{aligned}
K^+ &\sim -iQ_5 \sum_n a_n e^{-in(\tau/A+\sigma)} \\
K^- &\sim iQ_5 \sum_n a_n^\dagger e^{in(\tau/A+\sigma)} \\
J^+ &\sim -iQ_5 \sum_n c_n e^{-in(\tau/A+\sigma)} \\
J^- &\sim iQ_5 \sum_n c_n^\dagger e^{in(\tau/A+\sigma)} \ .
\end{aligned} \tag{9.10}$$

Each current may be mode expanded and using (9.9) the vacuum obeys

$$\begin{aligned}
n > A: &\quad J_n^+|0\rangle = 0 \ , \quad K_n^+|0\rangle = 0 \ , \\
n > -A: &\quad J_n^-|0\rangle = 0 \ , \quad K_n^-|0\rangle = 0 \ .
\end{aligned} \tag{9.11}$$

Notice that this is different from the familiar highest weight conditions, which state that, for example, $K_{n>0}^+$ should annihilate the vacuum. The highest weight conditions can be restored by the transformation

$$K_n^\pm = \tilde{K}_{n\mp w}^\pm \ , \qquad J_n^\pm = \tilde{J}_{n\mp w}^\pm \ , \tag{9.12}$$

with w an integer satisfying $w < A < w+1$. With respect to $\tilde{K}$ and $\tilde{J}$, the states created from $|0\rangle$ fill out the conventional highest weight representations. This shows that for J_0 not a multiple of Q_5, the states are in the discrete representations with spectral flow number equal to the integer part of J_0/Q_5.

On the other hand, when J_0/Q_5 is an integer, the $SL(2,R)$ part of the state is in the continuous representation with spectral flow number J_0/Q_5 [16].

The fact that spectral flow is necessary when J_0 is comparable to Q_5 should not be too surprising. In fact, the role of spectral flow is precisely to resolve the apparent conflict between the upper limit on

$SL(2,R)$ spin of the discrete representations (3.17) and the freedom to have arbitrarily high angular momentum on S^3. More generally, for spacetimes of the form $AdS_3 \times \mathcal{N}$, the analysis of [16] shows that the amount of spectral flow is determined by ratio of the conformal weight h coming from the operator of the $\mathcal{N}$ CFT to the $SL(2,R)$ level k,

$$w < \sqrt{\frac{4h}{k}} < w + 1 \; . \tag{9.13}$$

For the case at hand, we see that $4h$ can be approximated as J_0^2/k and using $k \sim Q_5$ this reproduces what we found above.

What is surprising, however, is that (9.7) and the arguments that follow it imply that spectral flow should also be taken in the $SU(2)$ theory, with the same amount as the $SL(2,R)$ part. To be sure, this is not to suggest that the Hilbert space of $SU(2)$ WZW model needs to be enlarged to include spectral flowed representations, similar to what was done in the case of $SL(2,R)$ model. Whereas the $\widehat{SL}(2,R)$ representations generated by spectral flow are new and distinct from the conventional representations, this is not true in the case of $\widehat{SU}(2)$ representations. But as we explained supersymmetry requires that spectral flow is taken in both WZW models. Due to the high number of supersymmetries possible on this background[22] it is not unreasonable to think that this peculiar feature of the supersymmetric theory manifests itself in the purely bosonic analysis presented here. Additionally, note that the action of spectral flow on the angular momentum generator,

$$J_0^3 \to J_0^3 + \frac{wk}{2}, \tag{9.14}$$

[22] String theory on $AdS_3 \times \mathcal{N}$ generically has $N = 2$ spacetime supersymmetry if $\mathcal{N}$ has an affine $U(1)$ symmetry and the coset $\mathcal{N}/U(1)$ admits a $N = 2$ superconformal algebra. In the case $\mathcal{N} = S^3 \times \mathcal{M}$, supersymmetry is enhanced to $N = 4$ [89,90,95].

has the right form to be useful in keeping track of states with $J \sim k$ while k is taken to infinity. This feature makes it worthwhile to introduce spectral flow in the $SU(2)$ WZW model[23]. In the next section, we will use this idea to obtain the large J spectrum of superstrings on $AdS_3 \times S^3$.

