Topics in Little Higgs Physics

A thesis presented

by

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to

The Department of Physics

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in the subject of

Physics

Harvard University

Cambridge, Massachusetts

May 2004

Abstract

The Standard Model is the currently accepted model of elementary particle interactions as determined by many years of particle physics experiments. It is both highly predictive and successful in its predictions, which is one of the great achievements of the last half century of science. However, there are indications that the Standard Model is incomplete with the main evidence being the so called hierarchy problem. In general terms, the hierarchy problem suggests that the parameters of the Standard Model have to be severely fine tuned in order to describe the real world. Motivated by the naturalness issues the hierarchy problem provokes, many theories of beyond the Standard Model physics have been proposed that alleviate the fine tuning. Continuing this approach, recently a new scenario called "Little Higgs" theories has been suggested as a viable alternative to the Standard Model, which has some novel features compared to the other competitive theories. The fact that these Little Higgs models may describe the real world is interesting and motivates their study.

In the first chapter, an introduction describing in elementary terms what the hierarchy problem is and how Little Higgs theories are a solution to this problem is presented. After this short introduction, two topics in Little Higgs theories are analyzed. The first subject is determining if Little Higgs theories are consistent with the precision experiments performed to date. This is covered in chapters two and three, where two different Little Higgs models with approximate custodial $SU(2)$ symmetry are presented. This symmetry increases the range of parameter space where these Little Higgs theories are both consistent with precision tests and are natural under the sense of the hierarchy problem. Thus, these theories are viable candidates for beyond the Standard Model physics. The fourth chapter is on the second subject, which is analyzing the constraints that unitarity places upon the Little Higgs theories. The important consequence is that requirements for unitarity suggest that there is new physics in these theories at lower scales than previously expected. This can

have implications for future experiments, as this physics can consist of particles light enough to be produced at the next generation of particle accelerators.

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Acknowledgements

First of all, I would like to thank my advisor, Howard Georgi, for accepting me as his student, for his mentoring and especially his unending patience. The most important lessons I've learned in graduate school I attribute to him: to always explain things as simply and clearly as possible and to always study what one finds most interesting.

I'm also thankful for the many wonderful professors I've had a chance to interact with here: in particular Sidney Coleman, John Huth, Markus Luty, Shiraz Minwalla, Lisa Randall, Andy Strominger, and Cumrun Vafa. I would especially like to thank Nima Arkani-Hamed for being a secondary advisor as well as for the many espressos. From my undergraduate years at Stanford, I would like to thank Robert Wagoner, his electrodynamics course has much to do with my love of theoretical physics, as well as Daniel Bump, Rick Pam, and Richard Schoen. From high school, I would like to thank Gary Hicks both for his excellent teaching and mentoring.

I would also like to thank the many postdoctoral researchers as well: Hsin-Chia Cheng, Paolo Creminelli, Ian Low, Lubos Motl, Shinji Mukohyama, Toby Wiseman deserve specific thanks.

Also deserving of thanks are classmates from past and present. Those in theoretical physics: Thomas Gregoire, Can Kilic, Rakhi Mahbubani, Joe Marsano, Andy Neitzke, Matthew Schwartz, Shiyamala Thambyahpillai, Dave Thompson, Jay Wacker, Devin Walker, the friends from TASI, and all the others. I would also like to thank the other graduate school classmates, especially Will Chen and Ming Hsu for partaking in many indian buffet binges. From before graduate school, I would especially like to thank Ryan Braner, Paul Chan, Hubie Chen, Arti Garg, Michelle Ngo, Richard Tan, Chris Tate, Ashwin Vasan, and Jason Yandell for continual friendship and support.

In addition, I must thank the wonderful administrative staff of the Physics Department. The heroes include Sheila Ferguson, Mary Lampros, Stuart McNeil, Rob Meyer, Duarte Moura and Nancy Partridge.

I would also like to thank the National Science Foundation for their three years of support through their Graduate Student Fellowship program. I am also grateful for the travel grant the Physics department provides which has enabled me to go to several enjoyable schools, conferences and workshops. In general, I am thankful for the overall support of basic research in this country.

Most importantly I would like to also thank my family for their unending support and love both during graduate school as well as throughout my life. My brother Alfred who has always been there with an open ear and helpful advice. My cousin Han who in his own way encourages my idealist tendencies. My parents, Henry and Karen, who instilled in me the importance of education, but most importantly have supported me in all my life decisions. Finally, I would like to thank my wife Sara, whose love and care over the past five years have been invaluable to me and are a blessing that I can never believe I fully deserve.

Chapter 1: Introduction

1 The Hierarchy Problem

Particle physics has been a vast overachievement in the last half century, where enormously courageous and ambitious experiments have been performed in step with the proposal of new imaginative theories. Through this endeavor, particle interactions have been probed down to 10^{-16} centimeters and one clear theory has emerged for what governs this known physics. That theory is called the Standard Model (SM) and is suprisingly accurate and seemingly complete.

Although there is no experimental reason to doubt this completeness, there is a more abstract reason to do so [1]. In the SM, there is a mass scale given by the masses of the electroweak gauge bosons (the mediators of the weak force). This mass scale is called the weak scale and is approximately $M_W \approx 100$ GeV. This scale is determined by minimizing the potential energy of a scalar field called the Higgs H. Schematically this is done by minimizing a potential of the form

$$V = -m_H^2 H^2 + H^4. \tag{1.1}$$

which has a minimum at $H = m_H/\sqrt{2}$ that is required by experiment to be near the weak scale M_W and thus forces $m_H \approx 100$ GeV. The hierarchy problem can be simply stated as the fact that this scenario requires fine tuning of the parameters of the theory. This problem occurs since there are much larger mass scales in the SM that can enter into the potential. For instance, interactions generate corrections to the mass parameter of the potential (called radiative corrections) and since the interaction strengths are given by dimensionless quantities, these corrections tend to scale with the largest masses in the theory. In the worst case scenario, the highest scale where the SM could be valid up to as a consistent theory is the Plank scale $M_P \approx 10^{19}$ GeV where quantum gravitational effects are expected to get strong (requiring a quantum theory of gravity). Thus, the main concern will be why there is this large hiearchy of 17 orders of magnitude between M_W and M_P.

Schematically what occurs is that the mass parameter in the above potential has the following form

$$m_H^2 = m_0^2 + m_{\rm rad}^2 \tag{1.2}$$

where m_0^2 is the mass parameter with the interactions turned off and $m_{\rm rad}^2$ is the radiative correction induced by the interactions. If the SM is complete up to the Planck scale, then we expect that $m_{\rm rad}^2 \approx M_P^2$. Therefore to get the experimental result $m_H^2 \approx M_W^2$ requires that $m_0^2 + M_P^2 \lesssim M_W^2$. This means that not only is the size of m_0 near the Planck scale but has to be *extremely* close to it, as seen by dividing by M_P^2 on both sides, giving

$$\frac{m_H^2}{M_P^2} = \frac{m_0^2}{M_P^2} + \frac{m_{\rm rad}^2}{M_P^2} \approx \frac{m_0^2}{M_P^2} + 1. \tag{1.3}$$

The left hand side has the size $(100\ \mathrm{GeV}/10^{19}\ \mathrm{GeV})^2 = 10^{-34}$ while we can see from the equation that $\frac{m_0^2}{M_P^2} = -1 + 10^{-34}$. Therefore, these two terms have to cancel to fantastic precision. If one varies m_0^2 alone, there is a need to have a parameter precisely tuned to its first 30 or so digits! It is in this sense that the Standard Model is considered to be unnaturally fine tuned, since only with non-generic parameters can it describe the real world.

Now, it is important to note that there is nothing wrong in principle with this fine tuning in terms of the theory being consistent. However the fact that there is a need to do such a large fine tuning suggests that an extension of the SM that does not require such a fine tuning is more natural and thus more attractive.

Using this abstract guideline to motivate new theories would be merely philosophy unless these new theories could be differentiated from each other. Thankfully the resolution of the hierarchy problem should be accessible to the Large Hadron Collider, the upcoming particle accelerator experiment at CERN. This is suggested by the above analysis since the theory is only natural if $m_{\rm rad}$ is near the weak scale, which in turn requires that there is new physics near the weak scale.

2 Little Higgs Theories

In the last two years, new solutions to the hiearchy problem, called "Little Higgs" theories were proposed. In these theories, each interaction preserves enough symmetry so that no single interaction gives radiative corrections to the Higgs mass (m_H in Eq. 1.2). However, two interactions together do generate a Higgs mass, but this is further suppressed since it is proportional to the two coupling strengths. In contrast, the Standard Model Higgs only requires a single interaction to generate a mass so it has larger radiative mass corrections compared to Little Higgses. This suppression of the radiative corrections is what improves the naturalness of the theory and helps alleviate the hierarchy problem.

In these theories there are two interesting subjects to consider, which will comprise the bulk of this thesis. The first subject is determining if Little Higgs theories are consistent with the high precision measurements that the SM already predicts well. In essence, it's important to determine whether the new physics that solves the hiearchy problem gives predictions in conflict with known data. To begin with, one of the reasons that the SM works so well is that there is an approximate symmetry called custodial $SU(2)$ which gives predictions confirmed by experiment. Thus, in models of beyond the Standard Model physics, it is beneficial to maintain this symmetry. To try to improve consistency with experiments, it is interesting to try and implement this symmetry into Little Higgs models. This is accomplished in chapters 2 and 3 in two different Little Higgs models. The implemented symmetry indeed allows Little Higgs theories to be consistent with the precision tests and thus confirms that these theories are viable candidates for what comes beyond the Standard Model.

The second subject is analyzing the constraints that unitarity impose on a Little Higgs theory. Unitarity is a strong constraint on quantum field theories and corresponds to the reasonable requirement that the sum of the probability of all possible outcomes must equal 1. In the Little Higgs theories above a certain energy scale unitarity is violated. Thus, if Little Higgs theories are indeed unitary, they must have new physics appearing at this new energy scale. The reason this can be interesting is the fact that this energy scale is much lower than what other considerations indicate, which has an impact on experiment. New particles are expected to appear at this scale, and a factor of 2 or 3 change in the mass of the particle can be crucial in determining whether or not it can be detected in future experiments. Using this as motivation, in chapter 4, the unitarity violation scales in some general Little Higgs theories are calculated. The scales that are calculated suggests that the new physics is borderline in terms of being visible at the Large Hadron Collider. Some speculation on the new particles, their detectability, and their physics signatures is given.

In summary, Little Higgs theories are very interesting theories of what might extend the Standard Model. They are not in conflict with up to date experiments, but will be more stringently tested in the near future. It will be exciting when the new data is analyzed, to see if Little Higgs theories have any part in describing the real world.

Note: The work in the chapters that follow has been published in separate papers before in the References [2–4].

Chapter 2: Little Higgs and Custodial $SU(2)$[1]

3 Introduction

Recently the Little Higgs mechanism has been proposed as a way to stabilize the weak scale from the radiative corrections of the Standard Model. In Little Higgs models the Standard Model Higgs boson is a pseudo-Goldstone and is kept light by approximate non-linear symmetries [5–11], see [12, 13] for summaries of the physics and [14–17] for more detailed phenomenology. The Little Higgs mechanism requires that two separate couplings communicate to the Higgs sufficient breaking of the non-linear symmetry to generate a Higgs mass. The weak scale is radiatively generated two loop factors beneath the cut-off $\Lambda \sim 10-30$ TeV. Little Higgs models predict a host of new particles at the TeV scale that cancel the low energy quadratic divergences to the Higgs mass from Standard Model fields. The Little Higgs mechanism has particles of the *same* spin cancel the quadratic divergences to the Higgs mass, i.e. a fermion cancels a quadratic divergence from a fermion. In models described by "theory space," such as the Minimal Moose, particles of the same spin and quantum numbers cancel quadratic divergences, for example a TeV scale vector that transforms as a $SU(2)_L$ triplet cancels the W quadratic divergence. To avoid fine-tuning the Higgs potential by more then $\mathcal{O}(20\%)$ the top quark one loop quadratic divergence should be cut off by roughly 2 TeV , the quadratic divergence from $SU(2)_L$ should be cut off by 5 TeV , while the quadratic divergence from the Higgs quartic coupling should be cut off by 8 TeV .

These TeV scale particles are heavier than the current experimental limits on direct searches, however these particles may have effects at low energy by contributing to higher dimension operators in the Standard Model after integrating them out. The effects of integrating out the TeV scale partners have been considered in [18–20] and have provided constraints on some Little Higgs models from precision electroweak observables. Understanding what constraints are placed on each Little Higgs model is a detailed question but their themes are the same throughout. The arguments for the most severe constraints on the

[1]This chapter is based on reference [2] done in collaboration with Jay Wacker.

"Littlest Higgs" model discussed in [19,20] arise from the massive vector bosons interactions because they can contribute to low energy four Fermi operators and violate custodial $SU(2)$. Consider the B' which cancels the quadratic divergence of the B, the gauge eigenstates are related to the physical eigenstates by:

$$B = \cos\theta' B_1 + \sin\theta' B_2 \qquad\qquad B' = \cos\theta' B_2 - \sin\theta' B_1 \tag{3.1}$$

where the mixing angles are related to the high energy gauge couplings through:

$$g_1' = \frac{g'}{\cos\theta'} \qquad\qquad g_2' = \frac{g'}{\sin\theta'} \tag{3.2}$$

where g' is the low energy $U(1)_Y$ gauge coupling. With the Standard Model fermions charged only under $U(1)_1$, the coupling to the B' is:

$$\mathcal{L}_{B'\text{F Int}} = g' \tan\theta' \; B'_\mu \; j^\mu_{U(1)_Y} \tag{3.3}$$

where $j^\mu_{U(1)_Y}$ is the $U(1)_Y$ current. The mass of the B' goes as:

$$m^2_{B'} \sim \frac{g'^2 f^2}{\sin^2 2\theta'} \tag{3.4}$$

where f is the breaking scale. After integrating out the B' there is a four Fermi coupling of the form:

$$\mathcal{L}_{4\text{ Fermi}} \sim \frac{\sin^4\theta'}{f^2}\left(j^\mu_{U(1)_Y}\right)^2 \tag{3.5}$$

The coefficient of this operator needs to be roughly less than $(6\text{ TeV })^{-2}$ and can be achieved keeping f fixed as $\theta' \to 0$.

The Little Higgs boson also couples to the B' through the current:

$$\mathcal{L}_{B'\text{ H Int}} \sim g' \cot 2\theta' \; B'_\mu \; (ih^\dagger \overleftrightarrow{D}^\mu h). \tag{3.6}$$

Integrating out the B' induces several dimension 6 operators including:

$$\mathcal{L}_{(h^\dagger Dh)^2} \sim \frac{\cos^2 2\theta'}{f^2}\big((h^\dagger Dh)^2 + \text{ h.c. }\big) \tag{3.7}$$

This operator violates custodial $SU(2)$ and after electroweak symmetry breaking it lowers the mass of the Z^0 and gives a positive contribution to the T parameter. This operator needs to be suppressed by (5 TeV $)^{-2}$. Thus the Higgs coupling prefers the limit $\theta' \to \frac{\pi}{4}$. There are additional contributions to the T parameter that can negate this effect, this argument shows the potential tension in Little Higgs models that could push the limits on f to $3-5$ TeV.

The reason why the B' contributes to an $SU(2)_C$ violating operator is because it, like the B, couples as the T^3 generator of $SU(2)_r$[2] and its interactions explicitly break $SU(2)_C$. The most straight-forward way of softening this effect is to complete the B' into a full triplet of $SU(2)_C$[3]. This modification adds an additional charged vector boson $W^{r\,\pm}$. By integrating out these charged gauge bosons there is another dimension 6 operator that gives a mass to the $W^\pm$ compensating for the effect from the B'. This can be implemented by gauging $SU(2)_r$ instead of $U(1)_2$. At the TeV scale $SU(2)_r \times U(1)_1 \to U(1)_Y$. With these additional vector bosons, it is possible to take the $\theta' \to 0$ limit without introducing large $SU(2)_C$ violating effects while simultaneously decoupling the Standard Model fermions from the B' and keeping the breaking scale f fixed. Thus the limits on the model will roughly reduce to limits on the $SU(2)_r$ coupling and the breaking scale.

It is not necessary to have a gauged $SU(2)_r$ for the Little Higgs mechanism to be viable because the constraining physics is not crucial for stabilizing the weak scale. The B' is canceling the $U(1)_Y$ quadratic divergence that is only borderline relevant for a cut-off $\Lambda \lesssim 10-15$ TeV but is providing some of the main limits through its interactions with the Higgs and the light fermions. The light fermions play no role in the stability of the

[2]Recall that in the limit that $g' \to 0$ there is an $SU(2)_l \times SU(2)_r$ symmetry of the Higgs and gauge sector. Only the T^3 generator is gauged inside $SU(2)_r$ and g' can be viewed as a spurion parameterizing the breaking. After electroweak symmetry breaking $SU(2)_l \times SU(2)_r \to SU(2)_C$.

[3]The W' transforms as a triplet of $SU(2)_C$ so no $SU(2)_C$ violating operators are generated by its interactions.

weak scale, therefore the limits from their interactions can be changed without altering the Little Higgs mechanism. It is straightforward to avoid the strongest constraints [21]. The easiest possibility is to only gauge $U(1)_Y$ and accept its quadratic divergence with a cut-off at $10 - 15$ TeV. Another way of dealing with this issue is to have the fermions charged equally under both $U(1)$ gauge groups. With this charge assignment the fermions decouple from the B' when $\theta' \to \frac{\pi}{4}$ which also decouples the Little Higgs from the B'. There are other ways of decoupling the B' by mixing the Standard Model fermions with multi-TeV Dirac fermions in a similar fashion as [11]. However having a gauged $SU(2)_r$ allows for a particularly transparent limit where TeV scale physics is parametrically safe and does not add significant complexity.

In this note a new Little Higgs model is presented that has the property that it has custodial $SU(2)$ as an approximate symmetry of the Higgs sector by gauging $SU(2)_r$ at the TeV scale. To construct a Little Higgs theory with an $SU(2)_C$ symmetry we can phrase the model building issue as: "Find a Little Higgs theory that has the Higgs boson transforming as a $\mathbf{4}$ of $SO(4)$." This is precisely the same challenge as finding a Little Higgs theory that has a Higgs transforming as a $\mathbf{2}_{\frac{1}{2}}$ of $SU(2)_L \times U(1)_Y$. In the latter case it was necessary to find a group that contained $SU(2) \times U(1)$ and where the adjoint of the group had a field transforming as a $\mathbf{2}_{\frac{1}{2}}$ and the simplest scenario is $SU(3)$ where $\mathbf{8} \to \mathbf{3_0} + \mathbf{2}_{\frac{1}{2}} + \mathbf{1_0}$. For a $\mathbf{4}$ of $SO(4)$ the simplest possibility is $SO(5)$ where an adjoint of $SO(5)$ decomposes into $\mathbf{10} \to \mathbf{6} + \mathbf{4}$. The generators of $SO(5)$ are labeled as T^l, T^r, and T^v for the $SU(2)_l$, $SU(2)_r$ and $SO(5)/SO(4)$ generators respectively.

The model presented in this paper is a slight variation of the "Minimal Moose" [7] that has four non-linear sigma model fields, X_i:

$$X_i = \exp(ix_i/f) \tag{3.8}$$

where x_i is the linearized field and f is the breaking scale associated with the non-linear sigma model. The Minimal Moose has an $[SU(3)]^8$ global symmetry associated with transformations on the fields:

$$X_i \to L_i X_i R_i^\dagger \tag{3.9}$$

with $L_i, R_i \in SU(3)$. To use the $SO(5)$ group theory replace the $SU(3) \to SO(5)$ keeping the "Minimal Moose module" of four links with an $[SO(5)]^8$. The Minimal Moose had an $SU(3)\times[SU(2)\times U(1)]$ gauged where the $[SU(2)\times U(1)]$ was embedded inside $SU(3)$ while this model has an $SO(5) \times [SU(2) \times U(1)]$ gauge symmetry, using the $T^{l\,a}$ generators for $SU(2)$ and $T^{r\,3}$ generator for $U(1)$.

The primary precision electroweak constraints arise from integrating out the TeV scale vector bosons. In this model there is a full adjoint of $SO(5)$ vector bosons. Under $SU(2)_l \times SU(2)_r$ they transform as:

$$W^l \sim (\mathbf{3}_l, \mathbf{1}_r) \quad W^r \sim (\mathbf{1}_l, \mathbf{3}_r) \quad V \sim (\mathbf{2}_l, \mathbf{2}_r) \tag{3.10}$$

Because only $U(1)_Y$ is gauged inside $SU(2)_r$ the $W^{r\,a}$ split into $W^{r\pm}$ and W^{r3}. The W^{r3} is the mode that is responsible for canceling the one loop quadratic divergence of the $U(1)_Y$ gauge boson and is denoted as the B'. Finally the V has the same quantum numbers as the Higgs boson but has no relevant interactions to Standard Model fields.

In the limit where the $SO(5)$ gauge coupling becomes large the Standard Model W and B gauge bosons become large admixtures of the $SU(2) \times U(1)$ vector bosons. This means that the orthogonal combinations, the W' and B', are dominantly admixtures of the $SO(5)$ vector bosons. The Standard Model fermions are charged only under $SU(2) \times U(1)$ which means that the TeV scale vector bosons decouple from the Standard Model fermions in this limit.

In the remaining portion of the paper the explicit model is presented and the spectrum is calculated along with the relevant couplings for precision electroweak observables in Section 4. This model has two light Higgs doublets with the charged Higgs boson being the heaviest of the physical Higgs states because of the form of the quartic potential. This potential is different than the quartic potential of the MSSM and has the property that it forces the

Higgs vacuum expectation values to be complex, breaking $SU(2)_C$ in the process. This will result in the largest constraint on the model. In Section 5 the TeV scale particles are integrated out and their effects discussed in terms of the dimension 6 operators that are the primary precision electroweak observables. For an $SO(5)$ coupling of $g_5 \sim 3$ and $f \sim 700$ GeV and for $\tan\beta \lesssim 0.3$ the model has no constraints placed on it. The limit on $\tan\beta$ ensures a light Higgs with mass in the $100-200$ GeV range. With the rough limits on the parameters, the masses for the relevant TeV scale fields are roughly 2.5 TeV for the gauge bosons, 2 TeV for the top partner, and 2 TeV for the Higgs partners. Finally in Section 6 the outlook for this model and the state of Little Higgs models in general is discussed.

4 $SO(5)$ Minimal Moose

Little Higgs models are theories of electroweak symmetry breaking where the Higgs is a pseudo-Goldstone boson and can be described as gauged non-linear sigma models. In this model there is an $SO(5) \times [SU(2) \times U(1)]$ gauge symmetry with standard gauge kinetic terms with couplings g_5 and g_2, g_1, respectively. There are four non-linear sigma model fields, X_i, that transform under the global $[SO(5)]^8 = [SO(5)_L]^4 \times [SO(5)_R]^4$ as:

$$X_i \to L_i X_i R_i^\dagger. \tag{4.1}$$

Under a gauge transformation the non-linear sigma model fields transform as:

$$X_i \to G_{2,1} X_i G_5^\dagger \tag{4.2}$$

where G_5 is an $SO(5)$ gauge transformation and $G_{2,1}$ is an $SU(2) \times U(1)$ gauge transformation with $SU(2) \times U(1)$ embedded inside $SO(4) \simeq SU(2)_l \times SU(2)_r$, see Appendix A for a summary of the conventions. The gauge symmetries explicitly break the global $[SO(5)]^8$ symmetry and the gauge couplings g_5 and $g_{2,1}$ can be viewed as spurions. Notice that g_5 only breaks the $[SO(5)_R]^4$ symmetry, while $g_{2,1}$ only breaks the $[SO(5)_L]^4$ symmetry.

