Week 18  (1/13/03)

Distribution of primes

Let \( P(N) \) be the probability that a randomly chosen integer, \( N \), is prime. Show that

\[
P(N) = \frac{1}{\ln N}.
\]

**Note:** Assume that \( N \) is very large, and ignore terms in your answer that are of subleading order in \( N \). Also, make the assumption that the probability that \( N \) is divisible by a prime \( p \) is exactly \( 1/p \) (which is essentially true, for a large enough sample size of numbers).

**Correction:** (Thanks to Bob Silverman for pointing this out.) The assumption, “the probability that \( N \) is divisible by a prime \( p \) is exactly \( 1/p \),” it is actually not valid. More precisely, when dealing with a sufficiently large number of primes, the probability that a prime \( p \) divides \( N \) is not independent of the probability that a prime \( q \) divides \( N \). (This is related to Mertens’ theorem, which you can look up.) The solution I posted for this problem is therefore incorrect, even though it does end up giving the correct result of \( P(N) = 1/\ln N \). (A correction factor ends up canceling out.) However, you might still find it interesting to solve the problem using the given (invalid) assumption.