10. The plane wave spectrum

We now turn to explaining how the plane wave spectrum arises from the exact $AdS_3 \times S^3$ results. The discussion will be limited to the NS sector, as the R sector can be obtained by similar methods, with the additional use of the spin fields.

10.1. Short strings

We start with the discrete $w = 0$ states, the holomorphic side of which is labeled by the quantum numbers

$$|\ell, n, N\rangle \otimes |j, m, M\rangle \otimes |h^{\mathcal{M}}\rangle \ . \tag{10.1}$$

The notation in labeling the $\widehat{SL}(2,R) \times \widehat{SU}(2)$ part of the state is the same as what was used in section 2, and $h^{\mathcal{M}}$ is the conformal weight coming from the CFT on $\mathcal{M}$. In order for (10.1) to be physical, it must satisfy

$$-\frac{\ell(\ell-1)}{Q_5} + \frac{j(j+1)}{Q_5} + N + M + h^{\mathcal{M}} = \frac{1}{2} \ . \tag{10.2}$$

Let us look for the ground state within a given j sector. First, we note that the GSO projection [88] requires the lowest excitation number to

[23] See [96] for an interesting application of spectral flow in the $SU(2)$ WZW model.

be one half, so from (10.2) we find $\ell = j+1$. Next, we see that the lowest value of energy (for fixed j) is obtained if this one half unit of excitation comes from the action of $\zeta^+_{-1/2}$ or $\chi^-_{-\frac{1}{2}}$. In the first case, the ground state is

$$|J/2, J/2\rangle \otimes \zeta^+_{-\frac{1}{2}}|J/2-1, J/2-1\rangle \otimes |0\rangle \ , \tag{10.3}$$

and in the second,

$$\chi^-_{-\frac{1}{2}}|J/2+1, J/2+1\rangle \otimes |J/2, J/2\rangle \otimes |0\rangle \ . \tag{10.4}$$

Combining with an identical state in the anti-holomorphic side, we see that there is a total of four states that carry angular momentum J and energy $E = J$, i.e. the light cone vacuum.

We will not discuss the Ramond sector in detail, but in order to complete the discussion of light cone ground states we briefly mention how many are found in the Ramond sector. The number of light cone ground states coming from the Ramond sector depends on whether $\mathcal{M}$ is T^4 or $K3$. For T^4, there are two ground states in the R sector, and one can construct the usual NS-NS, NS-R, R-NS, R-R sectors to find a total of 16 ground states [97]. When the internal manifold is $K3$, for the purposes of counting ground states we can think of T^4/Z_2 instead. Then, as explained in [72], the ground states in the NS-R and R-NS sectors are projected out, and the 16 twisted sectors each give a ground state in the R-R sector. Thus there are 24 ground states in all, as expected.

The excited states of $w = 0$ representations are obtained from the lowest weight of $SL(2,R)$ and the highest weight of $SU(2)$ by the action of negatively moded generators. Physical states do not carry excitations along the time direction. For example, in the $SL(2,R)$

Hilbert space those states satisfying the Virasoro conditions can be written

$$\prod_{r=1/2}^{\infty}(\chi^{+}_{-r})^{N^{+}_{r}}(\chi^{-}_{-r})^{N^{-}_{r}}\prod_{n=0}^{\infty}(k^{+}_{-n})^{N^{+}_{n}}(k^{-}_{-n})^{N^{-}_{n}}|\ell,n=\ell\rangle\ , \tag{10.5}$$

which has the grade

$$N=\sum_{n}n(N^{+}_{n}+N^{-}_{n})+\sum_{r}r(N^{+}_{r}+N^{-}_{r}) \tag{10.6}$$

and $n=\ell+q_{SL}$, with

$$q_{SL}=\sum_{n}(N^{+}_{n}-N^{-}_{n})+\sum_{r}(N^{+}_{r}-N^{-}_{r})\ . \tag{10.7}$$