The non-linear sigma model fields, X_i, can be written in terms of linearized fluctuations around a vacuum $\langle X_i \rangle = \mathbb{1}$:

$$X_i = \exp(i x_i / f) \tag{4.3}$$

where f is the breaking scale of the non-linear sigma model and x_i are adjoints under the diagonal global $SO(5)$. The interactions of the non-linear sigma model become strongly coupled at roughly $\Lambda \simeq 4\pi f$ where new physics must arise. The kinetic term for the non-linear sigma model fields is:

$$\mathcal{L}_{\text{nl}\sigma\text{m Kin}} = \frac{1}{2} \sum_i f^2 \,\text{Tr}\, D_\mu X_i D^\mu X_i^\dagger . \tag{4.4}$$

where the covariant derivative is:

$$D_\mu X_i \;=\; \partial_\mu X_i - i g_5 X_i T^{[mn]} W^{[mn]}_{SO(5)\mu} + i\big(g_2\, T^{la}\, W^{la}_\mu + g_1\, T^{r3}\, W^{r3}_\mu\big) X_i \tag{4.5}$$

where $W^{[mn]}_{SO(5)}$ are the $SO(5)$ gauge bosons, W^{la} are the $SU(2)$ gauge bosons and W^{r3} is the $U(1)$ gauge boson. One linear combination of linearized fluctuations is eaten:

$$\rho \propto x_1 + x_2 + x_3 + x_4 \tag{4.6}$$

leaving three physical pseudo-Goldstone bosons in adjoints of the global $SO(5)$ that decompose under $SU(2)_l \times SU(2)_r$ as:

$$\phi^l \sim (\mathbf{3}_l, \mathbf{1}_r) \qquad \phi^r \sim (\mathbf{1}_l, \mathbf{3}_r) \qquad h \sim (\mathbf{2}_l, \mathbf{2}_r) \tag{4.7}$$

Under $U(1)_Y$, ϕ^r splits into $\phi^{r\,0}$ and $\phi^{r\,\pm}$.

Radiative Corrections

There are no one loop quadratic divergences to the masses of the pseudo-Goldstone bosons from the gauge sector because all the non-linear sigma model fields are bi-fundamentals of the gauge groups. This occurs because the g_5 gauge couplings break only the $SO(5)_{R_i}$ global symmetries, while the $g_{2,1}$ couplings only break the $SO(5)_{L_i}$ symmetries. To generate a mass term it must arise from an operator $|\operatorname{Tr} X_i X_j^\dagger|^2$ and needs to simultaneously break both the left and right global symmetries. This requires both the g_5 and $g_{2,1}$ gauge couplings which cannot appear as a quadratic divergence until two loops. This can be verified with the Coleman-Weinberg potential [22]. In this case the mass squared matrix is:

$$\begin{pmatrix} W_5^A & W_{2,1}^{A'} \end{pmatrix} \begin{pmatrix} g_5^2 f^2 \operatorname{Tr} T^A X_i X_i^\dagger T^B & g_5 g_{2,1} f^2 \operatorname{Tr} T^A X_i T^{B'} X_i^\dagger \\ g_5 g_{2,1} f^2 \operatorname{Tr} T^{A'} X_i^\dagger T^B X_i & g_{2,1}^2 f^2 \operatorname{Tr} T^{A'} X_i^\dagger X_i T^{B'} \end{pmatrix} \begin{pmatrix} W_5^B \\ W_{2,1}^{B'} \end{pmatrix} \tag{4.8}$$

Because the fields are unitary matrices, the entries along the diagonal are independent of the background field, x_i, and so is the trace of the mass squared. Therefore:

$$V_{1\text{ loop CW }\Lambda^2} = \frac{3}{32\pi^2}\Lambda^2 \operatorname{Tr} M^2[x_i] = \text{Constant} \tag{4.9}$$

There are one loop logarithmically divergent, one loop finite and two loop quadratic divergences from the gauge sector. All these contributions result in masses for the pseudo-Goldstone bosons that are parametrically two loop factors down from the cut-off and are $\mathcal{O}(g^2 f/4\pi)$ in size.

4.1 Vector Bosons: Masses and Couplings

The masses for the vector bosons arise as the lowest order expansion of the kinetic terms for the non-linear sigma model fields. The $SO(5)$ and $SU(2)$ W^l vector bosons mix as do the $SO(5)$ and $U(1)$ $W^{r\,3}$ vector bosons. They can be diagonalized with the following transformations:

$$\begin{aligned} B &= \cos\theta' W^{r3} - \sin\theta' W^{r3}_{SO(5)} \qquad & B' = W'^{\,r3} = \sin\theta' W^{r3} + \cos\theta' W^{r3}_{SO(5)} \\ W^a &= \cos\theta W^{la} - \sin\theta W^{la}_{SO(5)} \qquad & W'^a = W'^{\,la} = \sin\theta W^{la} + \cos\theta W^{la}_{SO(5)} \end{aligned}$$

where the mixing angles are related to the couplings by:

$$\begin{aligned} \cos\theta' &= g'/g_1 \qquad \sin\theta' = g'/g_5 \\ \cos\theta &= g/g_2 \qquad \sin\theta = g/g_5 \end{aligned} \tag{4.10}$$

The angles θ and θ' are not independent and are related through the weak mixing angle by:

$$\tan\theta_{\rm w} = \frac{\sin\theta'}{\sin\theta} \tag{4.11}$$

and since $\theta_{\rm w} \simeq 30^\circ$, $\sin\theta \simeq \sqrt{3}\sin\theta'$.

The masses for the vectors can be written in terms of the electroweak gauge couplings and mixing angles:

$$m^2{}_{W'} = \frac{16g^2 f^2}{\sin^2 2\theta} \qquad m^2{}_{B'} = \frac{16g'^2 f^2}{\sin^2 2\theta'} \qquad m^2{}_{W^{r\,\pm}} = \frac{16g'^2 f^2}{\sin^2 2\theta'}\cos^2\theta' \tag{4.12}$$

These can be approximated in the $\theta' \to 0$ limit as:

$$m^2_{B'} \simeq m^2_{W'}(1 - \frac{2}{3}\sin^2\theta) \qquad m^2_{W^{r\,\pm}} \simeq m^2_{W'}(1 - \sin^2\theta) \tag{4.13}$$

Note that the B', the mode that is canceling the quadratic divergence of the B, is not anomalously light[4]. The $U(1)_Y$ quadratic divergence is borderline relevant for naturalness and could be neglected if the cut-off $\Lambda \lesssim 10-15$ TeV . The corresponding mode is contributing to electroweak constraints but doing little to stabilize the weak scale quantitatively.

The Higgs boson couples to these vector bosons through the currents:

$$\begin{aligned} j^{\mu\,a}_{W'} &= g\cot 2\theta j^{\mu\,a}_{\rm H} = \frac{g\cos 2\theta}{2\sin 2\theta}\, ih^\dagger\sigma^a \overleftrightarrow{D}^\mu h \\ j^{\mu}_{B'} &= g'\cot 2\theta' j^{\mu}_{\rm H} = -\frac{g'\cos 2\theta'}{2\sin 2\theta'}\, ih^\dagger \overleftrightarrow{D}^\mu h \end{aligned} \tag{4.14}$$

where D_μ is the Standard Model covariant derivative and $j^{\mu\,a}_{\rm H}$ is the $SU(2)_L$ current that

[4]The B' in the "Littlest Higgs" is a factor of $\sqrt{5}$ lighter and in the $SU(3)$ Minimal Moose it is a factor of $\sqrt{3}$ lighter.

the Higgs couples to and $j^\mu_{\rm H}$ for $U(1)_Y$.

The Higgs also couples to the charged $SU(2)_r$ vector bosons through:

$$
\begin{aligned}
j^\mu_{W^r+} &= -\frac{g' \cos\theta'}{\sqrt{2}\sin 2\theta'}\, ihD^\mu h \\
j^\mu_{W^r-} &= j^\mu_{W^r+}{}^\dagger .
\end{aligned}
\tag{4.15}
$$

where the $SU(2)_L$ indices are contracted with the alternating tensor. Notice that this interaction is not invariant under rephasing of the Higgs: $h \to e^{i\phi} h$ sends $j_{W^r+} \to e^{2i\phi} j_{W^r+}$.

4.2 Scalar Masses and Interactions

In order to have viable electroweak symmetry breaking there must be a significant quartic potential amongst the light fields. It is useful to define the operators:

$$
\mathcal{W}_i = X_i X^\dagger_{i+1} X_{i+2} X^\dagger_{i+3} \tag{4.16}
$$

where addition in i is modulo 4. There is a potential for the non-linear sigma model fields:

$$
\mathcal{L}_{\rm Pot.} \;=\; \lambda_1 f^4 \,{\rm Tr}\, \mathcal{W}_1 + \lambda_2 f^4 \,{\rm Tr}\, \mathcal{W}_2 + \text{ h.c.} \tag{4.17}
$$

There is a $\mathbb{Z}_4$ symmetry where the link fields cycle as $X_i \to X_{i+j}$ that forces $\lambda_1 = \lambda_2$. This is an approximate symmetry that is kept to $\mathcal{O}(10\%)$. This potential gives a mass to one linear combination of linearized fields:

$$
u_H = \frac{1}{2}(x_1 - x_2 + x_3 - x_4). \tag{4.18}
$$

The other two physical modes are the Little Higgs and are classically massless:

$$
u_1 = \frac{1}{\sqrt{2}}(x_1 - x_3) \qquad\qquad u_2 = \frac{1}{\sqrt{2}}(x_2 - x_4). \tag{4.19}
$$

The potential in Eq. 4.17 can be expanded out in terms of these physical eigenmodes using the Baker-Campbell-Hausdorff formula:

$$\begin{aligned}\mathcal{L}_{\text{Pot.}} &= \lambda_1 f^4 \,\text{Tr}\,\exp\Big(2i\frac{u_H}{f} + \frac{1}{2}\frac{[u_1,u_2]}{f^2} + \cdots\Big) \\ &\quad + \lambda_2 f^4 \,\text{Tr}\,\exp\Big(-2i\frac{u_H}{f} + \frac{1}{2}\frac{[u_1,u_2]}{f^2} + \cdots\Big) + \text{ h.c.}\end{aligned} \tag{4.20}$$

The low energy quartic coupling is related to the previous couplings through:

$$\lambda^{-1} = \lambda_1^{-1} + \lambda_2^{-1} \qquad \lambda_1 = \lambda/\cos^2\vartheta_\lambda \qquad \lambda_2 = \lambda/\sin^2\vartheta_\lambda$$

The approximate $\mathbb{Z}_4$ symmetry sets $\theta_\lambda \approx \frac{\pi}{4}$ and the symmetry breaking parameter is $\cos 2\vartheta_\lambda \sim \mathcal{O}(10^{-1})$. The mass of the heavy scalar is:

$$m_{u_H}^2 = \frac{16\lambda f^2}{\sin^2 2\vartheta_\lambda}. \tag{4.21}$$

After integrating out the massive mode the resulting potential for the Little Higgs is the typical commutator potential:

$$V(u_1,u_2) = -\lambda\,\text{Tr}\,[u_1,u_2]^2 + \cdots \tag{4.22}$$

In order to have stable electroweak symmetry breaking it is necessary to have a mass term $ih_1^\dagger h_2 +$ h.c. . This can arise from a potential of the form:

$$\mathcal{L}_{T^{r\,3}\text{ Pot.}} = i\epsilon f^4\,\text{Tr}\,T^{r\,3}\left(\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 + \mathcal{W}_4\right) + \text{ h.c.} \tag{4.23}$$

where $T^{r\,3}$ is the $U(1)$ generator. The size of the effects are radiatively stable and they are set to be a loop factor less than λ, $\epsilon \sim 10^{-2}\lambda$. The coefficients are taken to be pure imaginary because the imaginary coefficient will be necessary to ensure stable electroweak symmetry breaking while the real parts are small $SO(5)$ splittings amongst the various modes. Expanding this out to quadratic order:

$$V_{T^{r3}\ \mathrm{Pot.}} = 4\epsilon f^2 \operatorname{Tr} T^{r3} i[u_1, u_2] + \cdots \tag{4.24}$$

In terms of the Higgs doublets, $h_{1,2} \in u_{1,2}$, the potentials are:

$$V(h_1, h_2) \simeq \frac{\lambda}{2}\left(|h_1^\dagger h_2 - h_1^\dagger h_2|^2 + 4|h_1 h_2|^2\right) + (4i\epsilon f^2 h_1^\dagger h_2 + \text{ h.c. }) \tag{4.25}$$

where the $h_1 h_2$ term is contracted with the $SU(2)$ alternating tensor. This potential is not the same as the MSSM potential and will lead to a different Higgs sector[5]. There are radiative corrections to this potential whose largest effect gives soft masses of $\mathcal{O}(100\text{ GeV})$ to the doublets:

$$\begin{aligned} V_{\text{eff}} \simeq\ & \frac{\lambda}{2}\left(|h_1^\dagger h_2 - h_2^\dagger h_1|^2 + 4|h_1 h_2|^2\right) \\ & +\left((ib + m_{12}^2) h_1^\dagger h_2 + \text{ h.c. }\right) + m_1^2|h_1|^2 + m_2^2|h_2|^2 \end{aligned} \tag{4.26}$$

where $b \approx 4\epsilon f^2$. Typically m_{12}^2 is taken to be small to simplify the phenomenology so that the Higgs states fall into CP eigenstates.

Radiative Corrections

There are no one loop quadratic divergences to the Higgs mass from the scalar potential[6]. The symmetry breaking pattern in the potential is more difficult to see, but notice that if either λ_1 or λ_2 vanished then there is a non-linear symmetry acting on the fields:

$$\begin{aligned} &\delta_{\epsilon_1} u_1 = \epsilon_1 + \cdots \qquad \delta_{\epsilon_1} u_2 = \epsilon_1 + \cdots \qquad \delta_{\epsilon_1} u_H = -\frac{i}{4f}[\epsilon_1, u_1 - u_2] + \cdots \\ &\delta_{\epsilon_2} u_1 = \epsilon_2 + \cdots \qquad \delta_{\epsilon_2} u_2 = \epsilon_2 + \cdots \qquad \delta_{\epsilon_2} u_H = +\frac{i}{4f}[\epsilon_2, u_1 - u_2] + \cdots . \end{aligned} \tag{4.27}$$

[5]In the $SU(3)$ Minimal Moose the Higgs potential was identical to the the MSSM because of the close relation between Little Higgs theories and orbifolded extra dimensions, see [8] for the precise relation.

[6]More generally potentials that only contain any non-linear sigma model field at most once can only give a quadratically divergent contribution to themselves.

$\mathrm{Tr}\,\mathcal{W}_1$ preserves the first non-linear symmetry but breaks the second, while $\mathrm{Tr}\,\mathcal{W}_2$ preserves the second but breaks the first. Either symmetry is sufficient to keep u_1 and u_2 as exact Goldstones, this is why $\lambda \to 0$ as λ_1 *or* $\lambda_2 \to 0$.

There are one loop logarithmically divergent contributions to the masses of the Little Higgs as well as one loop finite and two loop quadratic divergences. These are all positive and parametrically give masses of the order of $\lambda^2 f/4\pi$.

4.3 Electroweak Symmetry Breaking

At this point electroweak symmetry can be broken. The Little Higgs are classically massless but pick up $\mathcal{O}(100\ \mathrm{GeV})$ masses from radiative corrections to the tree-level Lagrangian. The gauge and scalar corrections to the Little Higgs masses give positive contributions to the mass squared of the Little Higgs while fermions give negative contributions. The mass matrix for the Higgs sector is of the form:

$$\mathcal{L}_{\text{Soft Mass}} = \begin{pmatrix} h_1^\dagger & h_2^\dagger \end{pmatrix} \begin{pmatrix} m_1^2 & \mu^2 \\ \mu^{*2} & m_2^2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \tag{4.28}$$

where $\mu^2 = m_{12}^2 + ib$. To have viable electroweak symmetry breaking requires:

$$\begin{aligned} m_1^2 > 0 \qquad & m_2^2 > 0 \\ m_1^2 m_2^2 - m_{12}^4 \quad & > \quad 0 \\ m_1^2 m_2^2 - m_{12}^4 - b^2 \quad & < \quad 0. \end{aligned} \tag{4.29}$$

The vacuum expectation values are:

$$\langle h_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v\cos\beta \end{pmatrix} \qquad \langle h_2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v\sin\beta e^{i\phi} \end{pmatrix} \tag{4.30}$$

The potential has a flat direction when $\beta = 0, \frac{\pi}{2}$ and when $\phi = 0$. Unfortunately when $\phi \neq 0$ custodial $SU(2)$ is broken[7]. The phase can be solved for in terms of the soft masses as:

$$\cos\phi = \frac{m_{12}^2}{m_1 m_2}. \tag{4.31}$$

The breaking of $SU(2)_C$ by the Higgs sector provides one of the strongest limits on the model. For simplicity $\mu^2 = ib$ is taken to be pure imaginary forcing $\phi = \frac{\pi}{2}$. Taking $\phi = \frac{\pi}{2}$ is clearly the worst-case scenario for $SU(2)_C$ and not generic because there is no reason for m_{12} to be significantly smaller than any of the other masses.

The parameters of electroweak symmetry breaking can be solved for readily in the limit $\phi = \frac{\pi}{2}$ in terms of the masses:

$$\begin{aligned} 2\lambda v^2 &= (m_1^2 + m_2^2)\left(\frac{|b|}{m_1 m_2} - 1\right) \\ \tan\beta &= \frac{m_1}{m_2} \\ \tan 2\alpha &= \left(1 - \frac{2m_1 m_2}{|b|}\right)\tan 2\beta. \end{aligned} \tag{4.32}$$

where α is the mixing angle for the $h^0 - H^0$ sector. The soft masses should not be much larger than v otherwise it either requires some tuning of the parameters so that $b \simeq m_1 m_2$ or λ becoming large. These arguments will change when $m_{12}^2 \neq 0$. The masses for the five physical Higgs are:

$$\begin{aligned} m_{A^0}^2 &= m_1^2 + m_2^2 \\ m_{H^\pm}^2 &= m_1^2 + m_2^2 + 2\lambda v^2 \qquad = \qquad x\, m_{A^0}^2 \\ m_{h^0}^2 &= m_{H^\pm}^2 \frac{\left(1 - \sqrt{1 - m_0^2/m_{H^\pm}^2}\right)}{2} \\ m_{H^0}^2 &= m_{H^\pm}^2 \frac{\left(1 + \sqrt{1 - m_0^2/m_{H^\pm}^2}\right)}{2} = m_{H^\pm}^2 - m_{h^0}^2 \end{aligned} \tag{4.33}$$

[7]This can be seen by going back to the $SO(4)$ description. By having a phase it is the same as having two $SO(4)$ vectors acquire vacuum expectation values in different directions leaving only $SO(2) \simeq U(1)_Y$ unbroken.

where

$$x = |b|/m_1 m_2 \qquad\qquad m_0^2 = \frac{8\lambda v^2 \sin^2 2\beta}{x} \tag{4.34}$$

The heaviest Higgs is the charged $H^\pm$ and this has consequences for precision electroweak observables. The mass of the lightest Higgs is bounded by:

$$\frac{1}{4}m_0^2 \leq m_{h^0}^2 \leq \frac{1}{2}m_0^2 \tag{4.35}$$

where the lower bound is saturated as $m_{H^\pm}^2 \to \infty$ and the upper bound is saturated as $m_{H^\pm}^2 \to m_0^2$.

4.4 Fermions

The Standard Model fermions are charged only under the $SU(2)\times U(1)$ gauge group. Since all the fermions except the top quark couple extremely weakly to the Higgs sector, the standard Yukawa coupling to the linearized Higgs doublets can be used without destabilizing the weak scale. These small Yukawa couplings are spurions that simultaneously break flavor symmetries as well as the chiral symmetries of the non-linear sigma model. There are many ways to covariantize these couplings but they only differ by irrelevant operators.

$$\mathcal{L}_{\text{Yuk}} = y_u\, qhu^c + y_d\, qh^\dagger d^c + y_e\, lh^\dagger e^c \tag{4.36}$$

There is no symmetry principle that prefers type I or type II models. This can have significant implications for Higgs searches.

The couplings of the Standard Model fermions to the heavy gauge bosons is:

$$\mathcal{L}_{\text{Int}} = g\tan\theta\; W'^a_\mu\, j^\mu_{\text{F}a} + g'\tan\theta'\; B'_\mu\, j^\mu_{\text{F}} \tag{4.37}$$

where $j^{\mu\, a}_{\text{F}}$ is the $SU(2)_L$ electroweak current involving the Standard Model fermions and j^μ_{F} is the $U(1)_Y$ electroweak current involving the Standard Model fermions. In the limit

$g_5 \to \infty$ both $\theta, \theta' \to 0$ and the TeV scale gauge bosons decouple from the Standard Model fermions.

Top Yukawa

The top quark couples strongly to the Higgs and how the top Yukawa is generated is crucial for stabilizing the weak scale. The top sector must preserve some of the $[SO(5)]^8$ global symmetry that protects the Higgs mass. There are many ways of doing this but generically the mechanisms involve adding additional Dirac fermions. To couple the non-linear sigma model fields to the quark doublets it is necessary to transform the bi-vector representation to the bi-spinor representation, see Appendix A. The linearized fields are re-expressed as:

$$\tilde{x}_{i\alpha}{}^{\beta} = x_{i[mn]}\sigma^{[mn]}{}_{\alpha}{}^{\beta} \tag{4.38}$$

where m, n are $SO(5)$ vector indices running from 1 to 5, α, β are $SO(5)$ spinor indices running from 1 to 4 and $\sigma^{[mn]}{}_{\alpha}{}^{\beta}$ are generators of $SO(5)$ in the spinor representation. The exponentiated field, $\tilde{X}_i = \exp(i\tilde{x}_i/f)$, has well-defined transformation properties under the global $SO(5)$'s and the operator, $\mathcal{X} = (\tilde{X}_1 \tilde{X}_3^\dagger)$, transforms only under the $SU(2) \times U(1)$ gauge symmetry:

$$\mathcal{X} \quad \to \quad \tilde{G}_{2,1} \mathcal{X} \tilde{G}_{2,1}^\dagger \tag{4.39}$$

where $\tilde{G}_{2,1}$ is an $[SU(2) \times U(1)] \subset SO(5)$ gauge transformation in the spinor representation of $SO(5)$.

It is necessary to preserve some of the global $SO(5)$ symmetry in order to remove the one loop quadratic divergence to the Higgs mass from the top. As in the Minimal Moose, it is necessary to add additional fermions to fill out a full representation, in this case a **4** of $SO(5)$ for either the q_3 or the u_3^c. The large top coupling is a result of mixing with this TeV scale fermion. The most minimal approach is to complete the q_3 into:

$$\mathcal{Q} = (q_3, \tilde{u}, \tilde{d}) \qquad \mathcal{U}^c = (0_2, u_3^c, 0) \tag{4.40}$$

where $\tilde{u} \sim (\mathbf{3}_c, \mathbf{1}_{+\frac{2}{3}})$ and $\tilde{d} \sim (\mathbf{3}_c, \mathbf{1}_{-\frac{1}{3}})$ with charge conjugate fields $\tilde{u}^c$ and $\tilde{d}^c$ canceling the anomalies. The top Yukawa coupling is generated by:

$$\mathcal{L}_{\text{top}} = y_1 f \mathcal{U}^c \mathcal{X} \mathcal{Q} + y_2 f \tilde{u}\tilde{u}^c + \tilde{y}_2 f \tilde{d}\tilde{d}^c + \text{ h.c.} \tag{4.41}$$

The $\tilde{u}$ and u_3^c mix with an angle ϑ_y and after integrating out the massive combination the low energy top Yukawa is given by:

$$y_{\text{top}}^{-2} = 2(|y_1|^{-2} + |y_2|^{-2}) \qquad\qquad \tan\vartheta_y = \frac{|y_1|}{|y_2|}. \tag{4.42}$$

After electroweak symmetry breaking the top quark and the top partner pick up a mass:

$$m_t = \frac{y_{\text{top}} v \cos\beta}{\sqrt{2}} \qquad\qquad m_{t'} = \frac{2\sqrt{2} y_{\text{top}} f}{\sin 2\vartheta_y}\left(1 - \frac{v^2\cos^2\beta\sin^2 2\vartheta_y}{32 f^2}\right). \tag{4.43}$$

The decoupling limit is the $y_2 \to \infty$ limit where $\vartheta_y \to 0$.