Similar relations hold for the $SU(2)$ part. Now (10.2) is used to solve for ℓ, which then gives for the energy

$$\begin{aligned} E &= 1+q_{SL}+\bar{q}_{SL} \\ &\quad +\sqrt{(2j+1)^2+2Q_5(N+\bar{N}+M+\bar{M}+h^{\mathcal{M}}+\bar{h}^{\mathcal{M}}-1)}\ , \end{aligned} \tag{10.8}$$

with j related to J by $J=2j-q_{SU}-\bar{q}_{SU}$. Now we take the "Penrose limit" Q_5, $J\to\infty$ with J/Q_5 fixed, and expanding to terms of order one we find

$$E-J=2+q_{SL}+\bar{q}_{SL}+q_{SU}+\bar{q}_{SU}+\frac{Q_5}{J}(N+\bar{N}+M+\bar{M}+h^{\mathcal{M}}+\bar{h}^{\mathcal{M}}-1)\ . \tag{10.9}$$

Note that the vacuum states considered above corresponds to sum of the q's totalling -2 and total grade equal to 1. That the lowest energy state surviving the GSO projection in the NS sector has a half unit of excitation is similar to what happens in flat space. The difference in this case is that the various raising operators have different charges under E and J. Note also that the $w = 0$ continuous representations are projected out from the physical spectrum, since for those representations it is impossible to satisfy the physical state condition unless $N = 0$. Hence the spectrum is free of tachyons.

Having understood the $w = 0$ states, we now turn to the spectral flowed states. Consider a state in the spectral flowed representation of $\widehat{SL}(2, R) \times \widehat{SU}(2)$, tensored with an operator on $\mathcal{M}$,

$$|w, \tilde{\ell}, \tilde{n}, N\rangle \otimes |w, \tilde{j}, \tilde{m}, M\rangle \otimes |h^{\mathcal{M}}\rangle \ . \tag{10.10}$$

There is a similar state on the anti-holomorphic side. Using (7.20), the physical state condition determines $\tilde{\ell}$ to be

$$2\tilde{\ell} = 1 - Q_5 w + \sqrt{(2\tilde{j} + Q_5 w + 1)^2 + 2Q_5\left(N + \bar{N} + M + \bar{M} - 2w + h^{\mathcal{M}} + \bar{h}^{\mathcal{M}} - 1\right)} \ , \tag{10.11}$$

where we have used the second equation in (7.20) for the weight of the $SU(2)$ state. In this relation N and M are the grades measured by L_0, not $\tilde{L}_0$, of the $SL(2, R)$ and $SU(2)$ model respectively. Now we can use $J = 2\tilde{j} + Q_5 w - q_{SU} - \bar{q}_{SU}$ to substitute for $\tilde{j}$ in the expression above, and the energy is given by

$$E = 2\tilde{\ell} + Q_5 w + q_{SL} + \bar{q}_{SL} \ . \tag{10.12}$$

This result is an exact formula for the energy of a string state in $AdS_3 \times S^3 \times \mathcal{M}$ with angular momentum J around S^3.

Taking the limit Q_5, $J \to \infty$ and expanding to terms of order one,

$$\begin{aligned} E - J = {} & 2 + q_{SL} + \bar{q}_{SL} + q_{SU} + \bar{q}_{SU} \\ & + \frac{Q_5}{J}(N + \bar{N} + M + \bar{M} - 2w - 1) + \frac{Q_5}{J}(h^{\mathcal{M}} + \bar{h}^{\mathcal{M}}) \ . \end{aligned} \tag{10.13}$$

The states with $E = J$ again have the form (10.3) or (10.4), but now there is a slight difference due to spectral flow. For example, in the spectral flowed analogue of (10.3), fermionic generator is given by $\tilde{\zeta}^{+}_{-\frac{1}{2}}$, which has $M = \frac{1}{2} + w$ after taking into account the shift in moding from spectral flow. This serves to cancel the extra term in (10.13) compared to (10.9). As found in [88], the pattern of chiral states is relatively simple. Once the $w = 0$ chiral states are identified, spectral flow generates the chiral states with higher R-charge. In general a state similar to (10.5) in a spectral flowed representation has $\tilde{N}$ and $\tilde{q}_{SL}$ defined in the same manner as (10.6) and (10.7), respectively. They are related to what appear above as

$$\begin{aligned} N &= \tilde{N} - w\tilde{q}_{SL} \ , \\ q_{SL} &= \tilde{q}_{SL} \ . \end{aligned} \tag{10.14}$$

In the semiclassical discussion of the previous section we saw that the amount of spectral flow necessary is determined by the ratio J_0/Q_5, where J_0 is the angular momentum of the ground state, i.e. a state in the zero grade of a $\widehat{SU}(2)$ representation. In the fully quantum treatment, w is determined through the inequality $\frac{1}{2} < \tilde{\ell} < \frac{Q_5+1}{2}$, which becomes

$$\begin{aligned} w^2 < & \frac{(2\tilde{j} + Q_5 w + 1)^2}{Q_5^2} \\ & + \frac{2}{Q_5}(N + \bar{N} + M + \bar{M} - 2w + h^{\mathcal{M}} + \bar{h}^{\mathcal{M}} - 1) < (w+1)^2 \ . \end{aligned} \tag{10.15}$$

It should be remembered that N and M also depend on w, through (10.14) and an analogous relation for M. In (10.15) we can think of $\tilde{j} + Q_5 w/2$ as the highest weight of the $SU(2)$ representation from which the current algebra representation is constructed,

$$J_0 = 2\tilde{j} + Q_5 w \ , \tag{10.16}$$

and (10.15) reproduces the semiclassical result found previously.

10.2. Long strings and the "missing" chiral primaries

Let us now discuss what happens when the inequality in (10.15) is saturated, which in the semiclassical approximation corresponds to J_0/Q_5 becoming an integer. In this case we know from [16] that the state belongs to a continuous representation of $\widehat{SL}(2,R)$ with spectral flow number $w = J_0/Q_5$, i.e. it is a long string in AdS_3. Morever, the energy of the solution changes smoothly in the transition from a short string to a long string (and vice versa). The continuous representations do not have highest or lowest weights and for this reason the spectral flowed states are labelled by the eigenvalues of $\tilde{L}_0$. The plane wave spectrum of the long strings is therefore

$$E - J = 2 + \frac{Q_5}{J}(\tilde{N} + \bar{\tilde{N}} + \tilde{M} + \bar{\tilde{M}} - 2w - 1) + \frac{Q_5}{J}(h^{\mathcal{M}} + \bar{h}^{\mathcal{M}}) \ . \quad (10.17)$$

Sometimes it is possible for a long string to have zero light cone energy despite the fact that it is massive. If $|0, w\rangle$ denotes a state with $E = J$ then $k^+_w|0, w\rangle$ continues to have zero light cone energy because k^+_w's contribution to (10.17), proportional to $\tilde{N}$, vanishes. The physical mechanism responsible for this phenomenon is the same as in AdS_3. Namely, the coupling to the NS three form cancels the gravitational attraction. In the context of plane waves supported by NS field strengths it has already been observed that there are additional zero modes in the spectrum [50,73,96], which can be understood as the statement that states with special values of p^+ — integer multiples of $1/\mu\alpha'$ — do not feel the confining potential of the plane wave.

It is interesting to note that simplifying $AdS_3 \times S^3$ to the plane wave makes more apparent the presence of long strings in the spectrum. As we have just stated, some of the long strings correspond to chiral primaries in the dual CFT. It has been appreciated for a while now that there is a mismatch of chiral primaries in the AdS_3/CFT_2 correspondence when considering the AdS_3 with a purely NS background due to the fact that AdS_3 with vanishing R-R fields corresponds to a "singular" CFT [98,88]. The mismatch arises when one tries to compare the spectrum of chiral primary operators in the CFT to the spectrum of chiral string states based on the discrete representations of $\widehat{SL}(2,R)$. It was suggested in [98] that the chiral primaries that disappear when all the R-R fields are set to zero might be found among the continuum. We find explicitly that indeed there are chiral primaries belonging to the continuous representations.