Radiative Corrections

The top coupling respects a global $SO(5)$ symmetry. This ensures that there are no one loop quadratically divergent contributions to the Higgs mass and can be seen through the Coleman-Weinberg potential. The one loop quadratic divergence is proportional to $\operatorname{Tr} MM^\dagger$, where $M \sim P_{\mathcal{U}^c}\mathcal{X}$ is the mass matrix for the top sector in the background of the Little Higgs and $P_{\mathcal{U}^c} = \text{diag}(0,0,1,0)$ is a projection matrix from the $\mathcal{U}^c$. Expanding this out:

$$\begin{aligned} V_{1\text{ loop CW }\Lambda^2} &= -\frac{12\Lambda^2}{32\pi^2}\operatorname{Tr} P_{\mathcal{U}^c}\mathcal{X}\mathcal{X}^\dagger P_{\mathcal{U}^c} \\ &\sim \operatorname{Tr} P_{\mathcal{U}^c} = \text{Constant} \end{aligned} \tag{4.44}$$

which gives no one loop quadratic divergences to any of the x_i fields. One loop logarithmically divergent, one loop finite and two loop quadratically divergent masses are generated at the order $\mathcal{O}(y_{\text{top}}^2 f/4\pi)$. Since the top only couples to h_1 amongst the light fields, it only generates a negative contribution to m_1^2. This drives $\tan\beta$ to be small since this is the only interaction that breaks the $h_1 \leftrightarrow h_2$ symmetry explicitly.

Note that the $\tilde{d}$ can be decoupled without affecting naturalness. This is because there is an accidental $SU(3)$ symmetry that is identical to the $SU(3)$ symmetry of the Minimal Moose.

$$\mathcal{L}_{\rm Top} = y_1 f u^c \tilde{u} + \frac{i}{\sqrt{2}} y_1 u^c h_1 q - \frac{1}{4}\frac{y_1}{f} u^c h_1^\dagger h_1 \tilde{u} + \cdots \tag{4.45}$$

is invariant under:

$$\delta h_1 = \epsilon \qquad \delta q = \frac{i\sqrt{2}}{f}\epsilon^* \tilde{u} \qquad \delta\tilde{u} = \frac{i\sqrt{2}}{f}\epsilon q. \tag{4.46}$$

This can be seen by imagining an $SU(4)$ symmetry acting on $\mathcal{X}$. With only the $\tilde{u}$ there is an $SU(3)$ acting in the upper components. The $SU(4)$ symmetry is just the $SO(6) \supset SO(5)$. The $SU(3)$ is not exact but to quadratic order in h it is an accidental symmetry. This means that in principle it is possible to send $\tilde{y}_2 \to 4\pi$ without affecting naturalness and therefore it is safe to ignore this field. Performing the same calculation as above, the charged singlet, $\phi_1^{r\,\pm}$, gets a quadratically divergent mass and is lifted to the TeV scale.

5 Precision Electroweak Observables

Throughout this note the scalings of the contributions of TeV scale physics to precision electroweak observables have been discussed. The contributions to the higher dimension operators of the Standard Model are calculated in this section. The most physically transparent way of doing this is to integrate out the heavy fields and then run the operators down to the weak scale. The most difficult contribution to calculate is the custodial $SU(2)$ violating operator because there are several sources. Beyond that there are four Fermi operators and corrections to the Z^0 and $W^\pm$ interactions. There are no important contributions to the S parameter besides the contributions from the Higgs that turn out to be small. In Sec. 5.4 we summarize the constraints on the model from precision electroweak observables and state the limits on the masses.

5.1 Custodial $SU(2)$

Custodial $SU(2)$ provides limits on beyond the Standard Model physics. When written in terms of the electroweak chiral Lagrangian, violations of $SU(2)_C$ are related to the operator:

$$\mathcal{O}_4 = c_4 v^2 \big(\operatorname{Tr} T_3 \omega^\dagger D_\mu \omega \big)^2 \tag{5.1}$$

where ω are the Goldstone bosons associated with electroweak symmetry breaking. The coefficient of this operator is calculated in this section. This is directly related to $\delta\rho$. However, typically limits are stated in terms of the T parameter which is related to $\delta\rho_*$ which differs from $\delta\rho$ when there are modifications to the $W^\pm$ and Z^0 interactions with Standard Model fermions. In Sec 5.4 this difference is accounted for.

There are typically five new sources of custodial $SU(2)$ violation in Little Higgs models. The first is from the non-linear sigma model structure itself. By expanding the kinetic terms to quartic order there are operators that give the $W^\pm$ and Z^0 masses. If $SU(2)_C$ had not been broken by the vacuum expectation values of the Higgs, then there could not be any operators that violate $SU(2)_C$. Custodial $SU(2)$ is only broken with the combination of the two vacuum expectation values which means that the only possible operator that could violate $SU(2)_C$ must be of the form $(h_2^\dagger D h_1)^2$. However, the kinetic terms for the non-linear sigma model fields never contain h_1 and h_2 simultaneously meaning that any operator of this form is not present.

Vector Bosons

The second source of custodial $SU(2)$ violation is from the TeV scale gauge bosons. The massive W' never gives any $SU(2)_C$ violating contributions to the $W^\pm$ and Z^0 mass. The B' typically gives an $SU(2)_C$ violating contribution to the electroweak gauge boson masses but the additional contributions from the $W^{r\,\pm}$ vector bosons largely cancel this. Summing the various contributions:

$$\delta\rho = -\frac{v^2}{64f^2}\sin^2 2\theta' + \frac{v^2}{64f^2}\sin^2 2\beta \sin^2\phi. \tag{5.2}$$

The second term is a result of the phase in the Higgs vacuum expectation value that breaks the $SU(2)_C$ and arises because the $W^{r\pm}$ interactions are not invariant under rephasing of the Higgs. The phase is generally taken to be $\frac{\pi}{2}$ to have the Higgs states fall into CP eigenstates. This is not generic and requires tuning m_{12}^2 to be small. Numerically this contribution is:

$$\alpha^{-1}\delta\rho \simeq \frac{1}{8}\sin^2 2\beta \frac{(1\text{ TeV })^2}{f^2} \tag{5.3}$$

where the $\sin^2 2\theta'$ term has been dropped because it cancels in the conversion to ρ_* as will be shown in Sec. 5.4. This prefers β to be small which is the direction that is radiatively driven by the top sector. For instance at $\sin 2\beta \sim \frac{1}{3}$, this contribution to $\delta\rho$ is negligibly small for $f \sim 700$ GeV. By going to small $\tan\beta$ the mass of the lightest Higgs becomes rather light, for instance, for $\sin 2\beta \simeq \frac{1}{3}$ the mass of the lightest Higgs is bounded by $m_{h^0} \leq v$ with most of the parameter space dominated by $m_{h^0} \leq 150$ GeV.

Triplet VEV

Another possible source of $SU(2)_C$ violation is from a triplet vacuum expectation value. The form of the plaquette potential in Eq. 4.20 ensures that the tri-linear couplings are of the form:

$$h_1^\dagger \phi_H^l h_2 - h_2^\dagger \phi_H^l h_1. \tag{5.4}$$

There are two equivalent ways of calculating the effect, either integrating out ϕ_H^l to produce higher dimension operators or by calculating its vacuum expectation value. The operator appears as:

$$\mathcal{L}_{u_H u_1 u_2} = \lambda \cot 2\vartheta_\lambda f\ i\ \operatorname{Tr} u_H[u_1, u_2] \tag{5.5}$$

After integrating out u_H the leading derivative interaction is:

$$\mathcal{L}_{\text{eff}} = -\frac{\cos^2 2\vartheta_\lambda}{16f^2} \operatorname{Tr} D_\mu[u_1, u_2] D^\mu[u_1, u_2] \tag{5.6}$$

where D_μ are the Standard Model covariant derivatives. Expanding this out there is a term that gives a contribution to ρ:

$$\delta\rho = \frac{v^2}{4f^2} \cos^2 2\vartheta_\lambda \sin^2 2\beta \sin^2 \phi \tag{5.7}$$

The approximate $\mathbb{Z}_4$ symmetry of the scalar and gauge sectors sets $\vartheta_\lambda \simeq \frac{\pi}{4}$ with $\cos 2\vartheta_\lambda \sim 10^{-1}$ meaning that this contribution is adequately small.

One might also worry that the light triplets in $u_{1,2}$ get tadpoles after electroweak symmetry breaking (through radiatively generated $h^\dagger \phi h$ terms), which due to their relatively light masses could lead to phenomenologically dangerous triplet vevs.[8] However, these light scalars are not involved in canceling off the quadratic divergences to the Higgs masses. Thus these triplets can be safely raised to the TeV scale by introducing "Ω plaquettes" as described in [8], where $\Omega = \exp\left(2\pi i T^{r3}\right) = \operatorname{diag}(-1,-1,-1,-1,1)$. These operators suitably suppress the magnitudes of the light triplet vevs and do not affect naturalness.

Two Higgs Doublets

The ρ parameter also receives contributions from integrating out the Higgs bosons. It is known that this contribution can be either positive or negative. It is positive generically if the $H^\pm$ states are either lighter or heavier than all the neutral states, while it is negative if there are neutral Higgs states lighter and heavier than it. The Higgs potential of this theory generically predicts that the charged Higgs is the heaviest Higgs boson. There are

[8]We thank C. Csaki for pointing out that integrating out heavy quarks might generate these terms.

four parameters of the Higgs potential: m_1^2, m_2^2, b, and λ where one combination determines $v = 247$ GeV. If $\phi \neq \frac{\pi}{2}$ then this analysis becomes much more complicated. The contribution to ρ_* from vacuum polarization diagrams is:

$$\begin{aligned}\delta\rho_* \;=\; & \frac{\alpha}{16\pi\sin^2\theta_{\rm w} m_{W^\pm}^2}\Big(F(m_{A^0}^2, m_{H^\pm}^2) \\ & + \sin^2(\alpha-\beta)\big(F(m_{H^\pm}^2, m_{h^0}^2) - F(m_{A^0}^2, m_{h^0}^2) + \delta\hat\rho_{\rm SM}(m_{H^0}^2)\big) \\ & + \cos^2(\alpha-\beta)\big(F(m_{H^\pm}^2, m_{H^0}^2) - F(m_{A^0}^2, m_{H^0}^2) + \delta\hat\rho_{\rm SM}(m_{h^0}^2)\big)\Big)\end{aligned} \tag{5.8}$$

where

$$F(x,y) = \frac{1}{2}(x+y) - \frac{xy}{x-y}\log\frac{x}{y} \tag{5.9}$$

$$\begin{aligned}\delta\hat\rho_{\rm SM}(m^2) = \; & F(m^2, m_{W^\pm}^2) - F(m^2, m_{Z^0}^2) \\ & + \frac{4m^2 m_{W^\pm}^2}{m^2 - m_{W^\pm}^2}\log\frac{m^2}{m_{W^\pm}^2} - \frac{4m^2 m_{Z^0}^2}{m^2 - m_{Z^0}^2}\log\frac{m^2}{m_{Z^0}^2}\end{aligned} \tag{5.10}$$

In two Higgs doublet models setting an upper limit on the lightest Higgs mass from precision electroweak measurements is less precise. There can be cancellations but it appears as though the T parameter is quadratically sensitive to the mass of the heaviest Higgs. The spectrum of Higgs generated by the Higgs potential keeps the splittings between the masses of the Higgs bosons constant:

$$m_{H^\pm}^2 - m_{A^0}^2 = 2\lambda v^2 \qquad\qquad m_{H^\pm}^2 - m_{H^0}^2 = m_{h^0}^2$$

with $m_{h^0}^2 \leq 4\lambda v^2 \sin^2 2\beta$. This means that if λ is kept small then the T parameter is insensitive to the overall mass scale of the Higgs. With $\alpha - \beta = \frac{\pi}{4}$ the contribution to ρ_* goes as:

$$\begin{aligned}\alpha^{-1}\,\delta\rho_* \;\simeq\; & \frac{1}{10} - \frac{m_{h^0}^2}{(500\ \text{GeV})^2} - \frac{1}{4}\frac{m_{h^0}^2}{m_{H^\pm}^2} - \frac{1}{30}\log\frac{m_{H^\pm}^2}{(500\ \text{GeV})^2} \qquad \lambda = \frac{1}{2} \\ \simeq\; & \frac{1}{3} - \frac{m_{h^0}^2}{(500\ \text{GeV})^2} - \frac{1}{2}\frac{m_{h^0}^2}{m_{H^\pm}^2} - \frac{1}{30}\log\frac{m_{H^\pm}^2}{(500\ \text{GeV})^2} \qquad \lambda = 1.\end{aligned} \tag{5.11}$$

As λ becomes larger the contributions to the T parameter typically become larger, positive and favoring heavier Higgs with smaller mass splittings to satisfy precision electroweak fits. Notice that even for $\lambda = \frac{1}{2}$ where the contributions to $\delta\rho_*$ are quite small the mass of the lightest Higgs is only bounded by $m_{h^0} \leq 350$ GeV. However the contributions to ρ from the gauge boson sector prefer a small β to keep the contributions small, thus favoring a light Higgs.

Top Partners

The top partners provide another source of $SU(2)_C$ violating operators arising from integrating out the partners to the top quark: $\tilde{u}$ and $\tilde{u}^c$. Since this is a Dirac fermion it decouples in a standard fashion as y_2 becomes large [23]. The contribution after subtracting off the Standard Model top quark contribution is:

$$\begin{aligned} \delta\rho_{t'*} &= \frac{N_c \sin^2\theta_L}{8\pi^2 v^2}\left[\sin^2\theta_L F(m_{t'}^2, m_t^2) + F(m_{t'}^2, m_b^2) - F(m_t^2, m_b^2) - F(m_{t'}^2, m_t^2)\right] \\ &\simeq \frac{N_c \sin^2\theta_L}{16\pi^2 v^2}\left[\sin^2\theta_L m_{t'}^2 + 2\cos^2\theta_L \frac{m_{t'}^2\, m_t^2}{m_{t'}^2 - m_t^2}\log\frac{m_{t'}^2}{m_t^2} - (2-\sin^2\theta_L)m_t^2\right] \end{aligned} \quad (5.12)$$

where θ_L is the t' and t mixing angle after electroweak symmetry breaking and can be expressed in terms of the original Yukawa and the mixing angle ϑ_y:

$$\sin\theta_L \simeq \frac{v\sin^2\vartheta_y\cos\beta}{2f} \quad (5.13)$$

Using this and the expressions for the mass of the t and t' in Eq. 4.43 the expression for the $\delta\rho_{t'*}$ parameter reduces to:

$$\delta\rho_{t'*} \simeq \frac{3y_{\rm top}^2 v^2\sin^4\vartheta_y\cos^4\beta}{128\pi^2 f^2}\Big(\tan^2\vartheta_y - 2\big(\log\frac{v^2\sin^2\vartheta_y\cos^2\vartheta_y\cos^2\beta}{4f^2} + 1\big)\Big) \quad (5.14)$$

This contribution vanishes as $\vartheta_y \to 0$ which is the limit $y_1 \to 0$ while keeping $y_{\rm top}$ fixed. In the limit of $\vartheta_y = \frac{\pi}{4} - \delta\vartheta_y$ near where $m_{t'}$ is minimized, the contribution for small β goes as:

$$\alpha^{-1}\delta\rho_{t'*} \simeq \frac{(1 - 4.4\,\delta\vartheta_y + 7.5\,\delta\vartheta_y^2)}{25}\Big(1 - 1.8\sin^2\beta + 0.7\sin^4\beta\Big)\frac{(1\text{ TeV })^2}{f^2}. \quad (5.15)$$

This is adequately small for any β and the contribution quickly drops with $\delta\vartheta_y$. For instance, with $\delta\vartheta_y \simeq 0.1$, $\delta\rho_{t'*}$ drops by 40% while $m_{t'}$ only rises by 2%. This means that this contribution can be taken to be a subdominant effect.

5.2 S parameter

The main source for contributions to the S parameter is from integrating out the physical Higgs bosons. As for the case with the ρ parameter, a two Higgs doublet spectrum leaves a great deal of room for even a heavy spectrum where all the states are above 200 GeV. Generically the S parameter does not lead to any constraints in the Higgs spectrum because of cancellations:

$$\begin{aligned} S &= \frac{1}{12\pi}\Big(\sin^2(\beta-\alpha)\log\frac{m_{H^0}^2}{m_{h^0}^2} - \frac{11}{6} + \\ &\quad \cos^2(\beta-\alpha)G(m_{H^0}^2, m_{A^0}^2, m_{H^\pm}^2) + \sin^2(\beta-\alpha)G(m_{h^0}^2, m_{A^0}^2, m_{H^\pm}^2)\Big) \end{aligned} \tag{5.16}$$

where

$$G(x,y,z) = \frac{x^2+y^2}{(x-y)^2} + \frac{(x-3y)x^2\log\frac{x}{z} - (y-3x)y^2\log\frac{y}{z}}{(x-y)^3}. \tag{5.17}$$

This can be approximated by expanding around large $m_{H^\pm}^2$ masses and taking $\alpha - \beta = \frac{\pi}{4}$:

$$S = S_{\rm SM} - \frac{5}{144\pi} - \frac{1}{16\pi}\frac{2\lambda v^2}{m_{H^\pm}^2} + \frac{1}{48\pi}\frac{m_{h^0}^2}{m_{H^\pm}^2} + \frac{1}{24\pi}\log\frac{m_{H^\pm}^2}{m_{h^0}^2} \tag{5.18}$$

These are adequately small in general for all reasonable values of λ and $m_{h^0}^2$.

5.3 Electroweak Currents

The last source of electroweak constraints comes from the modifications to electroweak currents and four Fermi operators at low energies. These come from two primary sources, the Higgs-Fermion interactions from the current interactions in Eqs. 4.14 and 4.37:

$$\begin{aligned}\mathcal{L}_{\mathrm{H\,F}} &= -\frac{j^a_\mu{}_{W'\mathrm{H}}\, j^{\mu a}{}_{W'\mathrm{F}}}{M^2_{W'}} - \frac{j_{\mu B'\mathrm{H}}\, j^\mu{}_{B'\mathrm{F}}}{M^2_{B'}} \\ &= -\frac{\sin^2\theta\cos 2\theta}{8f^2} j_{\mathrm{H}}{}^{a\,\mu} j_{\mathrm{F}\,a\,\mu} - \frac{\sin^2\theta'\cos 2\theta'}{8f^2} j_{\mathrm{H}}{}^{\mu} j_{\mathrm{F}\mu}\end{aligned} \tag{5.19}$$

and the direct four Fermi interactions:

$$\begin{aligned}\mathcal{L}_{\mathrm{F\,F}} &= -\frac{(j^a_\mu{}_{W'\mathrm{F}})^2}{2M^2_{W'}} - \frac{(j_{\mu B'\mathrm{F}})^2}{2M^2_{B'}} \\ &= -\frac{\sin^4\theta}{8f^2} j_{\mathrm{F}}{}^{a\,\mu} j_{\mathrm{F}\,a\,\mu} - \frac{\sin^4\theta'}{8f^2} j_{\mathrm{F}}{}^{\mu} j_{\mathrm{F}\mu}.\end{aligned} \tag{5.20}$$

It requires a full fit to know what the limits on these interactions are, but to first approximation these interactions are fine if they are suppressed by roughly $\Lambda_{\mathrm{lim}} \sim 6$ TeV [24]. Since $\sin\theta \simeq \sqrt{3}\sin\theta'$, the biggest constraints come from the effects of the W'. The constraints reduce to a limit on the $g_5 - f$ plane of:

$$\frac{2\sqrt{2}f}{\sin\theta} \gtrsim \Lambda_{\mathrm{lim}}. \tag{5.21}$$

Clearly for $f \sim 2.5$ TeV there are no limits on g_5, for $f \sim 1.5$ TeV, $g_5 \sim 1.5$ and for $f \sim 0.7$ TeV, $g_5 \sim 3$. [9] These are clearly all in the natural regime for the Little Higgs mechanism to be stabilizing the weak scale. This limit is very closely related to the mass of the W':

$$M_{W'} \gtrsim \frac{g}{\sqrt{2}\cos\theta}\Lambda_{\mathrm{lim}} \tag{5.22}$$

Thus, the mass of the $W' \gtrsim \frac{2}{5}\Lambda_{\mathrm{lim}}$. This sets a lower limit on the mass of the W' of 2.5 TeV.

5.4 Summary of Limits

To state the limits it is necessary to convert ρ to ρ_* which is related to the T parameter. While ρ is related to custodial $SU(2)$, ρ_* is related to physical results and differs from ρ

[9] It is not possible to push g_5 much larger than 3 because perturbativity is lost when the loop factor suppression $T_2(A)g_5^2/8\pi^2$ becomes roughly 1. This requires $g_5 \lesssim 5$.

when there are modifications to electroweak current interactions. The difference is due to the discrepancy between the pole mass of the $W^\pm$ and the way that the mass of the $W^\pm$ is extracted through muon decay.

In this model the Standard Model fermions couple to the W' and B' and integrating out the heavy gauge bosons generates both four Fermi interactions and corrections to the J_Y, J_W fermionic currents after electroweak symmetry breaking. Following the analysis in [20, 25], the Fermi constant is corrected by:

$$\frac{1}{G_F} = \sqrt{2}v^2\left(1 + \frac{\delta M_W^2}{M_{W_0}^2} - \frac{v^2}{64f^2}\sin^2 2\theta\right). \tag{5.23}$$

To determine ρ_*, it is necessary to integrate out the Z^0 and express the four Fermi operators as

$$-\frac{4G_F}{\sqrt{2}}\rho_*(J_3 - s_*^2 J_Q)^2 + \alpha J_Q^2 \tag{5.24}$$

which gives us to order (v^2/f^2)

$$\begin{aligned} \delta\rho_* &= \alpha T = \frac{\delta M_W^2}{M_{W_0}^2} - \frac{\delta M_Z^2}{M_{Z_0}^2} + \frac{v^2}{64f^2}\sin^2 2\theta' \\ &= \delta\rho + \frac{v^2}{64f^2}\sin^2 2\theta'. \end{aligned} \tag{5.25}$$

Because all the other contributions to ρ are small, the primary limit on the theory comes from the $SU(2)_C$ violation in the gauge sector.

At this point the limits can be summarized for the masses of the particles. The limit on the breaking scale, f, is roughly 700 GeV from the contributions to T from the gauge bosons. The Higgs contributions to ρ_* could have been large, but because $\tan\beta$ is small it turns out to be subdominant. The mass of the lightest Higgs is bounded to be less than 250 GeV with most of the parameter space dominated by masses less than 150 GeV. The TeV scale vector bosons are all roughly degenerate with masses greater than 2.5 TeV. The mass of the top partner is roughly 2 TeV. While the mass of the heavy Higgs are roughly

2 TeV from the limits on f.

If we chose to exclude the A_b^{FB} measurement as an outlier, the implications for this model are significant. Discarding this measurement might be reasonable since it deviates from other Standard Model measurements by roughly 3σ. This model does not significantly alter the physics of A_b^{FB} from the Standard Model. This measurement is not generally excluded because doing so pulls the fit for the T parameter positive which favors a very light Higgs in the Standard Model and is excluded by direct searches. However there are additional positive contributions that mimic a light Higgs boson in this model. On a general principle, the connection between a light Higgs boson and a positive contribution to the T parameter does not hold in two Higgs doublet models and it is quite easy to have the Higgs sector produce $\delta T \sim 0.2$. By ignoring A_b^{FB} the best fit for the $S - T$ plane moves to $T \sim 0.15 \pm 0.1$. See [26, 27] for more details. This significantly reduces the constraints on this model because all TeV scale physics pulls towards positive T. The contribution from the gauge bosons becomes roughly about the best fit for T even with $\tan\beta \sim 1$ and $f \sim 700$ GeV. This in turn can lower the limit on $m_{t'}$ and also remove the preference for lighter Higgs.