11. The decomposition of the Hilbert space in the Penrose limit

We started with a unitary spectrum of string states in $AdS_3 \times S^3 \times \mathcal{M}$. This spectrum is obtained from the Hilbert space of the $SL(2,R)$ WZW model, tensored with the Hilbert spaces of the $SU(2)$ model and CFT on $\mathcal{M}$, and imposing the Virasoro constraints. In obtaining the results of previous section we have restricted our focus to a particular subsector of this physical Hilbert space. We now address the question of what happens to the remaining states in the Hilbert space. We find that the ratios J/Q_5, J^2/Q_5 determine where the state ends up.

As we take the limit $R \to \infty$, we expect that some of the states become strings in flat space, some become strings in the plane wave, and the rest with divergent $E-J$. The spectra in flat space and plane wave should form independent, unitary Hilbert spaces. Presumably, this means that the states with divergent $E-J$ should also, but with a different description. An example of such states would be those that have high angular momentum along a different circle on S^3. These states would be related to what we considered above by a global rotation on the sphere.

We have found the states on the plane wave. Which states correspond to strings in flat space? In any dimension, flat space is obtained from plane wave when [50]

$$\mu\alpha' p^+ << 1 \ . \tag{11.1}$$

But in our case, $\mu\alpha' p^+ = J/Q_5$, and we know that the integer part of J/Q_5 is related to the spectral flow parameter w in the large J, Q_5 limit. Thus we conclude that the flat space spectrum comes from the unflowed short strings in the original AdS_3 theory. We can indeed check that for $J/Q_5 \to 0$, J^2/Q_5 finite, the physical state condition for $w = 0$ short strings (10.2) reproduces the mass formula of superstrings in six flat dimensions times $\mathcal{M}$, because the terms in L_0 that involve the quadratic Casimirs become p^2 as the space becomes flat [82].

It is important to note that even though we have just identified the flat space spectrum as arising from the $w = 0$ sector of the original theory, this does not mean that none of the $w = 0$ short strings remain in the plane wave. Some of the states can still carry $J \sim Q_5$, and as the limit $Q_5 \to \infty$ is taken we find the result (10.9). However, the $w = 0$ plane wave states are generally farther from chiral than the spectral flowed states.

If $J^2/Q_5 \to 0$, then (10.9) tells us that the string modes have energy that diverges as $\sqrt{Q_5}$. Note, however, that even in this case the supergravity modes (i.e. states at grade 1/2 for both the right and left movers) remain, and they fall into the global $SL(2,R) \times SU(2)$ multiplets.

12. When the radius is small

An extremely interesting question one would like to address is what we can learn about string theory on $AdS \times S$ from string theory on plane waves. In the case of $AdS_3 \times S^3$ and its plane wave limit, we have a good understanding of both string theories, and we now turn to this question.

But first, we'd like to stress a small point, which is that a priori there are two distinct notions of "high curvature" one needs to keep in mind. When one speaks of a highly curved plane wave, that actually means

$$\mu\alpha' p^+ >> 1 \ . \tag{12.1}$$

In this case the string spectrum consists of highly spectral flowed states. We see from (10.13) that this means the low lying string modes become almost degenerate. This is similar to what happens in the $AdS_5 \times S^5$ plane wave.

Despite being "highly curved", the highly curved plane wave still involves taking the radii of $AdS \times S$ to infinity. Hence the GS superstring in highly curved plane waves is still amenable to quantization. The second, and more interesting, notion of "high curvature" is obtained by dropping the $R^2 \to \infty$ condition. Then clearly the geometry cannot be thought of as a plane wave. Since it is only after the Penrose limit is taken that the GS string can be solved, presently known results about the plane wave of $AdS \times S$ are not expected to remain valid in the case of small radius.

However, there have been some reasons to think that the plane wave spectrum (8.8) might continue to correctly describe the large J spectrum even outside the strict $Q_5 \to \infty$ limit. Authors of [92] studied various aspects of string theory on the plane wave (8.3) from the point of view of the dual $(\mathcal{M})^{Q_1 Q_5}/S_{Q_1 Q_5}$ CFT. One of the more interesting things they found in that work was that after extrapolating (8.8) to $Q_5 = 1$, the result surprisingly agrees with the spectrum predicted by the dual CFT at the orbifold point[24]. Since the CFT spectrum is believed to be reliable for arbitrary Q_5, whereas the string spectrum was found under the assumption that Q_5 is taken to infinity, this hints that perhaps (8.8) is true even when the spacetime geometry does not correspond to a plane wave. There have also been some work along this line for the $AdS_5 \times S^5$ plane wave [78,79], but with some differences, which we will discuss in the last section.

We can answer this question directly for the $AdS_3 \times S^3$ case since we worked out the string spectrum that is valid for all values of Q_5. Our results apply equally to small Q_5, when we should think of the geometry as $AdS_3 \times S^3 \times \mathcal{M}$ with the first two factors being highly curved. Thus, we can take (10.11), (10.12) and expanding for arbitrary fixed Q_5, large w, we find that, in fact, the large J spectrum is again given by (10.13). We conclude that the plane wave spectrum is actually the large J spectrum of strings on $AdS_3 \times S^3 \times \mathcal{M}$, for arbitrary values of the radius.

Actually, there are two special cases where the worldsheet description we have given so far could break down. These special cases occur for $Q_5 = 1$ or 2, whereby due to the shift in the level of the bosonic WZW models the $SU(2)$ model acquires a negative or zero level. However, the problem is not serious for the $Q_5 = 2$ case as we can understand it to mean that only the fermionic fields are present on the worldsheet for the S^3 part of the target space. The $Q_5 = 1$

[24] In fact, the NS $Q_5 = 1$ is the only case where a perfect agreement was found. Matching of the spectra in general requires g_s^2 corrections and on the CFT side involves moving away from the orbifold point in the moduli space.

case truly presents us with a difficulty since it is not clear how to make sense of the $SU(2)$ WZW model at level -1 as a physical theory. It is not known at present how to describe the $Q_5 = 1$ model, but arguments were presented in [98] to suggest that it is a sensible (albeit very special) system. We'd like to argue that the result (10.13) is valid even for the $Q_5 = 1$ case even though our starting point was not suited to describe it. For one, it would be rather unusual for the expression (10.13) to be true for $Q_5 = 2, 3, \ldots \infty$ and not be true for $Q_5 = 1$ when nothing special happens as we try to set $Q_5 = 1$. More importantly, the dual CFT is well defined at $Q_5 = 1$ and its prediction for the string spectrum [92] matches perfectly with (10.13). Perhaps $Q_5 = 1$ actually represents the zero radius limit of AdS_3, thus providing the reason behind perfect agreement with the symmetric orbifold. The orbifold point of the CFT corresponds to the free theory (analogous to setting $g_{YM} = 0$ in AdS_5/CFT_4), whose dual string theory would apparently be formulated on zero radius AdS_3. We will return to this issue in the Discussion.

13. Discussion

The two main objectives of this investigation have been

(a) To provide a CFT description of strings in a plane wave background, giving the necessary framework for a detailed study of BMN correspondence using the powerful tools of CFT.

(b) To investigate the relationship between string theory on $AdS \times S$ and string theory on plane waves, using the solvable $AdS_3 \times S^3$ case as a model.

We offer some comments on each of these issues.

It is worth emphasizing that we are now positioned to take advantage of the CFT techniques to study string interactions in the $AdS_3 \times S^3$ plane wave. This is in stark constrast to the much studied case of the ten dimensional plane wave that arises from $AdS_5 \times S^5$, where the RNS description of strings is lacking and interactions can only be studied using string field theory. In fact, correlation functions in AdS_3 have already been calculated [81], so together with the correlation functions of $SU(2)$ WZW model it should be possible to obtain scattering amplitudes in the plane wave by appropriately taking the large J, Q_5 limit. This should prove to be an useful area for study.

In regards to the AdS_3 correlation functions, we show in Appendix A that the spectral flow number violation rule found in [81] can be understood as the conservation of angular momentum in the plane wave.