6 Conclusions and Outlook

In this paper we have found a Little Higgs model with custodial $SU(2)$ symmetry that is easily seen to be consistent with precision electroweak constraints. This demonstrated that Little Higgs models are viable models of TeV scale physics that stabilize the weak scale and that the breaking scale, f, can be as low as 700 GeV without being in contradiction to precision electroweak observables. This theory is a small modification to the Minimal Moose having global $SO(5)$ symmetries in comparison to $SU(3)$. Most of the qualitative features of the Minimal Moose carried over into this model including that it is a two Higgs doublet model with a colored Dirac fermion at the TeV scale that cancels the one loop quadratic divergence of the top and several TeV scale vector bosons. By having custodial $SU(2)$ it is possible to take the simple limit where the g_5 coupling is large where the contributions from TeV scale physics to precision electroweak observables become small. In the model presented, a breaking scale as low as $f = 700$ GeV was allowed by precision electroweak observables. The limits on the W' and B' are around 2.5 TeV and the mass of the top partner is roughly 2 TeV . These are the states that cancel the one loop quadratic divergences from the Standard Model's gauge and top sectors and their masses are where naturalness dictates. The charged Higgs boson was typically the heaviest amongst the light Higgs scalars this resulted in a positive contribution to T. The limits from custodial $SU(2)$ violating operators favored a light Higgs boson coming not from the standard oblique corrections from the Higgs boson, but indirectly from integrating out the TeV scale gauge bosons. These already mild limits might be reduced by going away from a maximal phase. Changing this phase would also require recalculating the contributions to $\delta\rho$ from the Higgs sector when the states do not fall into CP eigenstates. There are additional scalars that could be as light as 100 GeV that came as the $SO(5)$ partners to the Higgs. As mentioned earlier in the section on triplet vevs, these states can be lifted by "Ω plaquettes" to the multi-TeV scale and therefore their relevance for phenomenology is model dependent.

This model predicts generically a positive contribution to T mimicking the effect of a light Higgs in the Standard Model. This is interesting because if one excludes the A_b^{FB} measurement as an outlier then the fit to precision electroweak observables favors a positive $T \sim 0.15 \pm 0.1$. This is generally stated as the Standard Model has a best fit for a Higgs mass of 40 GeV if the A_b^{FB} measurement is excluded.

There has been recent interest in the phenomenology of the Higgs bosons inside Little Higgs models. Most of the recent work we believe carries over qualitatively including the suppression of $h \to gg, \gamma\gamma$ [16,17]. The LHC should be able to produce copious numbers of the TeV scale partners in the top and vector sectors [14].

Another possible way of removing limits arising from the phase in the Higgs vacuum expectation value is to construct a model that has only one Higgs doublet. All "theory space" models automatically have two Higgs doublets so one possibility would be to follow the example of the "Littlest Higgs" and construct a coset model such as $SO(9)/(SO(5) \times SO(4))$ [3]. There may be other two Higgs doublet models that have a gauged $SU(2)_r$ that do not force the Higgs vacuum expectation value to break $SU(2)_C$.

To summarize the larger context of this model, it provides a simple realistic Little Higgs theory that is parametrically safe from precision electroweak measurements. While it is not necessary to have a gauged $SU(2)_r$, it allows for transparent limits to be taken where the TeV scale physics decouples from the physics causing constraints while still cutting off the low energy quadratic divergences. There are other ways of avoiding large contributions to electroweak precision observables without a gauged $SU(2)_r$. The important issue is that the physics that is stabilizing the weak scale from the most important interactions is not providing significant constraints on Little Higgs models. This is the deeper reason why the model presented worked in such a simple fashion. Precision electroweak constraints are coming from the interactions of either the B' or the interactions of the light fermions. The quadratic divergence from $U(1)_Y$ only becomes relevant at a scale of $10 - 15$ TeV and is oftentimes above the scale of strong coupling for Little Higgs models. The interactions of the light fermions with the TeV scale vector bosons is not determined by electroweak gauge symmetry and can be altered by either changing the charge assignments or by mixing the fermions with multi-TeV scale Dirac fermions.

In a broader view Little Higgs models offer a rich set of models for TeV scale physics that stabilize the weak scale. Each Little Higgs model has slightly different contributions to precision electroweak observables, but they do not have parametric problems fitting current experimental measurements. In the next five years the LHC will provide direct probes of TeV scale physics and determine whether Little Higgs models play a role in stabilizing the weak scale.

Acknowledgments

We would like to thank N. Arkani-Hamed, T. Gregoire, C. Kilic, R. Mahbubani, and M. Schmaltz for useful discussions and comments on this work. We would also like to thank C. Csaki for pointing out that radiatively generated tadpoles could give additional triplet vevs. S.C. is supported through an NSF graduate student fellowship.

A Generators

The $SO(5)$ commutation relations are:

$$[T^{mn}, T^{op}] = \frac{i}{\sqrt{2}}(\delta^{mo}T^{np} - \delta^{mp}T^{no} - \delta^{no}T^{mp} + \delta^{np}T^{mo}) \tag{A.1}$$

where m, n, o, p run from $1, \ldots, 5$. These generators can be broken up into

$$T^{l\,a} = \frac{1}{2\sqrt{2}}\epsilon^{abc}T^{bc} + \frac{1}{\sqrt{2}}T^{a4} \qquad T^{r\,a} = \frac{1}{2\sqrt{2}}\epsilon^{abc}T^{bc} - \frac{1}{\sqrt{2}}T^{a4}$$
$$T^{v\,0} = T^{45} \qquad T^{v\,a} = T^{a5} \tag{A.2}$$

The commutation relations in this basis are of $SO(5)$ are

$$[T^{l\,a}, T^{l\,b}] = i\epsilon^{abc}T^{l\,c}, \qquad [T^{r\,a}, T^{r\,b}] = i\epsilon^{abc}T^{r\,c}, \qquad [T^{l\,a}, T^{r\,b}] = 0,$$
$$[T^{v\,0}, T^{l\,a}] = -[T^{v\,0}, T^{r\,a}] = \frac{i}{2}T^{v\,a}, \qquad [T^{v\,0}, T^{v\,a}] = \frac{i}{2}(T^{r\,a} - T^{l\,a}),$$
$$\left[T^{v\,a}, T^{l\,b}\right] = -\frac{i}{2}T^{v\,0}\delta^{ab} + \frac{i}{2}\epsilon^{abc}T^{v\,c}, \qquad [T^{v\,a}, T^{r\,b}] = \frac{i}{2}T^{v\,0}\delta^{ab} + \frac{i}{2}\epsilon^{abc}T^{v\,c}, \tag{A.3}$$
$$\left[T^{v\,a}, T^{v\,b}\right] = \frac{i}{2}\epsilon^{abc}(T^{l\,c} + T^{r\,c}).$$

Vector Representation

The vector representation of $SO(5)$ can be realized as:

$$T^{mn\;op} = \frac{-i}{\sqrt{2}}(\delta^{mo}\delta^{np} - \delta^{no}\delta^{mp}) \tag{A.4}$$

where m, n, o, p again run over $1, \ldots, 5$ and m, n label the $SO(5)$ generator while o, p are the indices of the vector representation. In this representation:

$$\text{Tr } T^A T^B = \delta^{AB}. \tag{A.5}$$

Spinor Representation

The spinor representation is given by the form

$$\sigma^{l\,a} = \begin{pmatrix} \sigma^a/2 & 0 \\ 0 & 0 \end{pmatrix} \qquad \sigma^{r\,a} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^a/2 \end{pmatrix},$$

$$\sigma^{v\,0} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \sigma^{v\,a} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & i\sigma^a \\ -i\sigma^a & 0 \end{pmatrix} \tag{A.6}$$

In this representation

$$\mathrm{Tr}\ T^A T^B = \frac{1}{2}\delta^{AB}. \tag{A.7}$$

Chapter 3: A "Littlest Higgs" Model with Custodial $SU(2)$ Symmetry[10]

7 Introduction

In the near future, experimental tests at the LHC will begin to map out physics at the TeV energy scale. With this data, a determination of the Higgs sector, and more importantly, discovering the physics that stabilizes the weak scale from radiative corrections should be achievable goals. However, in the interim, the industry of precision electroweak observables has given us some indirect evidence on what the theory beyond the standard model must look like. And given the unreasonably good fit of the standard model to these observables, these constraints generically suggest a theory with perturbative physics at the TeV scale.

For many years, the only models that could stabilize the weak scale and be weakly perturbative were supersymmetric models, most notably the MSSM. In the past two years, it has been shown that there is a new class of perturbative theories of electroweak symmetry breaking, that of the "Little Higgs" [2, 5–11, 28]. For reviews of the physics, see [12, 13] and for more detailed phenomenology see [14–17]. Little Higgs theories protect the Higgs boson from one-loop quadratic divergences because each coupling treats the Higgs boson as an exact goldstone boson. However, two different couplings together can break the non-linear symmetries protecting the Higgs mass, and thus the Higgs is a pseudo-goldstone boson with quadratic divergences to its mass pushed to two-loop order. This allows a separation of scales between the cutoff and the electroweak scale, so that physics can be perturbative until the cutoff is reached at $\Lambda \approx 10$ TeV.

Having weakly perturbative physics at the TeV scale is probably necessary but definitely not sufficient to guarantee a theory is safe from precision electroweak constraints. Currently precision observables have been measured beyond one-loop order in the standard model, and since Little Higgs model corrections are parameterically of this order, these observables can put constraints on these theories [18–20, 29–31]. However, these constraints are not

[10] This chapter is based on reference [3].

unavoidable, and isolating the strongest constraints can point to the necessary features to make Little Higgs models viable theories of electroweak symmetry breaking. First of all, there are modifications of the original models which address these strongest constraints and greatly ammeliorate the issue [29]. However, just recently, a Little Higgs model was introduced containing a custodial $SU(2)$ symmetry, the $SO(5)\times SU(2)\times U(1)$ moose model [2]. In the limit of strong coupling for the $SO(5)$ gauge group, the precision electroweak constraints due to the T parameter were softened and in general, there is a large region of parameter space consistent with precision electroweak constraints and naturalness [32].

Let's briefly summarize the physics that gives the custodial $SU(2)$ symmetry. The important point is that in models with a gauged $U(1)\times U(1)$ subgroup and standard model fermions gauged under just one of the $U(1)$'s, the massive B' of these theories provides two constraints. The first constraint is that integrating out the B' generates a custodial $SU(2)$ violating operator that after electroweak symmetry breaking corrects the standard model formula for the mass of the Z gauge boson. This gives corrections to the ρ parameter, and vanishes as the two $U(1)$ gauge couplings become equal. However, the second constraint pulls in the opposite direction in gauge parameter space. This is because the coupling of the B' to standard model fermions generates corrections to low energy four-fermi operators and also to coefficients of the $SU(2)_W\times U(1)_Y$ fermion currents. These corrections vanish in the limit in which the $U(1)$ that the standard model fermions is not gauged under becomes strong. Thus, these two constraints prefer different limits in parameter space and can constrain the model.

As pointed out before [2], there are simple modifications that evade these two constraints, such as only gauging $U(1)_Y$, charging the SM fermions equally under both $U(1)$'s, or through fermion mixing. Another simple approach that gives custodial $SU(2)$ symmetry is to complete the B' into a custodial $SU(2)$ triplet. If the triplet is exactly degenerate in mass, integrating it out does not contribute to a custodial $SU(2)$ violating operator. To include these new states, instead of gauging two $U(1)$'s, $SU(2)_R\times U(1)$ is gauged. After being broken down to the diagonal $U(1)_Y$, B' and $W^{r\,\pm}$ are put into a "$SU(2)_R$" triplet. Integrating out the $W^{r\,\pm}$ generates an operator which only gives mass to the W giving a ρ

contribution of the opposite sign of the B' contribution. Numerically, the total ρ contribution from the gauge sector cancels in the strong $SU(2)_R$ coupling limit (where the triplet becomes degenerate), which is the same limit that reduces corrections to fermion operators.

However, this cancelation is not quite exact for the $SO(5)\times SU(2)\times U(1)$ moose model. The Higgs quartic potential of that theory has a flat direction when the two Higgs vevs have the same phase, thus viable electroweak symmetry breaking requires the Higgs vevs to have different phases. This phase difference changes the ρ contribution due to the $W^{r\,\pm}$ gauge bosons. The Higgs currents of the $W^{r\,\pm}$ are not invariant under a vev phase rotation, and thus the cancelation in the strong coupling limit only occurs if the phase is 0 or π. Indeed, this remnant of custodial $SU(2)$ violation puts the strongest constraint on the theory.

The situation can be easily resolved if the Little Higgs theory contains only a single light Higgs doublet. In this case, the $W^{r\,\pm}$ current just transforms by a phase under the vev phase rotation, which cancels out of the contribution. It turns out that the $SO(5)\times SU(2)\times U(1)$ moose's defect can be removed by imposing a $\mathbb{Z}_4$ symmetry inspired by orbifold models [33], which leaves only a single light Higgs doublet that still has an order one quartic coupling. In this paper, we will take a different approach and construct a "Littlest Higgs" model with custodial $SU(2)$ symmetry and just *one* higgs doublet.

This "Littlest Higgs" model will be based on an $\frac{SO(9)}{SO(5)\times SO(4)}$ coset space, with an $SU(2)_L\times SU(2)_R\times SU(2)\times U(1)$ subgroup of $SO(9)$ gauged. The pseudo-goldstone bosons are a single Higgs doublet, an electroweak singlet and a set of three $SU(2)_W$ triplets, precisely the content of one of the original custodial $SU(2)$ preserving composite Higgs models [34]. The global symmetries protect the Higgs doublet from one-loop quadratic divergent contributions to its mass. However, the singlet and triplets are not protected, and will be pushed to the TeV scale. Integrating out these heavy particles will generate an order one quartic coupling for the Higgs. To complete the theory with fermions, the minimal top sector contains two extra colored quark doublets and their charge conjugates.

Since the primary motivation of the model is to improve consistency with precision electroweak observables, the model's corrections to these observables will be calculated. First, we will see that aside from some third generation quark effects, a limit will exist where non-oblique corrections vanish. This limit was recently described as "near-oblique" [31] and we will continue to use this terminology. The existence of this limit allows a meaningful S and T analysis of the oblique corrections, which will be performed in this model to order (v^2/f^2). The dominant contributions come from the extended top sector and the Higgs doublet, which are quite mild in most of parameter space. In fact, this analysis will show that there is a wide range of Higgs masses allowed in a large region of parameter space consistent with naturalness.

The outline of the rest of the paper is as follows: in section 8 we describe the model's coset space, light scalars and the symmetries that protect the Higgs mass. We also analyze the gauge structure and then describe the minimal candidate top sectors. In section 9, we will show how the quartic Higgs potential is generated as well as describe the log enhanced contributions to the Higgs mass parameter. There will be vacuum stability issues, and we will point out ways which these can be resolved. Also as usual, the top sector contributions will generically drive electroweak symmetry breaking. In section 10, some precision electroweak observables will be calculated and the constraints on the theory will be detailed. In section 11, we conclude and finally in appendix B, we describe our specific generators and representations of $SO(4)$.

8 The Model

The first ingredient necessary for custodial $SU(2)$ symmetry is the breakdown of $SU(2)_L \times SU(2)_R \times SU(2) \times U(1)$ down to the diagonal $SU(2)_W \times U(1)_Y$ subgroup. Therefore the global symmetry group must be at least rank 4. Two rank 4 groups are easy to eliminate– $SU(5)$ does not contain the gauged group and the $SO(8)$ adjoint contains no Higgs doublets. This leaves $SO(9)$, $Sp(8)$ and F_4 as the only remaining rank 4 candidates. In this paper, we'll focus on the $SO(9)$ group as it is the easiest to analyze. However, we do mention here that it appears to be difficult to get a single light Higgs doublet in the $Sp(8), F_4$ groups.

Isolating our attention to $SO(9)$, it is straightforward to implement the "Little Higgs" construction. Using the vector representation, the top four by four block will contain the gauged $SO(4) \cong SU(2)_L \times SU(2)_R$ and the bottom four by four block will contain the gauged $SU(2) \times U(1) \subset SO(4)$. The coset space should break these two $SO(4)$'s down to their diagonal subgroup, which can be achieved by an off-diagonal vev for a two-index tensor of $SO(9)$. In order to have the largest unbroken global symmetry (and thus reduce the amount of light scalars), a symmetric two-index tensor should be chosen.

This construction can be described in the following way: take an orthogonal symmetric nine by nine matrix, representing a non-linear sigma model field Σ which transforms under an $SO(9)$ rotation by $\Sigma \to V\Sigma V^T$. To break the $SO(4)$'s to their diagonal, we take Σ's vev to be

$$\langle\Sigma\rangle = \begin{pmatrix} 0 & 0 & \mathbb{1}_4 \\ 0 & 1 & 0 \\ \mathbb{1}_4 & 0 & 0 \end{pmatrix} \tag{8.1}$$

which breaks the $SO(9)$ global symmetry down to an $SO(5) \times SO(4)$ subgroup.[11] This coset space guarantees the existence of $20 = (36 - 10 - 6)$ light scalars. Of these 20 scalars, 6 will be eaten in the Higgsing of the gauge groups down to $SU(2)_W \times U(1)_Y$. The remaining 14 scalars consist of a single Higgs doublet h, an electroweak singlet ϕ^0, and three triplets ϕ^{ab} which transform under the $SU(2)_L \times SU(2)_R$ diagonal symmetry as[12]

$$h : (\mathbf{2}_L, \mathbf{2}_R) \qquad \phi^0 : (\mathbf{1}_L, \mathbf{1}_R) \qquad \phi^{ab} : (\mathbf{3}_L, \mathbf{3}_R). \tag{8.2}$$

This spectrum is particularly nice as each set of scalars can have vacuum expectation values that preserve custodial $SU(2)$; we will see later that this is approximately true. These fields parameterize the direction of the Σ field and can be written in the standard way

$$\Sigma = e^{i\Pi/f}\langle\Sigma\rangle e^{i\Pi^T/f} = e^{2i\Pi/f}\langle\Sigma\rangle \tag{8.3}$$

[11] We could separate the trace from Σ to make it transform as an irreducible representation of $SO(9)$, however this equivalent vev is chosen so that Σ can be orthogonal.

[12] See appendix B for specific representation and generator conventions.

where

$$\Pi = \frac{-i}{4} \left(\begin{array}{ccc} 0 & \sqrt{2}\vec{h} & -\Phi \\ -\sqrt{2}\vec{h}^T & 0 & \sqrt{2}\vec{h}^T \\ \Phi & -\sqrt{2}\vec{h} & 0 \end{array} \right). \tag{8.4}$$

In Π, the would-be goldstone bosons that are eaten in the Higgsing down to $SU(2)_W \times U(1)_Y$ have been set to zero. The singlet and triplets are contained in the symmetric four by four matrix Φ where

$$\Phi = \phi^0 + 4\phi^{ab}\, T^{l\,a} T^{r\,b}. \tag{8.5}$$

It is now simple to determine the global symmetries that protect the Higgs mass at one loop. Under the upper five by five $SO(5)_1$ symmetry, the scalars transform as:

$$\delta\vec{h} = \vec{\alpha} + \cdots \qquad \delta\Phi = -\frac{1}{2f}\left(\vec{\alpha}\,\vec{h}^{\,T} + \vec{h}\,\vec{\alpha}^{\,T}\right) + \cdots \tag{8.6}$$

Similarly, under the lower five by five $SO(5)_2$, the scalars transform as:

$$\delta\vec{h} = \vec{\beta} + \cdots \qquad \delta\Phi = \frac{1}{2f}\left(\vec{\beta}\,\vec{h}^{\,T} + \vec{h}\,\vec{\beta}^{\,T}\right) + \cdots \tag{8.7}$$

Any interaction that preserves at least one of these $SO(5)$ symmetries treats the Higgs as an *exact* goldstone boson. Thus, if all interactions are chosen to preserve one of these symmetries, the Higgs mass will be protected from one loop quadratic divergences. In the next two subsections, that motivation is used to determine the requisite interactions of the theory.

8.1 Gauge Sector

The gauge group structure obviously follows the preserving $SO(5)$ symmetry logic. The gauged $SO(4) \cong SU(2)_L \times SU(2)_R$ is generated by

$$\tau^{l\,a} = \begin{pmatrix} T^{l\,a} & \\ & 0_5 \end{pmatrix} \qquad \tau^{r\,a} = \begin{pmatrix} T^{r\,a} & \\ & 0_5 \end{pmatrix} \tag{8.8}$$

and preserves $SO(5)_2$ whereas the gauged $SU(2) \times U(1)$ is generated by

$$\eta^{l\,a} = \begin{pmatrix} 0_5 & \\ & T^{l\,a} \end{pmatrix} \qquad \eta^{r\,3} = \begin{pmatrix} 0_5 & \\ & T^{r\,3} \end{pmatrix} \tag{8.9}$$

and preserves $SO(5)_1$. The kinetic term for the pseudo-goldstone bosons can now be written as

$$\mathcal{L}_{kin} = \frac{f^2}{4}\,\text{Tr}\,[D_\mu \Sigma D^\mu \Sigma] \tag{8.10}$$

where the covariant derivative is given by

$$D_\mu \Sigma = \partial_\mu \Sigma + i\,[A_\mu, \Sigma] \tag{8.11}$$

with the gauge boson matrix A_μ defined by

$$A \equiv g_L W^{la}_{SO(4)} \tau^{l\,a} + g_R W^{ra}_{SO(4)} \tau^{r\,a} + g_2 W^{la} \eta^{l\,a} + g_1 W^{r3} \eta^{r\,3}. \tag{8.12}$$

Due to the vev of Σ, the vector bosons mix and can be diagonalized with the following transformations:

$$\begin{aligned} B &= \cos\theta' W^{r3} - \sin\theta' W^{r3}_{SO(4)} & \qquad B' = W'^{\,r3} = \sin\theta' W^{r3} + \cos\theta' W^{r3}_{SO(4)} \\ W^a &= \cos\theta W^{la} - \sin\theta W^{la}_{SO(4)} & \qquad W'^a = W'^{\,la} = \sin\theta W^{la} + \cos\theta W^{la}_{SO(4)} \end{aligned} \tag{8.13}$$

where the mixing angles are related to the couplings by:

$$\begin{aligned}\cos\theta' &= g'/g_1 \qquad \sin\theta' = g'/g_R \\ \cos\theta &= g/g_2 \qquad \sin\theta = g/g_L.\end{aligned} \tag{8.14}$$

Notice that there is no relation between θ and θ' since $SO(4)$ has two arbitrary gauge couplings g_L and g_R. They could of course be set equal by imposing a $\mathbb{Z}_2$ symmetry, which we will choose to do when describing the limits on the model. In this L-R symmetric limit, the constraint on the angles in order to get the correct θ_W is $\sin\theta \approx \sqrt{3}\sin\theta'$. The masses for the heavy vectors can now be written in terms of the electroweak gauge couplings and mixing angles:

$$m^2{}_{W'} = \frac{4g^2 f^2}{\sin^2 2\theta} \qquad m^2{}_{B'} = \frac{4g'^2 f^2}{\sin^2 2\theta'} \qquad m^2{}_{W^{r\pm}} = \frac{4g'^2 f^2}{\sin^2 2\theta'}\cos^2\theta'. \tag{8.15}$$

8.2 Fermion Sector

For all fermions besides the top quark, the yukawa couplings are small, and thus it is not necessary to protect the Higgs from their one loop quadratic divergences. However, there is the requirement that low energy observables such as four-fermi operators do not receive large corrections. This can be achieved by gauging the light fermions only under $SU(2) \times U(1)$. In the strong $SO(4)$ coupling limit, these fermions will decouple from the W' and B' and will not give strong precision electroweak corrections.

To implement the Yukawa couplings for the light fermions, we add

$$\begin{aligned}\mathcal{L}_{LF} &= \sqrt{2}f\left[y_u\,(0_4\,u^c\,0_4)\,\Sigma\begin{pmatrix}0_5\\ \vec{q}_u\end{pmatrix} + y_d\,(0_4\,d^c\,0_4)\,\Sigma\begin{pmatrix}0_5\\ \vec{q}_d\end{pmatrix} + y_l\,(0_4\,e^c\,0_4)\,\Sigma\begin{pmatrix}0_5\\ \vec{l}\end{pmatrix}\right] \\ &\quad + \text{h.c.}\end{aligned} \tag{8.16}$$

In this expression, we have defined the "$SO(4)$" representations corresponding to the

$SU(2) \times U(1)$ representations by

$$\vec{q}_u \leftrightarrow Q_u = (q\ 0_2) \qquad \vec{q}_d \leftrightarrow Q_d = (0_2\ q) \qquad \vec{l} \leftrightarrow L = (0_2\ l) \tag{8.17}$$

where q and l are the standard quark and lepton doublets. The exact correspondence between the two equivalent representations is presented in appendix B. At first order, these interactions reproduce the standard yukawa interactions for the light fermions.