Additionally, one expects that the map between the CFT operators and plane wave string states is easier to establish than the ten dimensional case, owing to the fact that the AdS_3/CFT_2 duality is highly constrained by the infinite dimensional conformal symmetry. Thus, it becomes a technically simpler problem to study the BMN correspondence in situations where many interacting string modes are involved.

The other main point of this paper is that we have actually compared string theory on $AdS_3 \times S^3$ to string theory on the plane wave. We have found that the plane wave spectrum, which one might have thought to be the result of some simplification of the $AdS_3 \times S^3$ spectrum that occurs in the limit $Q_5, J \to \infty$, actually is the result of $J \to \infty$ only.

Recently it has been conjectured by Frolov and Tseytlin [76,77,80] that the semi-classical formula for the energy of strings carrying spins in multiple directions in $AdS_5 \times S^5$ continues to hold true at small values of the radius, provided that the spins take very large values[25]. Based on the findings of this paper, we feel strongly that their conjecture is true. Furthermore, if the relationship between string theory on $AdS_3 \times S^3$ and its plane wave limit applies to other $AdS \times S$ spaces, it suggests that the string spectrum on the plane wave limit of $AdS_5 \times S^5$ [51,50] is in reality the large J string spectrum on $AdS_5 \times S^5$.

Before leaving the subject of the Frolov-Tseytlin solution, let us note a curious fact. Frolov and Tseytlin found that the solution carrying two non-zero equal spins in S^5 has the energy

$$E = \sqrt{(2J)^2 + \frac{R^4}{\alpha'^2}} \ . \tag{13.1}$$

This bears striking resemblance to the energy of a low-lying short string state in $AdS_3 \times S^3$ with the single spin

$$E \sim \sqrt{J^2 + c\frac{R^2}{\alpha'}} \ , \tag{13.2}$$

where c is a number of order 1. Other than the difference in the power of R^2/α', which could be explained by the fact that the role of N in AdS_5/CFT_4 is played by both $Q_1 Q_5$ and $\sqrt{Q_1 Q_5}$ in AdS_3/CFT_2 [92], the two expressions are almost identical. It should be kept in mind that (13.1) is a classical result whereas (13.2) is a quantum one. It is not clear if Frolov-Tseytlin solution has an interpretation as giving arise to a simpler spacetime geometry in a manner similar to BMN. However, as we have seen, strings with large J in $AdS_3 \times S^3$ have a simple description even though it is only after the radius is taken

[25] See [99] for a discussion regarding the supersymmetry of the spinning strings.

to be large as well that they can be viewed as moving in the plane wave. At any rate, it would be extremely interesting to understand why these two expressions are so similar. Perhaps studying strings on $AdS_3 \times S^3 \times S^3 \times S^1$ [100], which makes multi-spin solutions possible, along the lines of this work will shed light on this issue[26].

Another topic of interest has been pursued recently[27], involving strings in the critical tension limit and the possibility of defining string theory in the zero radius limit of AdS. The hope is to take the $\lambda \to 0, N \to \infty$ limit of AdS/CFT at its face value and establish a duality between string theory in the zero radius AdS and a free field theory. We should mention from the start, however, that the approach has been to send R^2/α' to zero in the classical Hamiltonian and then quantize the resulting (simpler) theory. This by no means assures us that we will find the same results when we take the same limit in the quantum theory. Another point to keep in mind is that when the radius of the spacetime is comparable to the string scale, it is not clear whether one can even assign a definitive value to the radius.

Now we focus on the $AdS_3 \times S^3$ example and try to address this issue. Strictly speaking, one must set $Q_5 = 0$ to study the zero radius AdS_3. In this case we do not know how to make sense of the worldsheet theory. However, as stated above we do not believe that one should insist on being able to set R^2/α' exactly to zero in the quantum treatment. For the time being, we will be content with considering $R^2 \sim \alpha'$, which is still a nontrivial case. It is perhaps useful to recast the large J expansion of the exact energy formula using the radius of curvature in

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[26] The author would like to thank A. Adams for this point.