On the other hand, the top yukawa is the strongest one loop quadratic divergence of the standard model and therefore the top sector must be extended in order to stabilize the Higgs mass parameter. From the symmetry considerations given earlier, the top sector has to preserve either the $SO(5)_1$ or $SO(5)_2$ symmetry. The minimal approach is to preserve the $SO(5)_1$ symmetry, which can be accomplished by adding t^c to an $SO(4)$ gauge vector $\vec{\mathcal{X}}^c$. In addition to this new vector, we add its charge conjugate $\vec{\mathcal{X}}$ and add a Dirac mass for the two fermions. The interactions are:

$$\mathcal{L}_{top} = y_1 f\, (\vec{\mathcal{X}}^{c\,T}\ t^c\ 0_4)\, \Sigma \begin{pmatrix} 0_5 \\ \vec{q}_t \end{pmatrix} + y_2 f \vec{\mathcal{X}}^T \vec{\mathcal{X}}^c + \text{ h.c.} \tag{8.18}$$

Now, the choice is whether or not to make $\vec{q}_t$ a "full" $SO(4)$ vector. Since it is only charged under $SU(2) \times U(1)$, it does not have to be a full $SO(4)$ vector, but can contain just one doublet like $\vec{q}_u$ above. For the sake of simplicity, we will choose to analyze the most minimal case of one doublet.

In this minimal case, the gauge charges of the fermions are:

	$SU(3)_c$	$SU(2)_L$	$SU(2)_R$	$SU(2)$	$U(1)$
q	3	1	1	2	1/6
t^c	$\bar{3}$	1	1	1	$-2/3$
$\vec{\mathcal{X}}$	3	2	2	1	2/3
$\vec{\mathcal{X}}^c$	$\bar{3}$	2	2	1	$-2/3$

(8.19)

Under the diagonal $SU(2)_W \times U(1)_Y$, $\vec{\mathcal{X}}$ contains two doublets $\mathcal{X}_1, \mathcal{X}_2$ with hypercharge 1/6 and 7/6 respectively. Expanding the terms, we find a mass term linking $\mathcal{X}_1^c$ with a linear combination of q and $\mathcal{X}_1$. Integrating out the heavy fermion gives a top yukawa coupling

$$y_t = \frac{y_1 y_2^*}{\sqrt{2(|y_1|^2 + |y_2|^2)}}. \tag{8.20}$$

9 Potential and EWSB breaking

By construction, the interactions of the theory do not generate one loop quadratic divergences for the mass parameter of the Higgs. To demonstrate this explicitly, the Coleman-Weinberg potential will be computed. The one loop quadratic divergent piece will generate a potential for Φ and h, including a quadratically divergent mass for the Φ. Similar to the $SU(6)/Sp(6)$ model [10], the gauge interactions will introduce an instability in the vacuum. The problem is a bit more serious here because the gauge contributions are opposite in sign for the singlet and triplet masses; thus, the origin of the potential is a saddle point. However, as in the $SU(6)/Sp(6)$ paper, there are ways to cure this instability issue. Once the instability has been addressed, integrating out the massive Φ will generate an order one quartic coupling for h, but no mass term.

For the log divergent piece of the Coleman-Weinberg potential, we will only analyze the contributions to the Higgs mass parameter. As usual, gauge and scalar sectors will give positive contributions whereas the top sector gives a large negative contribution that drives electroweak symmetry breaking.

One Loop Quadratic Term

The one loop quadratically divergent piece of the Coleman-Weinberg Potential is given by

$$V_{\text{one loop }\Lambda^2} = \frac{\Lambda^2}{32\pi^2}\,\mathrm{Str}\,(M^\dagger M[\Sigma]) \tag{9.1}$$

By the symmetry arguments given earlier, the different $SO(5)_i$ preserving interactions can generate operators depending on

$$SO(5)_1: \quad \Phi + \frac{1}{2f}\vec{h}\,\vec{h}^T \qquad \text{or} \qquad SO(5)_2: \quad \Phi - \frac{1}{2f}\vec{h}\,\vec{h}^T. \tag{9.2}$$

It will also be convenient to introduce some notation, where

$$\frac{1}{2f}\vec{h}\,\vec{h}^T = \mathcal{H}^0 + 4\mathcal{H}^{ab}\,T^{l\,a}T^{r\,b}. \tag{9.3}$$

$\mathcal{H}^0$ and $\mathcal{H}^{ab}$ are quadratic in the h fields and their explicit expressions appear in appendix B.

The gauge contribution can be calculated from the kinetic term for Σ, which gives

$$\begin{aligned} V_{\text{gauge}} = \quad & -\frac{9f^2}{8}\left[(g_L^2+g_R^2)(\phi^0-\mathcal{H}^0)^2+(g_2^2+g_1^2/3)(\phi^0+\mathcal{H}^0)^2\right]+ \\ & \frac{3f^2}{8}\left[(g_L^2+g_R^2)(\phi^{ab}-\mathcal{H}^{ab})^2+(g_2^2+g_1^2)(\phi^{ab}+\mathcal{H}^{ab})^2-2g_1^2(\phi^{a3}+\mathcal{H}^{a3})^2\right] \end{aligned} \tag{9.4}$$

where we have ignored a constant term, expanded to second order in Φ and fourth order in h, and set $\Lambda = 4\pi f$. There are two important points to make about this result. First of all, there is a sign difference between the mass terms for the singlet and triplets. Thus, the gauge interactions introduce a saddle point instability in the vacuum. This is expected since the gauge groups would prefer the Σ vev to be proportional to the identity; at this vacuum, no gauge groups are broken and indeed the negative mass squared for the singlet attempts to rotate the vev to this non-breaking vacuum. However, as we will see later, the top sector gives equal sign contributions to both mass terms. Also from the point of view of the effective field theory, operators can be written down that give equal sign contributions to both masses or even just to the singlet. The second thing to note about the gauge contribution is that only the gauged $U(1)$ introduces explicit custodial $SU(2)$ violation into the potential. As a matter of fact, this will be the only interaction that can give the triplets a custodial $SU(2)$ violating vev. Since g_1 will be approximately equal to the standard model hypercharge coupling, the triplet vevs usually give suitably small contributions to ρ. We will analyze the triplet vevs in greater detail in section 10.

Now, analyzing the top sector, we find the contribution

$$V_{\text{fermion}} = 3|y_1|^2 f^2 \left[(\phi^0 + \mathcal{H}^0)^2 + (\phi^{ab} + \mathcal{H}^{ab})^2 \right] \tag{9.5}$$

where again we have ignored a constant piece and set $\Lambda = 4\pi f$. As noted earlier, the fermion sector gives equal sign contributions to singlet and triplet masses and does not introduce custodial $SU(2)$ breaking at this order.[13] Following the $SU(6)/Sp(6)$ Little Higgs [10], we could also extend the top sector with an interaction that preserves the $SO(5)_2$ symmetry. This would have the added benefit of giving equal sign contributions to the $(\phi - \mathcal{H})^2$ terms and could lift the saddle point into a local minimum. Another way to do this is through operators such as

$$\mathcal{L}_1 = a_1 f^2 \sum_{i=1,j=1}^{4,4} \Sigma_{ij}\Sigma_{ij} = a_1 f^2 \left[(\phi^0 - \mathcal{H}^0)^2 + (\phi^{ab} - \mathcal{H}^{ab})^2 \right] \tag{9.6}$$

or

$$\mathcal{L}_2 = a_2 f^2 (\sum_{i=1}^{4} \Sigma_{ii})^2 = 4 a_2 f^2 \left[(\phi^0 - \mathcal{H}^0)^2 \right] \tag{9.7}$$

which respect the $SO(5)_2$ symmetry and give contributions to both singlet and triplet masses or just masses for the singlets. Depending on the UV completion of the model, these operators can be generated; for instance, they might appear naturally in an extended technicolor like completion.

These radiative corrections tell us that we must put in these operators with coefficients of their natural size of the form

$$\begin{aligned} V = \quad & \lambda_{\mathbf{1}}^- f^2 (\phi^0 - \mathcal{H}^0)^2 + \lambda_{\mathbf{1}}^+ f^2 (\phi^0 + \mathcal{H}^0)^2 + \\ & \lambda_{\mathbf{3}}^- f^2 (\phi^{ab} - \mathcal{H}^{ab})^2 + \lambda_{\mathbf{3}}^+ f^2 (\phi^{ab} + \mathcal{H}^{ab})^2 + \Delta\lambda_{\mathbf{3}} f^2 (\phi^{a3} + \mathcal{H}^{a3})^2. \end{aligned} \tag{9.8}$$

[13] If we had chosen $\vec{q}_t$ to contain two doublets, there would be no custodial $SU(2)$ breaking at any order.

As mentioned before, since g_1 will be small, $\Delta\lambda_{\mathbf{3}} \ll \lambda_{\mathbf{3}}^{\pm}$ is expected, which leads to approximately custodial $SU(2)$ preserving triplet vevs. We will assume that the singlet and triplet masses are positive; integrating out these heavy particles then leads to a quartic coupling of the Higgs (ignoring $\Delta\lambda_{\mathbf{3}}$ for simplicity):

$$\lambda|h|^4 \quad \text{where} \quad 4\lambda = \lambda_{\mathbf{1}} + 3\lambda_{\mathbf{3}} \tag{9.9}$$

and we've defined $1/\lambda_{(\mathbf{1},\mathbf{3})} = 1/\lambda^-_{(\mathbf{1},\mathbf{3})} + 1/\lambda^+_{(\mathbf{1},\mathbf{3})}$. Requiring a positive order one λ puts some mild constraints on the $\lambda^{\pm}_{(\mathbf{1},\mathbf{3})}$ parameters.

Log Contributions to the Mass Parameter

Even though the Little Higgs mechanism protects the Higgs from one-loop quadratic divergences, there are finite, one loop logarithmically divergent, and two loop quadratically divergent mass contributions, all of the same order of magnitude. Here we will analyze the logarithmically enhanced pieces as given by the one loop log term in the Coleman-Weinberg potential

$$V_{\text{one loop log}} = \frac{1}{64\pi^2}\,\mathrm{Str}\left[(M^\dagger M)^2 \ln\frac{M^\dagger M}{\Lambda^2}\right]. \tag{9.10}$$

The gauge contribution to the mass squared is positive

$$m^2_{\text{gauge}} = \frac{3}{64\pi^2}\left[3g^2 m^2_{W'} \ln\frac{\Lambda^2}{m^2_{W'}} + g'^2 m^2_{B'} \ln\frac{\Lambda^2}{m^2_{B'}}\right] \tag{9.11}$$

but the fermion contribution is negative

$$m^2_{\text{fermion}} = -\frac{3|y_t|^2}{8\pi^2} m^2_{t'} \ln\frac{\Lambda^2}{m^2_{t'}} \tag{9.12}$$

where we have defined $m^2_{t'} = (|y_1|^2 + |y_2|^2)f^2$.[14] This large top contribution generically dominates and drives electroweak symmetry breaking. We've chosen not to consider the

[14] The heavy t' quark is the only heavy quark whose mass shifts when the Higgs vev is turned on. Thus, it is the new heavy state that appears and cuts off the top yukawa quadratic divergence, which is also why the fermionic contribution to the Higgs mass parameter only depends on $m_{t'}$.

scalar contribution since it depends on the specifics behind the generation of the potential (Eq. 9.8). However, we mention that it is typically positive and subdominant to the fermion contribution.

10 Precision Electroweak Observables

Now that we have described the model's content and interactions, the contributions to electroweak observables can be calculated. In general, we will work to leading order in $O(v^2/f^2)$ and neglect any higher order effects. First in section 10.1, we will focus on non-oblique corrections to electroweak fermion currents and four-fermi interactions. We will demonstrate how the limit of strong g_L, g_R coupling is a "near-oblique" limit as discussed recently in [31]. This limit validates the usefulness of an S and T analysis and in sections 10.2 and 10.3 we will calculate the model's contributions to these parameters. We will choose to keep the S and T contributions from the Higgs sector, but will subtract out all other standard model contributions. Finally in section 10.4, the results of the full S-T analysis will be presented.

10.1 Electroweak Currents

First of all, there are non-oblique corrections due to the exchange of the heavy gauge bosons. Specifically, integrating out the heavy vectors generates four Fermi operators and Higgs-Fermi current current interactions (the Higgs-Higgs interactions give oblique corrections and will be considered in the Higgs contribution to T in section 10.2). The former are constrained by tests of compositeness and the latter after electroweak symmetry breaking induce corrections to standard model fermionic currents which are constrained by Z-pole observables. As pointed out recently by Gregoire, Smith, and Wacker [31], the S and T analysis is reliable when there exists a "near-oblique" limit where most of the non-oblique corrections vanish. The limit is called "near-oblique" since third generation quark physics still has non-vanishing effects. In this model this limit turns out to be the strong $g_L, g_R \to \infty$ limit that decouples the light generations from the heavy gauge bosons. A discussion of the non-decoupling third generation effects is outside the scope of this paper and thus they will not be analyzed. However, for a preliminary discussion of the important operators in such an analysis, see reference [31].

To calculate these induced effects, we first write down the relevant currents to the heavy gauge bosons starting with the Higgs (leaving off Lorentz indices for readability)

$$\begin{aligned} j^a_{W'H} &= g\cot 2\theta j^a_H = \frac{g\cos 2\theta}{2\sin 2\theta}\, ih^\dagger \sigma^a \overleftrightarrow{D} h \\ j_{B'H} &= g'\cot 2\theta' j_H = -\frac{g'\cos 2\theta'}{2\sin 2\theta'}\, ih^\dagger \overleftrightarrow{D} h \end{aligned} \tag{10.1}$$

and also for the standard model fermions (aside from the third generation quarks)

$$j^a_{W'F} = g\tan\theta\ j^a_F \qquad\qquad j_{B'F} = g'\tan\theta'\ j_F \tag{10.2}$$

where they are given in terms of the standard model $SU(2)_W, U(1)_Y$ currents $j^a_{(HF)}$ and $j_{(HF)}$. The one heavy gauge boson current left out is the Higgs current to $W^{r\,\pm}$, but since there is no corresponding fermionic current, integrating out $W^{r\,\pm}$ does not generate four-fermi operators or standard model current corrections.

Integrating out the heavy gauge bosons generates the Higgs-Fermi interactions

$$\begin{aligned} \mathcal{L}_{\mathrm{H\,F}} &= -\frac{j^a_\mu{}_{W'\mathrm{H}}\, j^{\mu a}{}_{W'\mathrm{F}}}{M^2_{W'}} - \frac{j_{\mu B'\mathrm{H}}\, j^\mu{}_{B'\mathrm{F}}}{M^2_{B'}} \\ &= -\frac{\sin^2\theta\cos 2\theta}{2f^2} j_{\mathrm{H}}{}^{a\,\mu} j_{\mathrm{F}\,a\,\mu} - \frac{\sin^2\theta'\cos 2\theta'}{2f^2} j_{\mathrm{H}}{}^{\mu} j_{\mathrm{F}\,\mu} \end{aligned} \tag{10.3}$$

and the four Fermi interactions

$$\begin{aligned} \mathcal{L}_{\mathrm{F\,F}} &= -\frac{(j^a_\mu{}_{W'\mathrm{F}})^2}{2M^2_{W'}} - \frac{(j_{\mu B'\mathrm{F}})^2}{2M^2_{B'}} \\ &= -\frac{\sin^4\theta}{2f^2} j_{\mathrm{F}}{}^{a\,\mu} j_{\mathrm{F}\,a\,\mu} - \frac{\sin^4\theta'}{2f^2} j_{\mathrm{F}}{}^{\mu} j_{\mathrm{F}\,\mu}. \end{aligned} \tag{10.4}$$

As a rough guide, these operators have to be suppressed by about $(4\ \mathrm{TeV})^2$ to be safe [24,31]. To simplify the analysis, we will take the $SO(4)$ symmetric limit $g_L = g_R$. In this restricted case, in order to get the correct $\sin\theta_W$ requires the relation $\sqrt{3}\theta' \approx \theta$ at small θ's. Thus, the $SU(2)$ operators give the tightest bound. Of these, the Higgs-Fermi $SU(2)$ operator turns out to be the most constrained giving a constraint

$$m_{W'} \gtrsim 1.8 \text{ TeV} . \tag{10.5}$$

For the value $f = 700$ GeV, this corresponds to a limit $\theta \lesssim 1/4$. However, to be safe we'll later take as a benchmark value $\theta' = 1/5\sqrt{3}, \theta \approx 1/5$ from which to compare with experiment. Note that for this near-oblique limit to exist, it was crucial that the light generations could be decoupled from the heavy gauge bosons. Again, only in this limit is an analysis of the oblique corrections S and T meaningful.

10.2 Custodial $SU(2)$

Custodial $SU(2)$ violating effects are highly constrained by precision electroweak tests and this model's primary motivation is to minimize any such violation. Custodial $SU(2)$ violation is conveniently parameterized by corrections to the ρ parameter (or equivalently the T parameter). In a Little Higgs model, there are potentially five sources of custodial $SU(2)$ violation. The first possible contribution is that of expanding out the kinetic term in terms of the Higgs field. The non-linear sigma model kinetic term contains interactions at high order that could give custodial $SU(2)$ violating masses to the W and Z. However, in this model there is no violation at any order. This is due to the fact that the kinetic term is invariant under a global $SO(4)_D$ that is broken down by the Higgs vev to custodial $SU(2)$. As a matter of fact, all terms in the expansion of the kinetic term just shift the value of the Higgs vev v, which gives $\delta\rho = 0$.

Vector Bosons

The second possibility is that integrating out the TeV scale gauge bosons (the $W', B', W^{r\,\pm}$) can generate a custodial $SU(2)$ violating operator. This is typically denoted as

$$\mathcal{O}_4 = |h^\dagger D_\mu h|^2. \tag{10.6}$$

In all previous Little Higgs theories, integrating out the W' gauge bosons does not generate this operator at $O(v^2/f^2)$ and this holds true for this model as well. On the other hand, integrating out the B' and $W^{r\,\pm}$ *does* generate this operator, but with opposite sign! There is a cancelation with the total contribution

$$\delta\rho_{\text{Gauge Boson}} = -\frac{v^2}{16 f^2}\sin^2 2\theta'. \tag{10.7}$$

Note that as advertised this vanishes in the limit $\theta' \to 0$, which is the same limit where the standard model fermions decouple from the B'. At the benchmark values $\theta' = 1/5\sqrt{3}, f =$ 700 GeV, this gives a contribution $\text{T}_{\text{gauge}} = -.056$. One can see that the addition of the extra $W^{r\,\pm}$ gauge bosons has provided an extra suppression factor of $\sin^2 2\theta' \approx 1/20$.

Triplet Vev

The third contribution to custodial $SU(2)$ violation comes from the triplet vevs. The key point is that the potential (Eq. 9.8) is custodial $SU(2)$ invariant except for the $\Delta\lambda_{\mathbf{3}}$ term generated by the gauged $U(1)$. The non-oblique corrections already prefer small θ' and thus small g_1. Therefore custodial $SU(2)$ violation in the potential should be small, and we should expect that $\Delta\lambda_{\mathbf{3}} \ll \lambda_{\mathbf{3}}^{\pm}$. Calculating the triplet vev contribution, we find

$$\delta\rho_{\text{triplet}} = \frac{v^2}{16 f^2}\left[\left(\frac{\lambda_{\mathbf{3}}^- - \lambda_{\mathbf{3}}^+ - \Delta\lambda_{\mathbf{3}}}{\lambda_{\mathbf{3}}^- + \lambda_{\mathbf{3}}^+ + \Delta\lambda_{\mathbf{3}}}\right)^2 - \left(\frac{\lambda_{\mathbf{3}}^- - \lambda_{\mathbf{3}}^+}{\lambda_{\mathbf{3}}^- + \lambda_{\mathbf{3}}^+}\right)^2\right] \approx \frac{v^2}{4 f^2}\frac{\lambda_{\mathbf{3}}^-(\lambda_{\mathbf{3}}^+ - \lambda_{\mathbf{3}}^-)}{(\lambda_{\mathbf{3}}^- + \lambda_{\mathbf{3}}^+)^3}\Delta\lambda_{\mathbf{3}}. \tag{10.8}$$

where we have expanded to first order in $\Delta\lambda_{\mathbf{3}}/\lambda_{\mathbf{3}}^{\pm}$ to get the end result. In comparison with the “Littlest Higgs”, there is now a beneficial $\Delta\lambda_{\mathbf{3}}/\lambda_{\mathbf{3}}^{\pm}$ suppression. We cannot really

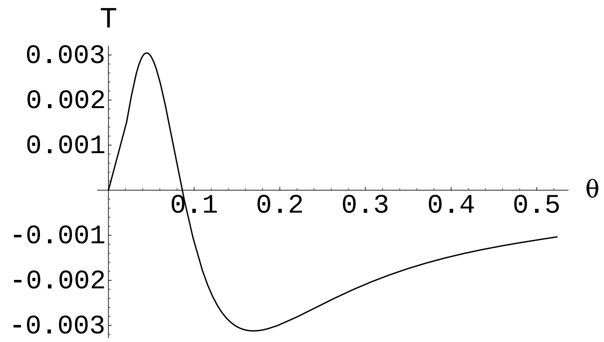

Figure 1: Triplet Contribution to T as a function of θ' with $y_1 = 2$, $g_L = g_R$, and $f =$ 700 GeV.

say anything more in the effective field theory since there are unknown order one factors in the relation between the $\lambda_{\mathbf{3}}$'s and the coefficients as calculated in the Coleman-Weinberg potential. However, to get a feel for the expected size of the contribution, we can take the Coleman-Weinberg coefficients at face value which for $y_1 = 2$, $g_L = g_R$, and $f = 700$ GeV, gives the plot T vs. θ' as shown in figure 1. In the limit $g_L = g_R$, there is an upper bound $\theta' \lesssim \pi/5$ (in order to get the correct $\sin\theta_W$) which is why the graph is cut off on the right. Order one factors aside, it is obvious that the triplet contribution to T is negligibly small due to the extra suppression described above.

Top Sector

The fourth source of custodial $SU(2)$ violation is the introduction of new fermions in the top sector. To calculate the effects of the extra fermions, it is easiest to compute the contributions to T through vacuum polarization diagrams by the definition

$$\mathrm{T} \equiv \frac{e^2}{\alpha \sin^2\theta_W \cos^2\theta_W m_Z^2}[\Pi_{11}(0) - \Pi_{33}(0)]. \tag{10.9}$$

If we ignore the small mixing effects induced by the b quark mass, the contribution is parameterized by the single parameter θ_t where

$$y_1 = \frac{\sqrt{2}y_t}{\sin\theta_t} \qquad y_2 = \frac{\sqrt{2}y_t}{\cos\theta_t}. \tag{10.10}$$

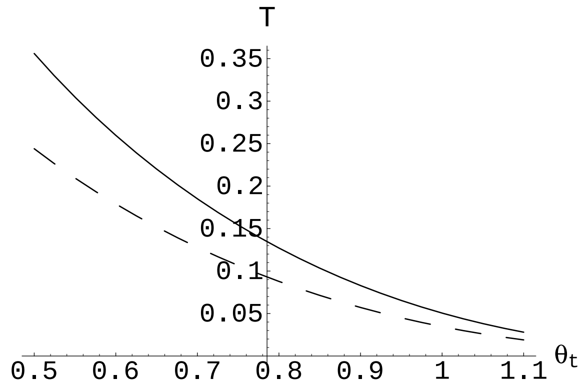

Figure 2: Top Sector Contribution to T as a function of θ_t. The solid line is for $f = 700$ GeV while the dashed line is for $f = 900$ GeV.

In figure 2, plots of the T contribution versus θ_t are plotted for $f = 700$ GeV and $f = 900$ GeV, centered around $\theta_t = \pi/4$. Note that the standard model contribution to T has already been subtracted off from the total top sector contribution, in order to give the final plotted results. A good fit to the T contribution in the range of θ_t plotted is $\cos^2\theta_t \cot\theta_t$, where the fit gets bad in the $\theta_t \leq 1/2$ region. As we change f, the constant of proportionality roughly scales as $1/f^2$. In figure 3, the dependence of $m_{t'}$ on θ_t is also plotted. An important point is that naturalness puts an upper bound constraint on the mass $m_{t'}$. By the standard given in [9], for a 200 GeV Higgs, 10% fine-tuning restricts $m_{t'} \lesssim 2$ TeV. Fortunately, as the figures show, it appears possible to get corrections to T within the 1-σ bound at f scales consistent with this amount of fine tuning. It is also important to keep in mind that these T contributions are quite mild (this appears to be a generic feature of Little Higgs models). For instance, the standard model top quark contribution is T ≈ 1.2 which is quite larger than the largest value in the plot of 0.35. It is also well known that moderate positive values of T increase the upper bound on the Higgs mass [35]. Since θ_t will be varied during the fit, this will dramatically change the allowed Higgs masses.

Higgs

Finally, the Higgs itself will contribute to T. The contribution is well known and we will use the explicit formula contained in [36]. For the purposes of this paper, we will take the

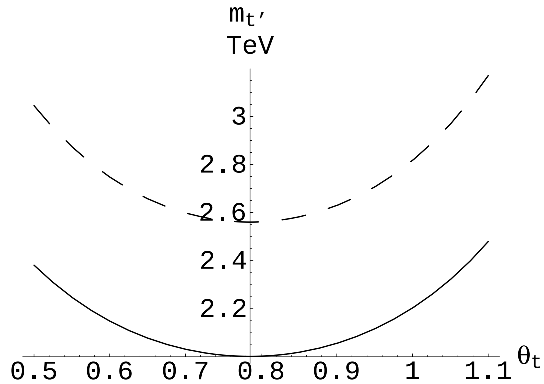

Figure 3: The heavy top mass $m_{t'}$ as a function of θ_t. The solid line is for $f = 700$ GeV while the dashed line is for $f = 900$ GeV.

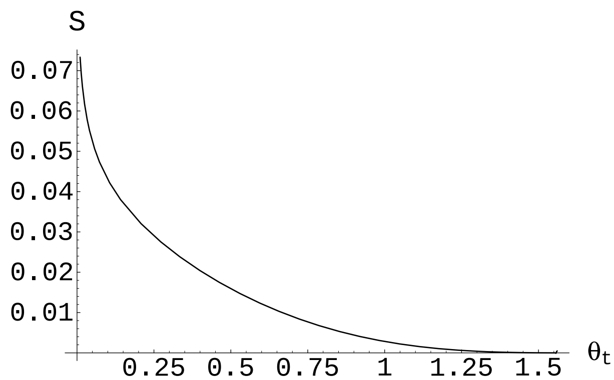

Figure 4: Top Sector Contribution to S for $f = 700$ GeV as a function of θ_t.

S,T origin when $m_{\text{Higgs, ref}} = 115$ GeV. As we increase the Higgs mass, this T contribution gets large and negative.

10.3 S Parameter

The S parameter along with the T parameter gives a good handle on the oblique contributions of any new physics. To the order at which we have been calculating (i.e. v^2/f^2), there are only two sources of S contributions. The first contribution is that of the Higgs. Again, we use the result in [36]. This gives a positive S contribution for Higgs masses larger than our chosen reference mass.

The second contribution to S comes from the extended top sector and again is best

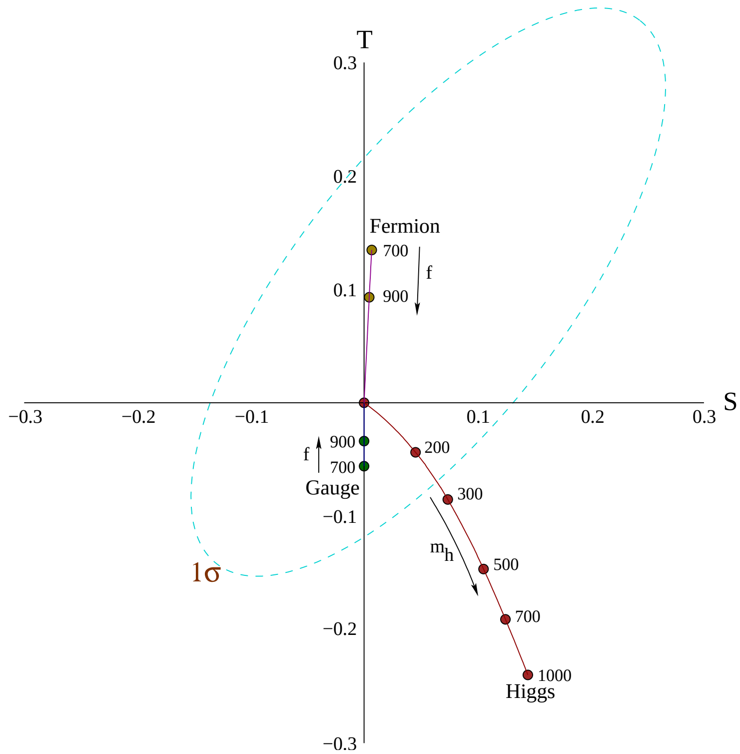

Figure 5: The approximate 1σ ellipse in the S-T plane. The origin (S=0,T=0) corresponds to the reference values $m_h = 115$ GeV and $m_t = 174.3$ GeV. The Higgs contribution for increasing Higgs mass is plotted for the values (115,200,300,500,700,1000) GeV which slopes down and to the right. Two representative points of the extra fermion and gauge contributions have also been plotted for $\theta_t = \pi/4$, $\theta' = 1/5\sqrt{3}$ and $f = 700, 900$ GeV.

calculated via vacuum polarization diagrams using the definition

$$S \equiv -16\pi\Pi'_{3Y}(0). \tag{10.11}$$

In figure 4, we have plotted the beyond the standard model S contribution from the top sector for $f = 700$ GeV. For the region of naturalness, the contributions to S are quite small and do not measurably affect the fit of the model.

10.4 Summary of Limits

Now, the fit to S and T can be performed. In figure 5, the approximate 1σ ellipse in the S-T plane as given in [37] has been plotted. Note that the (S=0,T=0) origin has been set to the reference values $m_h = 115$ GeV and $m_t = 174.3$ GeV. Sloping down and to the right, the exact contribution due to the Higgs has been plotted for the masses $m_h = (115, 200, 300, 500, 700, 1000)$ GeV. To represent the other two contributions, two points of the beyond the standard model fermion and gauge contributions have also been plotted for the values $\theta_t = \pi/4$, $\theta' = 1/5\sqrt{3}$, and $f = 700, 900$ GeV. The fermionic contribution generically points up and slightly to the right whereas the gauge contribution points downward. To find where the model is on the S-T plane, these three contributions should be added.

To be specific, we'll focus on the value $f = 700$ GeV as this limits the amount of fine-tuning in the model. In figure 6, the S and T contributions for the Higgs and fermions are summed for $\theta_t = \pi/4$ and $\theta' = 1/5\sqrt{3}$. From the graph there appears to be a generous range of m_h that falls within the 1σ limits, at least 115 GeV $\leq m_h \lesssim$ 400 GeV. If θ_t is changed, larger Higgs mass can be attained. Although changing θ_t from the equal mixing value $\pi/4$ increases fine-tuning, as m_h increases the Higgs mass parameter also increases which will reduce the fine-tuning, and thus θ_t can be manipulated as the Higgs gets heavier. This freedom helps since reducing θ_t will increase the fermion contribution to T (and only slightly increase S) as required to stay within the ellipse [35]. For instance, at $\theta_t = \pi/6$, we can still tolerate a 1 TeV Higgs mass. Changing θ' produces less of an effect, but as it goes to zero, it can also help improve the fit at large Higgs mass. Of course, at Higgs masses about a TeV, the Little Higgs mechanism is not even required if the cutoff is taken to be 10 TeV. However, within our model, we see that a large region of parameter space is allowed by both the S-T fit and naturalness. In figure 7, θ' and f have been fixed while θ_t and m_h are scanned; all points in the shaded region fit within the 1σ S-T ellipse, while all points above a dashed line are consistent with that percentage of fine-tuning (again using the fine-tuning definition of [9]). There is quite a large range of Higgs masses allowed by precision constraints, and most of it is within ten percent fine-tuning or better.

As a brief comment on more general f values, the positive fermion contribution to T decreases as f increases at a given θ_t, so it is more difficult to get within the ellipse for very large Higgs mass at large f. For instance, going up to $f = 900$ GeV pushes the range for

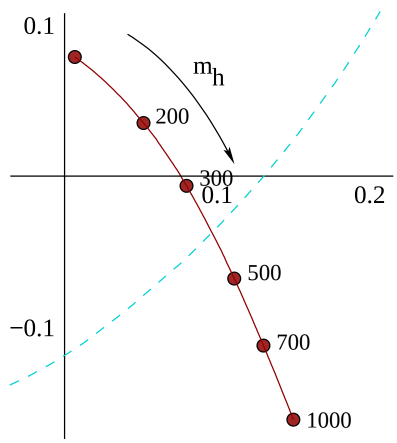

Figure 6: A closeup of the S-T plot with the summed contributions of the Higgs, gauge bosons, and fermions. In this plot, the values $f = 700$ GeV, $\theta_t = \pi/4$, $\theta' = 1/5\sqrt{3}$ are fixed, but Higgs mass is allowed to vary.

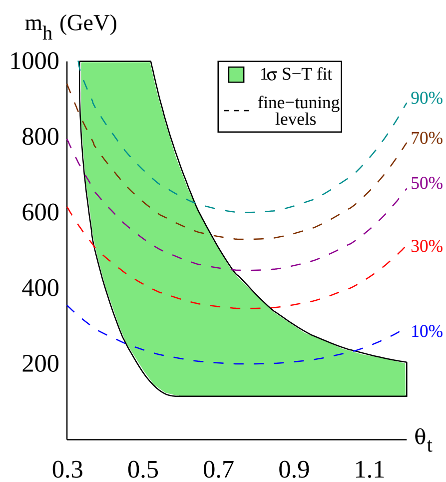

Figure 7: A scan of m_h vs. θ_t with the fixed values $\theta' = 1/5\sqrt{3}$ and f= 700 GeV. The shaded region is allowed by the 1σ fit while points above a dashed line are consistent with that percentage of fine-tuning.

$\theta_t = \pi/4$ and $\theta' = 1/5\sqrt{3}$ down to about 115 GeV $\leq m_h \lesssim$ 350 GeV. However, it is our hope that naturalness will help to keep f low, so that the TeV scale particles can still be discovered at the LHC.

Two more comments on this fit should be made. First of all, the experimental error in the top mass gives an uncertainty in the standard model contribution to S and T. With the current error of ± 5 GeV, this introduces an unknown $\pm .07$ contribution to T (the change in S is small), which can significantly affect the fit. The other thing to note is to remember that $O(v^4/f^4)$ effects and higher have been neglected. For instance, as seen in [31], S contributions from $O(v^4/f^4)$ and dimension 6 operators suppressed by Λ^2 are of the order $\pm .02$. Thus, to go beyond the $O(v^2/f^2)$ analysis as presented here will require some assumptions about the UV completion.

As a conclusion to this section, we plot some sample spectrums for an allowed region of parameter space in figure 8, with $f = 700$ GeV. For the heavy quark sector, we have allowed θ_t to vary. The t' quark is the heavy charge 2/3 quark that cuts off the top yukawa quadratic divergence, and is nearly degenerate with the charge $-1/3$ b' quark (the b' is usually about 5-10 GeV heavier). The T and Ψ quark (charge 2/3 and 5/3 respectively) are exactly degenerate at tree level and are not important in cutting off the top quadratic divergence. In general, the (T, Ψ) pair is lighter than the (t', b') pair. In the heavy gauge boson sector, the W' is generally the heaviest and the B' and $W_r^{\pm}$ are nearly degenerate (with the B' heavier). As θ decreases, all of the states get heavier and more degenerate. Finally, in the scalar sector, the simplifying assumption that naturalness in the gauge contribution sets $\lambda_{\mathbf{1}}^{\pm} = 3\lambda_{\mathbf{3}}^{\pm}$ has been assumed. This naturalness condition sets the scalars to be heavier than the triplets by a factor of $\sqrt{3}$. To simplify it even further, we've also assumed that $\lambda_{\mathbf{3}}^{-} = \lambda_{\mathbf{3}}^{+}$ and picked a Higgs mass of 200 GeV. All these particles have TeV scale masses and should be searched for at the LHC.

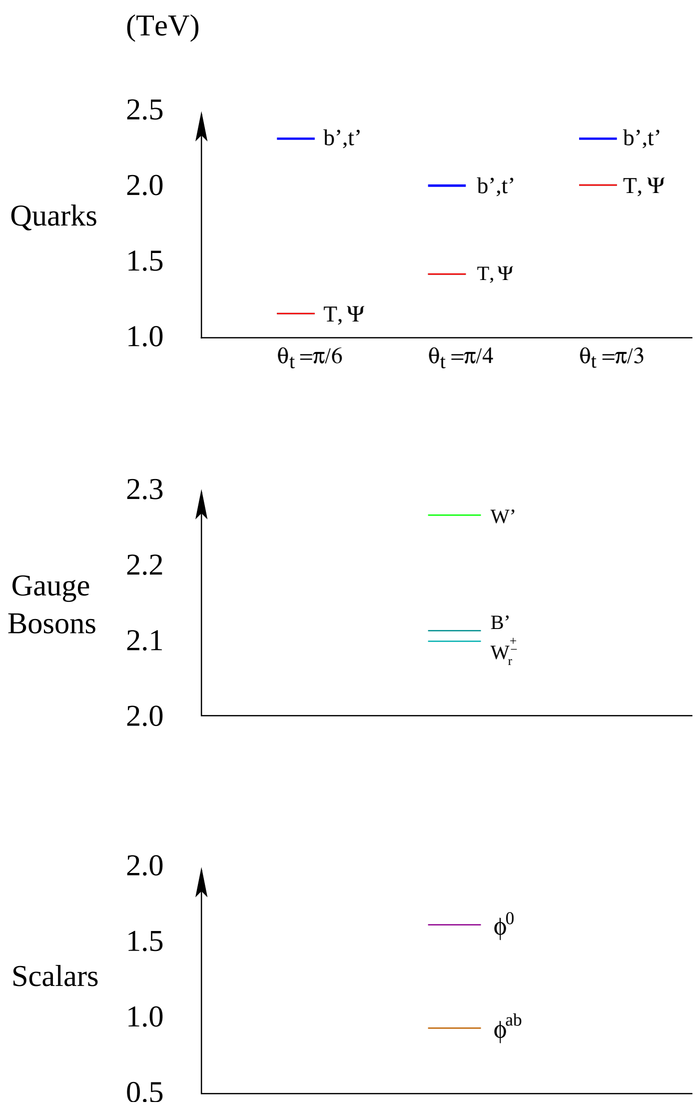

Figure 8: A sample spectrum for the values $f = 700$ GeV, $\theta = 1/5, \theta' = 1/(5\sqrt{3})$. In the scalar sector, $m_h = 200$ GeV and $\lambda_{\mathbf{1}}^{\pm} = 3\lambda_{\mathbf{3}}^{\pm} = 4/3$ has been assumed.

11 Conclusion

In this paper, a new "Littlest Higgs" model with custodial $SU(2)$ symmetry has been analyzed. Precision electroweak analyses of Little Higgs models has given suggestions on what features Little Higgs theories should realize, and with this motivation, the $\frac{SO(9)}{SO(5)\times SO(4)}$ model has been proposed in order to be easily compatible with precision constraints. Some of the unique features of the model include:

- Psuedo-goldstone bosons with custodial $SU(2)$ preserving vevs, comprised of a single light Higgs doublet and at the TeV scale, a singlet and three $SU(2)_W$ triplets, similar to [34].
- At $O(v^2/f^2)$, the 1σ S-T fit allows a generous range of Higgs masses in a large region of parameter space that is consistent with naturalness.

The other features follow that of the original "Littlest Higgs", including the generation of the Higgs potential through gauge and fermion interactions as well as the fermion sector driving electroweak symmetry breaking. At low energies, the effective theory is the standard model, with extra states at the TeV scale to cut off the quadratic divergences to the Higgs. Once again, we emphasize that the precision constraints are mild and a large region of Higgs masses is allowed in parameter space where the Higgs mass is natural.

In analyzing the one loop quadratically divergent term in the Coleman-Weinberg potential, we discovered that the gauge interactions introduced a saddle point instability in the vacuum. Two solutions to stabilize the vacuum were presented, either through extending the top sector or writing down operators that could give same sign contributions to the singlet and/or triplet masses. As an aside, we mention here briefly two "Littlest Higgs" models that also contain custodial $SU(2)$ where the preferred vacuum is stable. Firstly, changing the global symmetry from $SO(9)$ to $SU(9)$ changes the breaking pattern to $SU(9) \to SO(9)$. In this model, the upper $SU(4)$ gauge group can be gauged instead of $SO(4)$. The $SU(4)$ gauge interactions prefer the off-diagonal vacuum and stabilize the $(\phi - X)^2$ terms. This along with the top sector given earlier can stabilize the vacuum. This theory contains 2

Higgs doublets, 3 singlets, and 6 triplets and should preserve custodial $SU(2)$ in the same way as the $SO(9)$ model.

Recently, the idea of UV completing the "Littlest Higgs" via strong interactions giving rise to composite fermions and composite Higgs was introduced [38]. This idea requires the top sector to be comprised of full multiplets of the global symmetry. A model with custodial $SU(2)$ symmetry that conceivably could be UV completed in this manner is one based on the coset $\frac{SU(8)}{Sp(8)}$, where the upper 4 components of the 8 is $4 \equiv (2_L + 2_R)$ and $SU(2)_L \times SU(2)_R \times SU(2) \times U(1)$ is gauged. This time the gauge interactions naively make the vacuum a local maximum, but the sign could depend on the UV completion and is just a discrete choice. The spectrum of this theory turns out to be 4 Higgs doublets and 5 singlets! However, as one can see from these other examples, the model presented in this paper has the simplest spectrum, displays all the important physics, and is a complete and realistic model.

In summary, Little Higgs models are exciting new candidates for electroweak symmetry breaking. They contain naturally light Higgs boson(s) that appear as pseudo-goldstone bosons through the breaking of an approximate global symmetry. With perturbative physics at the TeV scale, these models produce relatively benign precision electroweak corrections. In this paper, one model that realizes custodial $SU(2)$ symmetry has been described, which may give some insight into why the standard model has worked so well for so long. In the near future, experiments at the LHC should start giving indications whether or not these candidate theories play a role in what comes beyond the standard model.

Acknowledgements

We would like to thank Nima Arkani-Hamed and Thomas Gregoire for many helpful discussions and especially Jay Wacker both for discussions and for suggesting the beginning framework of the model. We also thank our anonymous JHEP referee for many insightful comments. This work was supported by an NSF graduate student fellowship.

B Generators and Notation

The $SO(4)$ commutation relations are:

$$[T^{mn}, T^{op}] = \frac{i}{\sqrt{2}}(\delta^{mo}T^{np} - \delta^{mp}T^{no} - \delta^{no}T^{mp} + \delta^{np}T^{mo}) \tag{B.1}$$

where m, n, o, p run from $1, \ldots, 4$. These generators can be broken up into

$$T^{l\,a} = \frac{1}{2\sqrt{2}}\epsilon^{abc}T^{bc} + \frac{1}{\sqrt{2}}T^{a4} \qquad T^{r\,a} = \frac{1}{2\sqrt{2}}\epsilon^{abc}T^{bc} - \frac{1}{\sqrt{2}}T^{a4} \tag{B.2}$$

where a, b, c run from $1, \ldots, 3$. The commutation relations in this basis of $SO(4)$ are equivalent to $SU(2)_L \times SU(2)_R$:

$$[T^{l\,a}, T^{l\,b}] = i\epsilon^{abc}T^{l\,c}, \qquad [T^{r\,a}, \tau^{r\,b}] = i\epsilon^{abc}T^{r\,c}, \qquad [T^{l\,a}, T^{r\,b}] = 0.$$

Vector Representation

The vector representation of $SO(4)$ can be realized as:

$$T^{mn\;op} = \frac{-i}{\sqrt{2}}(\delta^{mo}\delta^{np} - \delta^{no}\delta^{mp}) \tag{B.3}$$

where m, n, o, p again run over $1, \ldots, 4$ and m, n label the $SO(4)$ generator while o, p are the indices of the vector representation. In this representation:

$$\text{Tr } T^A T^B = \delta^{AB}. \tag{B.4}$$

Higgs Representations

For the Higgs doublet, we have three equivalent forms of the representation. First, there is the $SO(4)$ vector representation, denoted as

$$\vec{h} \equiv \begin{pmatrix} h^a \\ h^4 \end{pmatrix} \text{where } a = 1, 2, 3, \tag{B.5}$$

the $SU(2)_W$ doublet representation (with Y $= -\frac{1}{2}$)

$$h \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} h^4 + ih^3 \\ -h^2 + ih^1 \end{pmatrix} \tag{B.6}$$

and the two by two matrix

$$H \equiv (h^4 + \sigma^a\, h^a)/\sqrt{2} = \begin{pmatrix} h & -\epsilon h^* \end{pmatrix} \tag{B.7}$$

where the antisymmetric tensor $\epsilon = i\sigma^2$. Under $SU(2)_L \times SU(2)_R$, this matrix transforms as

$$H \to L\, H\, R^\dagger \tag{B.8}$$

and thus a vev in the h^4 direction breaks $SU(2)_L \times SU(2)_R$ to custodial $SU(2)$.

The Coleman-Weinberg potential depends on the fields $\mathcal{H}^0$ and $\mathcal{H}^{ab}$ as defined by $\frac{1}{2f}\vec{h}\,\vec{h}^{\,T} = \mathcal{H}^0 + 4\mathcal{H}^{ab}\, T^{l\,a} T^{r\,b}$. These are given in terms of the Higgs fields as:

$$\mathcal{H}^0 = \frac{1}{4f}|h|^2 \qquad\qquad \mathcal{H}^{ab} = \frac{1}{8f}\left[\left(h^c h^c - h^4 h^4\right)\delta^{ab} - 2h^a h^b - 2\epsilon^{abc} h^c h^4\right] \tag{B.9}$$

Singlet and Triplet Representations

In this theory, there are TeV scale scalars transforming as a singlet and as triplets under $SU(2)_W$, which appear in the symmetric product of two $SO(4)$ vectors, *i.e.* $(\mathbf{4} \times \mathbf{4})_S = \mathbf{1} + \mathbf{9} = (\mathbf{1}_L, \mathbf{1}_R) + (\mathbf{3}_L, \mathbf{3}_R)$. In the non-linear sigma model field Σ, these appear in the symmetric four by four matrix Φ and can be written as

$$\Phi = \phi^0 + 4\phi^{ab}\, T^{l\,a} T^{r\,b}. \tag{B.10}$$

Note that since the left and right generators commute, this is a symmetric matrix. These fields are canonically normalized and for the triplets, $SU(2)_W$ acts on the a index in the triplet representation and $U(1)_Y$ acts on the b index by $T^{r\,3}$ in the triplet $SU(2)_R$ representation.

Fermion representation

The $SO(4)$ vector representation fits nicely for the scalars, but is a bit cumbersome for the fermion sector. However, taking inspiration from the $\vec{h}, H$ transformation properties, it isn't hard to see the correct correspondence. Let's first start with a set of doublets q_1, q_2 in a two by two matrix

$$Q \equiv (q_1\; q_2) \tag{B.11}$$

which transforms under $SU(2)_L \times SU(2)_R$ as

$$Q \to L\, Q\, R^\dagger. \tag{B.12}$$

Thus the q's are $SU(2)_L$ doublets and the $SU(2)_R$ rotates them into each other. Now, Q can be transformed into an $SO(4)$ vector by tracing

$$\vec{q}^{\,T} = (q^a, q^4) \equiv \frac{1}{\sqrt{2}} \left(\mathrm{Tr}\,(-i\sigma^a Q),\, \mathrm{Tr}\,(Q) \right) \tag{B.13}$$

where $\vec{q}$ transforms under the T^l, T^r generators of the $SO(4)$ representation. The normalization out front is important if this is to be completed into an $SO(5)$ vector by the addition of a singlet fermion Ψ (this is just in order to keep canonical normalization under group action).