[27] See, for example, [78,79,101,102,103,104,105].

string units (we ignore the internal space $\mathcal{M}$ for this discussion, whose contribution is suppressed anyway):

$$E - J = 2 + q_{SL} + \bar{q}_{SL} + q_{SU} + \bar{q}_{SU} + \frac{R^2}{\alpha' J}(N + \bar{N} + M + \bar{M} - 2w - 1) \ . \tag{13.3}$$

We should note that the last terms in parantheses is what gives this expression its stringy nature. If for some reason (such as simply taking the "tensionless" limit $R^2/\alpha' = 0$ while continuing to trust (13.3)) they were absent, what remains would resemble a field theory spectrum. It might seem at first that the last terms would be negligible for large J, finite R^2/α'. But in fact this is not the case, because the excited string modes generically have level of order $\alpha' J/R^2$ due to spectral flow. The only way in which the last terms in (13.3) disappear is in strict $R^2/\alpha' J = 0$ case[28]. When that happens the spectrum can be schematically written

$$H_{lc} \sim \sum_{all\ modes} a^\dagger a \ , \tag{13.4}$$

which looks like a free field theory[29]. This suggests that the theory with $R^2/\alpha' = 0$ (whatever its proper description might be) is not continuously connected to the $R^2 \sim \alpha'$ cases at finite J.

In a related topic, authors of [78], [79] found evidence that the string spectrum on the plane wave limit of $AdS_5 \times S^5$ may be extrapolated down to finite J after setting g_s to zero, which has the effect of reducing the spectrum to the form (13.4). The agreement with the

[28] Note that the combination $R^2/\alpha' J$ is the square root of the coupling constant λ' identified in the BMN limit of $AdS_5 \times S^5$ [53,55].

[29] However, not all information about string excitation numbers seems to be lost since the $L_0 = \bar{L}_0$ constraint still needs to be imposed.

SYM prediction (which was done in [79] for conformal weights upto 10) as well as considerations of this paper lend support to the claim that in fact the entire string spectrum on $AdS_5 \times S^5$ reduces to (13.4) at $g_s = 0$.

There have also been some work on computing R^{-2} corrections to the plane wave spectrum as a way of approximating the $AdS \times S$ spectrum [106,107,71]. The results of this paper might be useful as a guide in checking higher order calculations. It is important to note, however, that in computing corrections to the plane wave one does not have the freedom to choose R^2 and J independently. The advantage we had in the $SL(2,R) \times SU(2)$ model was being able to vary Q_5 and J in an independent manner.

In conclusion, strings in $AdS_3 \times S^3$ and its plane wave or its large J limit seem to be very useful models to study and it is hoped that they will lead to a better understanding of the more complicated plane wave/CFT and AdS/CFT dualities.

Appendix A. The spectral flow number violation rule

In [81] it was found that the N-point function of vertex operators with spectral flow numbers w_i, viewed as describing the interaction of $i = 2, \ldots, N$ incoming strings and $i = 1$ outgoing string, vanishes unless[30]

$$w_1 \leq \sum_{i=2}^{N} w_i + N - 2 \ . \tag{A.1}$$

This result was derived using representation theory of $\widehat{SL}(2,R)$ algebra, irrespective of what the spacetime consists of besides AdS_3, and does not rely on any particular physical picture.

We now show that when considering the plane wave limit of $AdS_3 \times S^3 \times \mathcal{M}$, (A.1) can be understood as enforcing the conservation of J. In order to find a non-zero correlation function the J_i must satisfy

$$J_1 = \sum_{i=2}^{N} J_i \ . \tag{A.2}$$

We now divide both sides of this equation by Q_5 and identify w_i as the integer part of J_i/Q_5 (see the footnote below and also note that we are in the $J, Q_5 \to \infty$ regime). On the RHS, there will be $N-1$ terms, each of the form $w_i + \Delta_i$ where $0 \leq \Delta_i < 1$. The sum of Δ_i's will therefore be less than $N-1$. Hence the spectral flow numbers will satisfy (A.1).

[30] The discrete states are taken to be in the ground states of their representations, i.e. $\tilde{n}_i = \tilde{\ell}_i$.

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