Finally, the generalization of this correspondence to a fermion transforming under an $SU(2) \times U(1)$ gauge group is straightforward.

Chapter 4: Unitarity and Little Higgs Models[15]

12 Introduction

The Standard Model (SM) with an elementary Higgs scalar is a remarkably simple theory, but despite the simplicity, it still successfully accommodates all known experimental data (aside from neutrino oscillations). However, the hierarchy problem [1] puts the naturalness and completeness of this theory in doubt. Already at one-loop level, quadratic radiative corrections to the Higgs mass parameter destabilize the weak scale, pulling it up to the intrinsic ultraviolet (UV) cutoff. At best, the SM is an effective field theory behaving naturally only up to an UV cutoff $\Lambda_{\rm SM}$ that could be higher than the weak scale by merely a loop factor, $\Lambda_{\rm SM} \sim 4\pi v \simeq 3\,\text{TeV}$.

This hierarchy problem (or naturalness problem) has motivated most of the major extensions of the SM since the seventies. The two earliest and best known directions are dynamical symmetry breaking [39] and the addition of supersymmetry [40]. More recently, theories with large or small extra dimensions [41] have been used to eliminate the hierarchy problem. These avenues are quite rich and have been explored in depth.

The newest addition to this list of candidates is an attractive idea called the "Little Higgs" [2, 3, 5–7, 9–11, 28, 42]. Little Higgs theories seek to solve a *little hierarchy,* by only requiring the Higgs mass be safe from one-loop quadratic divergences. In this mechanism, the extended global symmetries enable each interaction to treat the Higgs particle as a Goldstone boson. However, once all interactions are turned on, the Higgs becomes a pseudo-Goldstone boson [43]. Thus quadratic divergences in the mass parameter can only appear at two-loops and higher. This allows the theory to be natural with an UV cutoff up to two-loop factors above the weak scale, roughly $\Lambda \sim (4\pi)^2 v \sim 10 - 30\,\text{TeV}$. The required particle content and interactions are usually quite economical; there may be new heavy gauge bosons (W', Z' and B' for instance), new heavy quarks (t' and possible exotics), and new heavy scalars (electroweak singlets, triplets and/or extended Higgs doublet sector).

[15]This chapter is based on reference [4] done in collaboration with Hong-Jian He.

Many Little Higgs models have been constructed, most of which take just the minimal solution towards stabilizing the little hierarchy. This approach requires a very minimal addition of extra particles and interactions. At first glance, both experimentalists and theorists might find this approach depressing, since this just predicts a sparsely filled little desert at the LHC. However, as we will show in this Letter, the situation luckily seems much better. In fact, a new scale in the multi-TeV range is found to demand new physics beyond that required by the minimal Little Higgs mechanism.

To begin, we can take inspiration from our knowledge of the SM. After observing the W and Z gauge bosons, we could wonder whether their mere existence predicts any new physics to be discovered. The lesson here is well known. Since the scattering amplitudes for longitudinal weak bosons grow with energy, perturbative unitarity would be violated at a critical energy $E = \Lambda_{\text{U}}$ in the absence of Higgs boson [44–49]. The classic unitarity analysis determines this energy scale as $\Lambda_{\text{U}} \simeq 1.2\,\text{TeV}$ [46–51]. Note that this is noticeably lower than the cutoff scale for strong dynamics, $\Lambda \sim 4\pi v \simeq 3\,\text{TeV}$, as estimated by naive dimensional analysis (NDA) [52, 53].

The possible resolutions to this unitarity crisis are well known. If a Higgs scalar exists, the Higgs-contributions to the scattering amplitude cut off the growth in energy. Alternatively, if strong dynamics breaks the electroweak symmetry, possible new vector particles (such as techni-ρ's) will save unitarity. Imposing perturbative unitarity, these new states must appear below or around the scale $\Lambda_{\text{U}} \simeq 1.2\,\text{TeV}$ for the high energy theory to make sense. Independent of details in the UV completion, this bound ensures new physics to be seen at LHC energies.

Essentially the same lesson can be relearnt for the Little Higgs models. The low energy dynamics of the Little Higgs theories are described by the leading Lagrangian under the momentum expansion, which is the analog of the two-derivative operator in the usual chiral Lagrangian. Due to the two derivatives, the scattering amplitude of these scalars is expected to grow as E^2, and will eventually violate unitarity at an energy $E = \Lambda_{\text{U}}$. So far, the only difference from the SM case is the symmetry breaking structure. The different effective chiral Lagrangians will predict different interaction strengths and relations which determine the unitarity bound. Most importantly, the bound $\Lambda_{\rm U}$ points to the UV completion scale of the Little Higgs mechanism, and in analogy with the SM, is expected to be at accessible energy scales, lower than the NDA cutoff $\Lambda \sim 4\pi f \sim 10\,\text{TeV}$. Moreover, because the breaking of extended global symmetries of the Little Higgs models results in a large number of additional (pseudo-)Goldstones in the TeV range, we expect the collective effects of the Goldstone boson scatterings in a coupled channel analysis to further push down the unitarity bound $\Lambda_{\rm U}$.

The rest of this Letter is organized as follows. We first perform a generic unitarity analysis for a class of Little Higgs models in Sec. 12, and then carry out an explicit unitarity study for the Littlest Higgs model of $SU(5)/SO(5)$ in Sec. 13. We discuss the potential new physics signals in Sec. 14, which is not intended to be exhaustive, but just gives a flavor of the possible phenomenology at the LHC. This section ends with a discussion of the interpretation and implications for the unitarity violation scale versus the NDA cutoff scale. Finally, we conclude in Sec. 15.

13 Unitarity of Little Higgs Models: A Generic Analysis

As described in the introduction, Little Higgs models predict new physics in the TeV range, such as new gauge bosons and new fermions. However, there can be substantial variation in these extra ingredients and thus their analysis is usually model dependent. On the other hand, the symmetry breaking structure of a given Little Higgs theory is completely determined. For instance, the scalars in the Littlest Higgs model [9] arise from the global symmetry breaking $SU(5) \to SO(5)$. This guarantees the existence of 14 "light" (pseudo-)Goldstone bosons, most of which are expected in the TeV range. At leading order in the momentum expansion, the interactions of these Goldstones are completely fixed by the global symmetry breaking pattern. This allows us to perform a generic analysis of the Goldstone boson scatterings and the corresponding unitarity bounds. Note that the local symmetries (as well as the fermion sector) in the Little Higgs theories can vary, but according to the power counting [54, 55] they do not affect our analysis of the leading

Goldstone scattering amplitudes. So we can apply our generic unitarity formula to each given theory and derive the predictions.

The setup is rather simple. As mentioned above, a Little Higgs model is defined by breaking its global symmetry $\mathcal{G}$ down to a subgroup $\mathcal{H}$. This guarantees the existence of $|\mathcal{G}| - |\mathcal{H}| = \mathcal{N}$ Goldstone bosons, denoted by π^a $(a = 1, \cdots, \mathcal{N})$. At the lowest order of the derivative expansion [54], the Goldstone interactions are fully fixed by the symmetry breaking structure,

$$\mathcal{L}_{\mathrm{KE}} = \frac{f^2}{8}\,\mathrm{Tr}\,|\partial_\mu \Sigma|^2 \,. \tag{13.1}$$

In this expression, we define the nonlinear field $\Sigma \equiv \exp\,[2i\pi^a T^a/f]$, where $\mathrm{Tr}\,(T^a T^b) = \delta^{ab}$ ensures the canonical normalization for the π^a's. The specific form of the broken generators T^a depends on the particular model under consideration. The scale f is the Goldstone decay constant and is usually taken to be order $0.7 - 1\,\mathrm{TeV}$ for naturalness. Note that the factor of $1/8$ is a consequence of the normalization $\mathrm{Tr}\,(T^a T^b) = \delta^{ab}$ and the definition for Σ. Changing the factor $1/8$ will correspond to a simple rescaling of f. We note that in general the ∂_μ's should be raised to covariant derivatives by gauge invariance. However, since we will be concerned only with the leading Goldstone scatterings (instead of the more involved gauge boson scatterings), it is enough to include the partial derivatives. This restriction also does not weaken the analysis because power counting [55] shows that the leading energy growth of the Goldstone scattering amplitudes completely arises from the derivative terms and is independent of the gauge couplings. Finally, we note that the only Little Higgs models which cannot be described by this Lagrangian are the Simple Group Little Higgses [11]. This is due to the fact that in those models, the vacuum expectation value $\langle\Sigma\rangle$ is not unitary and leads to a different structure.

Expanding Eq. (13.1) up to quartic Goldstone interactions, we arrive at

$$\mathcal{L}_{\mathrm{KE}} = \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a + \frac{\Gamma^{abcd}}{3f^2}(\partial^\mu \pi^a)\pi^b(\partial_\mu \pi^c)\pi^d + O(\pi^5) \tag{13.2}$$

where we have defined

$$\Gamma^{abcd} \equiv \text{Tr}\left[T^aT^bT^cT^d - T^aT^cT^bT^d\right]. \tag{13.3}$$

To proceed with a coupled channel analysis, we will consider a canonically normalized singlet state under $\mathcal{H}$, consisting of $\mathcal{N}$ pairs of Goldstone bosons,

$$|S\rangle = \sum_{a=1}^{\mathcal{N}} \frac{1}{\sqrt{2\mathcal{N}}}\,|\pi^a\pi^a\rangle\ , \tag{13.4}$$

where the factor $1/\sqrt{2}$ is conventionally used to account for the identical particle states. The state $|S\rangle$ is a singlet since the π^a's form a real representation of the $\mathcal{H}$ symmetry in non-Simple Group models. Since the π^a's also form an irreducible representation of $\mathcal{H}$, this is the only singlet formed from two π^a's. The scattering amplitude $\mathcal{T}\,[S \to S]$ will contain $\mathcal{N}^2$ number of individual $\pi\pi \to \pi\pi$ channels, and is expected to be the largest amplitude for deriving the optimal unitarity bound. For instance, experience with the QCD $SU(2)$ chiral Lagrangian or the SM Higgs sector shows that the isospin singlet channel of $\pi\pi$ scattering results in the strongest unitarity bound [47,49–51]. We also note that among the π^a's there are would-be Goldstone bosons whose scattering describes the corresponding scattering of the longitudinal gauge bosons [such as $(W_L,\, Z_L)$ and $(W'_L,\, Z'_L,\, B'_L)$] in the high energy range ($s \gg m_W^2, m_{W'}^2$) via the equivalence theorem [45,47,49,56]. So, at high energies our analysis is equivalent to a unitary gauge analysis.

Using the interaction Lagrangian in Eq. (13.2), we can readily determine the singlet scattering amplitude at tree level,

$$\mathcal{T}\,[S \to S] = \frac{\mathcal{C}}{\mathcal{N} f^2}\; s\,, \tag{13.5}$$

where we have defined the group-dependent coefficient

$$\mathcal{C} = \sum_{a,b=1}^{\mathcal{N}} \Gamma^{aabb}\ . \tag{13.6}$$

To derive this result, we have used the relation for Mandelstam variables $s+t+u \approx 0$ after ignoring the small pion masses relative to the large energy scale $\sqrt{s}$. Here we note that because $\Gamma^{aaaa} = 0$, only the $\mathcal{N}(\mathcal{N}-1)$ inelastic channels, $\pi^a\pi^a \to \pi^b\pi^b$ $(a \neq b)$, contribute.

It is now straightforward to compute the 0th partial wave amplitude from Eq. (13.5),

$$a_0\,[S \to S] = \frac{1}{32\pi}\int_{-1}^{1} dz\; P_0(z)\mathcal{T}(s,z) = \frac{\mathcal{C}}{16\pi\mathcal{N}f^2}\,s\,, \tag{13.7}$$

which, as expected, grows quadratically with the energy and is subject to the unitarity constraint,

$$|\mathfrak{Re}\,a_0| \;<\; \frac{1}{2}\,. \tag{13.8}$$

Hence, we find that perturbative unitarity holds for energy scales

$$\sqrt{s} \;<\; \sqrt{\frac{8\pi\mathcal{N}}{|\mathcal{C}|}}\,f \;\equiv\; \Lambda_{\rm U}\,. \tag{13.9}$$

Since $\mathcal{C}$ tends to scale as $\mathcal{N}^{3/2}$ for large $\mathcal{N}$, the unitarity bound should scale as $\mathcal{N}^{-1/4}$ [57]. Hence, we expect the unitarity bound to be quite low since $\mathcal{N}$ is reasonably large in the Little Higgs models.

Using this general formula, we can readily compute the coefficient $\mathcal{C}$ and determine the unitarity bounds on the various Little Higgs theories. We compile our results in Table 1. Note that for moose models, there is a four times replicated non-linear sigma model structure. But, we have chosen to analyze only one of the non-linear sigma model fields. Any interaction between the different non-linear sigma model fields is model-dependent, so this restriction is consistent with our approach.

Table 1 shows that indeed the Little Higgs models generically contain a large number of Goldstone bosons, $\mathcal{N} = O(10-20)$, and our unitarity bound $\Lambda_{\rm U}$ is significantly lower than the conventional cutoff of the theory, $\Lambda \sim 4\pi f \simeq 12.6f$, as estimated by NDA. The observation that the unitarity violation scale turns out much lower than Λ is an encouraging sign, indicating that aspects of the Little Higgs UV completions may be possibly explored at the LHC. We will discuss more about the interpretations of our results and highlight the

Table 1: Summary of unitarity bounds in various Little Higgs theories.

Little Higgs Model	G	H	$\mathcal{N}$	$\lvert\mathcal{C}\rvert$	$\Lambda_{\rm U}/f$	$m_{W'}/f$	$m_{t'}/f$
Minimal Moose [5]	$SU(3)^2$	$SU(3)$	8	24	2.89	2.37	1
Littlest Higgs [9]	$SU(5)$	$SO(5)$	14	35	3.17	1.67	2
Antisymmetric Condensate [10]	$SU(6)$	$Sp(6)$	14	26	3.68	1.67	2
$SO(5)$ Moose [2]	$SO(5)^2$	$SO(5)$	10	15	4.09	3.35	$\sqrt{2}$
$SO(9)$ Littlest Higgs [3]	$SO(9)$	$SO(5)\otimes SO(4)$	20	35	3.79	2.37	2

possible collider signatures in Sec. 14.

To add a reference frame for the unitarity bounds in Table 1, we also give the masses of the W' gauge boson and the t' quark (using our current normalization of f). For the gauge boson, the mixing angle between the two $SU(2)$ gauge couplings has been set to $\theta = 1/5$. To scale to a different angle $\theta_{\rm new}$, just multiply by $\sin(2/5)/\sin 2\theta_{\rm new}$. A relatively small mixing angle is required since electroweak precision analysis restricts $m_{W'} \gtrsim 1.8$ TeV [2, 29, 31]. For the t' quark, we have minimized its mass, corresponding to maximizing the naturalness; in the particular case of two Higgs doublet models we have set $\sin\beta = 1$ (for other β values, just divide by $\sin\beta$).

A striking feature of Table 1 is that $2m_{W'} > \Lambda_{\rm U}$ holds for almost all Little Higgs models except the Antisymmetric Condensate model [10] where $\Lambda_{\rm U}$ is only slightly higher than the corresponding value of $2m_{W'}$. Such a low $\Lambda_{\rm U}$ means that for the center of mass energy $\sqrt{s} < \Lambda_{\rm U}$, the $W'W'$ scattering processes will not be kinematically allowed. From the physical viewpoint, this strongly suggests that additional new particles (having similar mass range) have to co-exist with W''s in the same effective theory so that their presence can properly restore the unitarity. But these new states should enter the Little Higgs theory in such a way as to ensure the cancellation of one-loop quadratic divergences [58]. From the technical viewpoint, this obviously implies the equivalence theorem no longer holds for predicting the $W'_L W'_L$ scattering amplitude by that of the corresponding Goldstone scattering. But the exact $W'_L W'_L$ scattering amplitude could only differ from the Goldstone amplitude by $m^2_{W'}/s = O(1)$ terms at most, and thus are not expected to significantly affect our conclusion.

14 Unitarity of the Littlest Higgs Model: An Explicit Analysis

In this section we will explicitly analyze the Littlest Higgs model of $SU(5)/SO(5)$ [9] by writing all Goldstone fields in the familiar electroweak eigenbasis of the SM gauge group. Then we will extract the leading Goldstone scattering amplitudes and derive the unitarity bounds, in comparison with our generic analysis of Sec. 12.

As mentioned earlier, the Littlest Higgs model has the global symmetry breaking structure $SU(5) \to SO(5)$, resulting in 14 Goldstone bosons which decompose under the SM gauge group $SU(2)_W \otimes U(1)_Y$ as

$$\mathbf{1}_0 \oplus \mathbf{3}_0 \oplus \mathbf{2}_{\pm 1/2} \oplus \mathbf{3}_{\pm 1}\,. \tag{14.1}$$

Here the $\mathbf{1}_0 \oplus \mathbf{3}_0$ denotes a real singlet χ^0_y and a real triplet $\chi^{\pm,0}$. They will become the longitudinal components of gauge bosons (B', W', Z') when the gauged subgroups $[SU(2)\otimes U(1)]^2$ are Higgsed down to the diagonal subgroup $G_{\rm SM}$. The $\mathbf{2}_{\pm 1/2}$ includes a Higgs doublet H and $\mathbf{3}_{\pm 1}$ a complex Higgs triplet Φ, defined as

$$H^T = \begin{pmatrix} \pi^+ \\ \dfrac{v + h^0 + i\pi^0}{\sqrt{2}} \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \phi^{++} & \dfrac{\phi^+}{\sqrt{2}} \\ \dfrac{\phi^+}{\sqrt{2}} & \phi^0 - iv' \end{pmatrix}, \tag{14.2}$$

where the would-be Goldstones $\pi^{\pm,0}$ will be absorbed by the light gauge bosons ($W^{\pm}$, Z^0) when electroweak symmetry breaking is triggered by the Yukawa and gauge interactions via the Coleman-Weinberg mechanism [22]. There will be some small mixings between the scalars in H and Φ due to the nonzero triplet VEV v', but the condition $M_\Phi > 0$ requires [15]

$$v' < \frac{v^2}{4f} \ll v\,, \tag{14.3}$$

so that for the current purpose it is enough to expand the tiny ratio v'/v and keep only its zeroth order at which the two sets of Goldstone bosons do not mix. This greatly simplifies our explicit analysis.

Collecting all the 14 Goldstone bosons we can write the nonlinear field $\Sigma = \exp\left[i2\Pi/f\right]\Sigma_0$ for the $SU(5)/SO(5)$ model where the 5×5 Goldstone matrix is given by

$$\Pi = \begin{pmatrix} \frac{1}{2}X & \frac{1}{\sqrt{2}}H^\dagger & \Phi^\dagger \\ \frac{1}{\sqrt{2}}H & \frac{2}{\sqrt{5}}\chi_y^0 & \frac{1}{\sqrt{2}}H^* \\ \Phi & \frac{1}{\sqrt{2}}H^T & \frac{1}{2}X^* \end{pmatrix}, \tag{14.4}$$

and

$$X = \begin{pmatrix} \chi^0 - \frac{\chi_y^0}{\sqrt{5}} & \sqrt{2}\chi^+ \\ \sqrt{2}\chi^- & -\chi^0 - \frac{\chi_y^0}{\sqrt{5}} \end{pmatrix}, \quad \Sigma_0 = \begin{pmatrix} & & \mathbf{1}_{2\times 2} \\ & 1 & \\ \mathbf{1}_{2\times 2} & & \end{pmatrix}. \tag{14.5}$$

Similar to Eq. (13.1), we derive the leading order Goldstone boson Lagrangian

$$\begin{aligned} \mathcal{L}_{\rm KE} &= \frac{f^2}{8}\,\mathrm{Tr}\,|\partial_\mu\Sigma|^2 \\ &= \frac{1}{2}\mathrm{Tr}\,(\partial^\mu\Pi)^2 + \frac{1}{3f^2}\mathrm{Tr}\left[(\Pi\partial^\mu\Pi)^2 - (\partial^\mu\Pi)^2\Pi^2\right] + O(\Pi^5), \end{aligned} \tag{14.6}$$

where the the first dimension-4 operator gives the canonically normalized kinetic terms for all Goldstone fields in Π, and the second term gives the quartic Goldstone interactions.

To derive the optimal unitarity limit from the Goldstone scatterings, we will consider a canonically normalized $SO(5)$ singlet state consisting of 14 pairs of Goldstone bosons,

$$\begin{aligned}
|S\rangle \;=\; \frac{1}{\sqrt{28}}\Big[\, & 2\left|\pi^+\pi^-\right\rangle + \left|\pi^0\pi^0\right\rangle + \left|h^0h^0\right\rangle + 2\left|\chi^+\chi^-\right\rangle \\
& + \left|\chi^0\chi^0\right\rangle + \left|\chi^0_y\chi^0_y\right\rangle + 2\left|\phi^{++}\phi^{--}\right\rangle \\
& + 2\left|\phi^+\phi^-\right\rangle + \left|\phi^0_1\phi^0_1\right\rangle + \left|\phi^0_2\phi^0_2\right\rangle \Big]\,,
\end{aligned} \tag{14.7}$$

where we have defined $\phi^0 \equiv \phi^0_1 + i\phi^0_2$. This is essentially a re-expression of our general formula (13.4) with all $\mathcal{N} = 14$ Goldstone fields in the electroweak eigenbasis. But the expanded form of the quartic interactions in (14.6) is extremely lengthy in the electroweak eigenbasis, making the explicit calculation of the whole amplitude $\mathcal{T}[S \to S]$ tedious. Before giving a full calculation of $\mathcal{T}[S \to S]$, we will *explicitly* expand Eq. (14.6) and illustrate the unitarity limits for the two sub-systems (χ^a, χ^0_y) and ($\pi^{\pm,0}$, h^0). From Eq. (14.6), we derive the corresponding interaction Lagrangians

$$\begin{aligned}
\mathcal{L}^{\pi h}_{\rm int} =& \frac{1}{12f^2}\Big\{ \big[-(2vh+h^2)(\partial_\mu\pi^a\partial^\mu\pi^a) - \\
& -(\partial_\mu h)^2\pi^{a2} + 2(v+h)(\partial_\mu h)(\pi^a\partial^\mu\pi^a)\big] \\
& + \Big[(\partial_\mu\pi^+)^2\pi^{-2} - [(\partial_\mu\pi^0)^2 + \partial_\mu\pi^+\partial^\mu\pi^-]\pi^+\pi^- \\
& + 2(\pi^0\partial_\mu\pi^0)(\pi^+\partial^\mu\pi^-) - \pi^{02}(\partial_\mu\pi^+\partial^\mu\pi^-) + \text{H.c.}\Big]\Big\}\,, \\
\mathcal{L}^{\chi}_{\rm int} =& \frac{1}{6f^2}\Big\{(\partial_\mu\chi^+)^2\chi^{-2} - [(\partial_\mu\chi^0)^2 + \partial_\mu\chi^+\partial^\mu\chi^-]\chi^+\chi^- \\
& + 2(\chi^0\partial_\mu\chi^0)(\chi^+\partial^\mu\chi^-) - \chi^{02}(\partial_\mu\chi^+\partial^\mu\chi^-) + \text{H.c.}\Big\}\,,
\end{aligned} \tag{14.8}$$

where the $U(1)$ Goldstone χ^0_y does not enter $\mathcal{L}^{\chi}_{\rm int}$ at this order. The Goldstones ($\pi^{\pm,0}$, h^0) form the SM Higgs doublet H which also has a renormalizable Coleman-Weinberg potential. But unlike $\mathcal{L}^{\pi h}_{\rm int}$, this potential only contributes constant terms to the Goldstone amplitudes and thus do not threaten the unitarity, especially when the pseudo-Goldstone Higgs h^0 is relatively light as favored by the electroweak precision data.

The Lagrangian $\mathcal{L}^{\pi h}_{\rm int}$ describes the leading derivative interactions of the Higgs doublet

H, characterized by the Goldstone decay constant f and originated from the global symmetry breaking $SU(5) \to SO(5)$. In analogy with the SM case [47], we find that $(\pi^{\pm,0},\, h^0)$ form an electroweak singlet state $|S_H\rangle = \frac{1}{\sqrt{8}}\left[2\left|\pi^+\pi^-\right\rangle + \left|\pi^0\pi^0\right\rangle + \left|h^0h^0\right\rangle\right]$. The corresponding s-wave amplitude is $a_0[S_H \to S_H] = (3s/64\pi f^2)$, where we have dropped small terms suppressed by the extra factor $(v/f)^2 \ll 1$. Imposing the condition (13.8), we deduce the unitarity limit

$$\sqrt{s} \;<\; \Lambda_{\rm U} = \sqrt{\frac{32\pi}{3}}\, f \;\simeq\; 5.79f\,, \tag{14.9}$$

which is lower than the NDA cutoff $\Lambda \sim 4\pi f$ by a factor of 2.2. Note that contrary to the scatterings of Goldstone π^a's (or W_L/Z_L's) in the SM, the $\pi\pi$ scatterings in the Littlest Higgs model grow with energy due to the derivative interactions in $\mathcal{L}_{\rm int}^{\pi h}$. Next, we turn to the $(\chi^{\pm},\, \chi^0)$ system. The Lagrangian $\mathcal{L}_{\rm int}^{\chi}$ for the Goldstone triplet is the same as the familiar $SU(2)$ chiral Lagrangian. So we define the normalized isospin singlet state $|S_{\chi^a}\rangle = \frac{1}{\sqrt{6}}\left[2\left|\chi^+\chi^-\right\rangle + \left|\chi^0\chi^0\right\rangle\right]$, and derive its s-partial wave amplitude $a_0\left[S_{\chi^a} \to S_{\chi^a}\right] = s/(16\pi f^2)$. Using the condition (13.8), we arrive at

$$\sqrt{s} \;<\; \Lambda_{\rm U} = \sqrt{8\pi}\, f \;\simeq\; 5.01f\,, \tag{14.10}$$

which is lower than $\Lambda \sim 4\pi f$ by a factor of 2.5.

After the above explicit illustrations, we will proceed with a full analysis of this model in the electroweak eigenbasis. The key observation is that the $SO(5)$ singlet state $|S\rangle$ in Eq. (14.7) can be decomposed into 4 smaller orthonormal states formed from two π^a's,

$$|S\rangle = \sqrt{\frac{2}{7}}\,|S_H\rangle + \sqrt{\frac{3}{14}}\,|S_{\chi^a}\rangle + \frac{1}{\sqrt{14}}|S_{\chi^0_y}\rangle + \sqrt{\frac{3}{7}}\,|S_\Phi\rangle\,, \tag{14.11}$$

each of which is an *electroweak singlet* state, defined as

$$\begin{aligned}
|S_H\rangle &\equiv \frac{1}{\sqrt{8}}\sum_{a=1}^{4}|\pi^a\pi^a\rangle \\
&= \frac{1}{\sqrt{8}}\left[2\left|\pi^+\pi^-\right\rangle + \left|\pi^0\pi^0\right\rangle + \left|h^0h^0\right\rangle\right], \\
|S_{\chi^a}\rangle &\equiv \frac{1}{\sqrt{6}}\sum_{a=5}^{7}|\pi^a\pi^a\rangle = \frac{1}{\sqrt{6}}\left[2\left|\chi^+\chi^-\right\rangle + \left|\chi^0\chi^0\right\rangle\right], \\
|S_{\chi_y^0}\rangle &\equiv \frac{1}{\sqrt{2}}\left|\pi^8\pi^8\right\rangle = \frac{1}{\sqrt{2}}\left|\chi_y^0\chi_y^0\right\rangle, \qquad (14.12)\\
|S_{\Phi}\rangle &\equiv \frac{1}{\sqrt{12}}\sum_{a=9}^{14}|\pi^a\pi^a\rangle \\
&= ;5\ \frac{1}{\sqrt{12}}\left[2\left|\phi^{++}\phi^{--}\right\rangle + 2\left|\phi^+\phi^-\right\rangle + \left|\phi_1^0\phi_1^0\right\rangle + \left|\phi_2^0\phi_2^0\right\rangle\right].
\end{aligned}$$

Now we will perform a full coupled-channel analysis for the Goldstone scatterings among these 4 electroweak singlet states and prove that the maximal eigenchannel just corresponds to the amplitude $\mathcal{T}[S \to S]$ in Sec. 12 with $|S\rangle$ given by Eq. (14.11) [equivalently, Eq. (14.7) or (13.4)]. There are 16 such individual scattering channels in total. Denoting each singlet state in Eq. (14.12) as $|S_j\rangle \equiv \frac{1}{\sqrt{2\mathcal{N}_j}}\sum_{a=a_j^{\min}}^{a_j^{\min}-1+\mathcal{N}_j}|\pi^a\pi^a\rangle$ with $j = H, \chi^a, \chi_y^0, \Phi$, we can now readily derive any amplitude $\mathcal{T}[S_j \to S_{j'}]$ by using the general formulas (13.5)-(13.6),

$$\mathcal{T}[S_j \to S_{j'}] \quad = \quad \frac{\mathcal{C}_{jj'}}{\sqrt{\mathcal{N}_j\mathcal{N}_{j'}}f^2}\, s\,, \qquad (14.13)$$

where $\mathcal{C}_{jj'} = \sum_{a=a_j^{\min}}^{a_j^{\min}-1+\mathcal{N}_j}\sum_{c=c_{j'}^{\min}}^{c_{j'}^{\min}-1+\mathcal{N}_{j'}}\mathcal{C}^{aacc}$ will be explicitly evaluated for $SU(5)/SO(5)$.

So, with all the singlet states $|S_j\rangle$, we deduce a 4×4 matrix of the leading s-wave amplitudes

$$\mathcal{A}_0 \;=\; \frac{s}{16\pi f^2}\begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{5}{\sqrt{2}} & \sqrt{\frac{3}{8}} \\ \frac{\sqrt{3}}{4} & 1 & 0 & \frac{1}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} & 0 & 0 & 0 \\ \sqrt{\frac{3}{8}} & \frac{1}{\sqrt{2}} & 0 & \frac{3}{2} \end{pmatrix} . \tag{14.14}$$

It has the eigenvalues $a_{0j} = \dfrac{s}{16\pi f^2}\left(-1,\, \dfrac{1}{2},\, \dfrac{5}{4},\, \dfrac{5}{2}\right)$, where the maximum channel $a_0^{\max} = 5s/(32\pi f^2)$ corresponds to a normalized eigenvector $(\sqrt{2/7},\ \sqrt{3/14},\ \sqrt{1/14},\ \sqrt{3/7})$, which in this basis is precisely the singlet state in Eq. (14.11)! Imposing the condition (13.8), we derive the best unitarity limit for the Littlest Higgs model,

$$\sqrt{s} \quad < \quad \Lambda_{\rm U} = \sqrt{\frac{16\pi}{5}}\, f \;\simeq\; 3.17 f\,, \tag{14.15}$$

in perfect agreement with the optimal bound in Table I.

With the information in Eq. (14.14), we can also analyze the optimal unitarity limits for all *sub-systems* via partial coupled-channel analysis, as summarized below.

Subsystem	$\Lambda_{\rm U}$	Subsystem	$\Lambda_{\rm U}$
$\{H\}$:	$5.79f$	$\{H,\,\chi^a\}$:	$4.35f$
$\{\chi^a\}$:	$5.01f$	$\{H,\,\Phi\}$:	$3.69f$
$\{\Phi\}$:	$4.09f$	$\{\chi^a,\,\Phi\}$:	$3.45f$
$\{H,\chi^a,\chi^0_y\}$:	$3.71f$	$\{H,\chi^0_y,\Phi\}$:	$3.45f$
$\{\chi^a,\chi^0_y,\Phi\}$:	$3.45f$	$\{H,\chi^a,\Phi\}$:	$3.27f$

(14.16)

It clearly shows that as more states are included into the coupled channel analysis, the unitarity limit $\Lambda_{\rm U}$ becomes increasingly stronger and approaches the best bound (14.15) in the full coupled-channel analysis. It also demonstrates the limit $\Lambda_{\rm U}$ to be fairly robust since omitting a few channels does not significantly alter the result. Finally, for the subsystems $\{H\} = \{\pi^{\pm,0},\, h^0\}$ and $\{\chi^a\}$, we see that Eq. (14.16) nontrivially agrees with Eqs. (14.9)-(14.10) derived from explicitly expanding (14.6).

In summary, taking the Littlest Higgs model as an example, we have explicitly analyzed the unitarity limits from the Goldstone scatterings via both partial and full coupled-channel analyses, with the Goldstone fields defined in the familiar electroweak eigenbasis. These limits are summarized in Eqs. (14.16) and (14.15). We find that the best constraint (14.15) indeed comes from the *full coupled-channel analysis* including all 14 Goldstone fields in the $SO(5)$ singlet channel (Eq. (13.4) or (14.11)), in complete agreement with Table I (Sec. 12). We have also systematically analyzed the smaller subsystems where some channels are absent. Most of the resulting unitarity limits in Eq. (14.16) are fairly close to the best limit, so Eq. (14.15) is relatively robust.

15 Implications for New Physics Signals

As shown in Sec. 12-13, the unitarity constraints already indicate that Little Higgs theories have an important intermediate scale $\Lambda_{\rm U}$, which is in the multi-TeV region and below the conventional NDA cutoff $\Lambda \sim 4\pi f$. Somewhere below $\Lambda_{\rm U}$, new particles should appear in order to unitarize the Goldstone scattering of π^a's. In particular, the longitudinal W_LW_L/Z_LZ_L scattering (or the corresponding Goldstone scattering $\pi\pi \to \pi\pi, hh$) will be measured by experiments. This process should start to exhibit resonance behavior at least by the scale $\Lambda_{\rm U}$, although what actually unitarizes the amplitude depends upon the UV completion. For the case of the Minimal Moose [5], we can rely on our intuition from the QCD-type dynamics. If it is dynamical symmetry breaking that generates the $SU(3)^2 \to SU(3)$ breaking, the new states should be the analogous vector meson multiplet, i.e., TeV scale (ρ, K^*, ω, ϕ) particles. On the other hand, we could envision a linear sigma model completion (with/without supersymmetry). As an example, there could be a scalar

Σ that transforms as a $(3, \bar{3})$ and gets a VEV proportional to the 3×3 unit matrix. In this case, we can expect new singlets and heavy octet scalars to appear in addition to the octet of Little Higgs bosons. If the Little Higgs theory respects T-parity (cf. second reference in [2]), these new states would have to be even under this parity. This means they can be singly produced and also have restricted decay channels, allowing only an even number of T-odd particles in the final state. So, selecting a specific UV completion can predict a very interesting phenomenology. This direction will be pursued further [58]. In order to investigate the phenomenology of these new states, realistic UV completions should be searched for. For instance, Ref. [38] provides an interesting dynamical UV completion, but more constructions should also be actively sought.

One might also wonder if small mixing angles or coupling constants would render these new states hard to observe experimentally. We clarify this by noting that the approximate global symmetry $\mathcal{H}$ relates the scattering of the $\mathcal{H}$ singlet to the scattering of light longitudinal W/Z bosons in the following manner. Neglecting $\mathcal{H}$ breaking effects, the general amplitude of $\pi\pi$ scattering is given by

$$\mathcal{T}(\pi^a\pi^b \to \pi^c\pi^d) \;=\; \sum_j c_j^{abcd} A_j(s,t,u)\,, \tag{15.1}$$

where j is a finite integer, c_j^{abcd} is a constant tensor invariant under $\mathcal{H}$, and $A_j(s,t,u)$ is a kinematic function depending on the Mandelstam variables. The $\mathcal{H}$ singlet amplitude is a specific linear combination of the kinematic functions. At the lowest order, we have seen that these functions grow with s and this specific combination needs to be altered at least by Λ_{U}. However, longitudinal W/Z scattering is just another linear combination of these kinematic functions. Thus, at the scale Λ_{U}, unitarizing only the $\mathcal{H}$ singlet scattering but keeping the SM-type scattering channels unaffected will require an accidental cancellation in the group theory space. So, generically any new resonance should be shared among all allowed individual scattering channels even though an amplitude for the SM-type channel alone violates unitarity at a relatively higher scale [57]. At worst, a possibly suppressed coefficient should only arise from the projection into the SM-type channel, rather than a small mixing or coupling (up to $\mathcal{H}$ breaking effects).

The scale $\Lambda_{\rm U}$ certainly opens up encouraging possibilities at the LHC, not only to test the minimal Little Higgs mechanism, but also to start probing possible new signs of its UV completion dynamics. We note that the unitarity bound $\Lambda_{\rm U} \sim (3-4)f$ puts an *upper limit* on the scale of new states which are going to restore the unitarity of the Little Higgs effective theory up to the UV scale $\sim 10\,\text{TeV}$ or above. So the masses of these new states can be naturally at anywhere between $\sim f$ and $\Lambda_{\rm U}$, but their precise values must depend on the detailed dynamics of a given UV completion. For instance, QCD-like UV dynamics would predict the lowest new resonance to be a ρ-like vector boson which is expected to be relatively heavy and close to our upper limit $\Lambda_{\rm U}$. But when the UV dynamics invokes supersymmetry, the lowest new state that unitarizes the $W_L W_L$ scattering would be scalar-like and can be substantially below $\Lambda_{\rm U}$, say $\sim 0.5f$ according to the lesson of supersymmetric SM. (Note that the classic unitarity bound for the Higgsless SM only requires $\sqrt{s} < \Lambda_{\rm U} = \sqrt{8\pi}v \simeq 5.0v \simeq 1.2\,\text{TeV}$ [46–51], but the minimal supersymmetric SM unitarizes the $W_L W_L$ scattering by adding 2-Higgs-doublets with the lightest Higgs boson mass $M_h \lesssim 130\,\text{GeV} \simeq 0.5v$ [40], which is typically a factor ~ 10 below $\Lambda_{\rm U}$.) So, it is legitimate to expect the lightest new state in the UV completion of Little Higgs models to lie anywhere in the range $0.5f \lesssim M_{\rm new}^{\rm min} \leqslant \Lambda_{\rm U}$, though its precise mass value is highly model-dependent. The natural size for the scale f is $\sim 1\,\text{TeV}$ [5]– [3]. The updated precision analyses [3, 29, 31, 59] showed that the Little Higgs models are readily consistent with the current data which constrain $f \gtrsim 0.5 - 1\,\text{TeV}$ at 95%C.L. (depending on details of the parameter space in each given model) [16], so f is allowed to be around its natural size $\sim 1\,\text{TeV}$. Taking $f \sim 1\,\text{TeV}$ for instance, we expect the lightest new state to be around $0.5\,\text{TeV} \lesssim M_{\rm new}^{\rm min} \lesssim 3-4\,\text{TeV}$. So, if lucky, the LHC may produce the lightest new resonance, or if it is too heavy, detect the effect of its resonance-tail (via higher order model-dependent contributions in the low energy derivative expansion) [60]. But a quantitative conclusion has to be highly model-dependent. To be conservative, we warn that the limited LHC center-of-mass energy does not guarantee the discovery for such state, especially when $M_{\rm new}^{\rm min}$ is

[16] E.g., it was shown [59] that the early precision bound in the Littlest Higgs model is essentially relaxed by just gauging the subgroup $SU(2) \times SU(2) \times U(1)$.

close to the upper limit $\Lambda_{\rm U}$. Further precision probe may be done at future e^+e^- Linear Colliders, and the proposed CERN CLIC with $E_{\rm cm} = 3-5\,{\rm TeV}$ and $\mathcal{L} = 10^{35}{\rm cm}^{-2}{\rm s}^{-1}$ [62] is particularly valuable. The definitive probe of the Little Higgs UV dynamics is expected at the future VLHC ($E_{\rm cm} = 50-200\,{\rm TeV}$ and $\mathcal{L} \gtrsim 10^{35}{\rm cm}^{-2}{\rm s}^{-1}$) [61]. Incorporating the new signatures of UV completion into relevant collider analyses will expand upon the existing phenomenological studies [14–17, 59, 63–67].

Next, we discuss the meanings of the two estimated UV scales, $\Lambda_{\rm U}$ and Λ, and their implications for an effective field theory analysis in the Little Higgs models. We note that these UV scales are determined by two different measures of perturbativity breakdown. Our lowest unitarity limit $\Lambda_{\rm U}$ is derived from the Goldstone scatterings in the singlet channel via the s-partial wave. (Weaker bounds may be obtained for the non-singlet channels via the higher order partial waves.) On the other hand, the NDA estimate of the UV cutoff is based on the consistency of the chiral perturbation expansion, i.e., one estimates the coefficient of an operator (counter term) of dimension-D from its renormalization-group running induced by one-loop contributions of an operator of dimension-$(D-2)$ and so on [52, 54], because the former's size should be at least of the same order as the latter's one-loop contribution (about $O(1)/16\pi^2$ multiplied by an $O(1)$ logarithm) barring an accidental cancellation. So one obtains the *original* NDA result [52],

$$\frac{f^2}{\Lambda^2} \gtrsim \frac{O(1)}{16\pi^2}, \qquad \Rightarrow \qquad \Lambda \lesssim 4\pi f\,, \tag{15.2}$$

which is a conservative *upper bound* on the UV cutoff. The true cutoff for the effective theory should be $\min(\Lambda_{\rm U},\,\Lambda)$. From low energy QCD, the chiral perturbation theory breaks down as the energy reaches the ρ-resonance at $M_\rho = 0.77\,{\rm GeV}$ which is below but still close to the upper limit $4\pi f \simeq 1.2\,{\rm GeV}$. So we know this original NDA upper bound $4\pi f$ describes the UV scale of the low energy QCD quite well [17]. But, the dynamics of Little Higgs UV

[17]The best unitarity limit of the low energy QCD $\pi\pi$ scattering comes from the $I=0$ isospin-singlet channel, $\Lambda_{\rm U} \simeq \sqrt{8\pi}f \simeq 0.47\,{\rm GeV}$. This lies significantly below the upper limit $4\pi f$ by a factor of 2.5. It is interesting to note that in the physical spectrum, besides the ρ meson, there are good evidences for a relatively light and broad σ meson in the $\sim 0.5\,{\rm GeV}$ range [68] which unitarizes the $I=0$ channel and agrees well with the unitarity limit $\Lambda_{\rm U} \simeq 0.47\,{\rm GeV}$. The fact that QCD chiral Lagrangian works quite well is largely because σ is a very broad $I=0$ resonance and hard to detect [68].

completions can of course be very different from QCD dynamics (or even supersymmetric). In fact, for an underlying gauge interaction with large color N_c and flavor N_f, a Generalized Dimensional Analysis (GDA) [53, 57] gives

$$\Lambda \lesssim \min\left(\frac{a}{\sqrt{N_c}}, \frac{b}{\sqrt{N_f}}\right) 4\pi f \,, \tag{15.3}$$

where a and b are constants of order 1. So we see that as long as N_c or N_f is much larger than that of QCD, the GDA cutoff will indeed be lower than the original NDA estimate. Furthermore, the observation that the unitarity of Goldstone scatterings indicates a lower UV cutoff for the chiral perturbation was made in [57], where it was shown that for a symmetry breaking pattern $SU(N)_L \otimes SU(N)_R \to SU(N)_V$ $(N \geqslant 2)$, the $\pi\pi$ scattering in the $SU(N)_V$-singlet and spin-0 channel would impose a unitarity violation scale

$$\Lambda \lesssim \frac{4\pi f}{\sqrt{N}} \,, \tag{15.4}$$

signaling a significantly lower UV scale for new resonance formation in comparison with the original NDA estimate. This is consistent with our current unitarity analysis for the Little Higgs models.

Finally, in an effective field theory analysis of the Little Higgs models, which UV cutoff is more relevant for suppressing the higher-dimensional operators? The precise answer has to be very model-dependent, relying on what type of heavy state(s) is integrated out when generating a given effective operator. Without knowing the true UV dynamics, the original NDA estimate $\Lambda \sim 4\pi f$ could be considered as a conservative analysis where the UV scale is the highest possible. So far all the electroweak precision analyses [18, 19, 29, 31, 59] adopted the NDA estimate of Λ. But we should keep in mind that the actual UV cutoff Λ could be significantly lower, as suggested by $\Lambda_{\rm U}$, although Λ has to be fixed by the underlying dynamics [cf. GDA estimate in Eq. (15.3)]. Hence it will be instructive to take the two UV scales $\Lambda_{\rm U}$ and $\Lambda \sim 4\pi f$ as guidelines and allow the predictions to vary in between. The ultimate determination of the UV scale can only come from future experiments.

16 Conclusions

In this Letter, we systematically studied the unitarity constraints in various Little Higgs models using a general formalism in Sec. 12. Our analysis of the Goldstone scatterings is rather generic and mainly independent of the choices of parameters, gauge groups and fermion interactions, etc. This is because the leading Goldstone interactions in the derivative expansion are completely governed by the structure of global symmetry breaking, allowing us to perform a coupled channel analysis for the *full* Goldstone sector in a universal way. We observed that because the global symmetry breaking in the Little Higgs theories generically predict a large number of (pseudo-)Goldstone bosons, their collective effects via coupled channel analysis of Goldstone scatterings tend to push the unitarity violation scale $\Lambda_{\rm U}$ significantly below the conventional NDA cutoff $\Lambda \sim 4\pi f \simeq 12.6f$. Specifically, $\Lambda_{\rm U} \sim (3-4)f$ (cf. Table I), which puts an *upper limit* on the mass of the lightest new state, i.e., $M_{\rm new}^{\rm min} \leqslant \Lambda_{\rm U} \sim (3-4)\,\text{TeV}$ for $f \sim 1\,\text{TeV}$.

As a comparison, in Sec. 13 we took the Littlest Higgs model of $SU(5)/SO(5)$ as an example and explicitly analyzed the Goldstone scatterings in their electroweak eigenbasis. We performed both partial and full coupled-channel analyses. We derived various unitarity violation limits for this minimal model and demonstrated that as more Goldstone states are included into the coupled channel analysis, the unitarity limit $\Lambda_{\rm U}$ becomes increasingly stronger, close to the best bound [cf. Eqs. (14.16) and (14.15)]. This concrete analysis shows that the optimal unitarity limits in Sec. 12 are fairly robust.

We stress that these tight unitarity limits strongly suggest the encouraging possibility of testing the precursors of the Little Higgs UV completion at the upcoming LHC (although no guarantee is implied). A definitive test is expected at the future VLHC [61]. In Sec. 14 we discussed some implications for the UV completions and the related collider signatures. Finally, we concluded Sec. 14 by discussing the meanings of the two estimated UV cutoff scales $\Lambda_{\rm U}$ (from unitarity violation) and Λ (from NDA/GDA). Deciding which estimate to be more sensible in an effective field theory analysis of Little Higgs models is unclear before knowing the precise UV dynamics. Only future experiments can provide an ultimate, definitive answer.

Note added: As this work was being completed, a related preprint [69] appeared which did an explicit unitary-gauge calculation of only light W_L/Z_L scattering in the Littlest Higgs model. Unfortunately its result is incorrect due to, for instance, mistaking the upper bound on the Higgs triplet VEV which leads to erroneously large gauge-Higgs triplet couplings.

Acknowledgments

It is our pleasure to thank H. Georgi, M. Luty, M. E. Peskin, J. Wacker, and especially N. Arkani-Hamed and R. S. Chivukula for valuable discussions. We also thank the organizer of SUSY-2003 for the hospitality and the stimulating enviroment during June 5-10, 2003 when this work was initiated. SC was supported by an NSF graduate student fellowship, and HJH by the DOE under grant DE-FG03-93ER40757.

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