Solution

Week 1 (9/16/02)

Basketball and tennis ball

(a) For simplicity, assume that the balls are separated by a very small distance, so that the relevant bounces happen a short time apart. This assumption isn't necessary, but it makes for a slightly cleaner solution.

Just before the basketball hits the ground, both balls are moving downward with speed (using $mv^2/2 = mgh$)

$$v = \sqrt{2gh}. \tag{1}$$

Just after the basketball bounces off the ground, it moves upward with speed v, while the tennis ball still moves downward with speed v. The relative speed is therefore $2v$. After the balls bounce off each other, the relative speed is still $2v$. (This is clear if you look at things in the frame of the basketball, which is essentially a brick wall.[1]) Since the upward speed of the basketball essentially stays equal to v, the upward speed of the tennis ball is $2v + v = 3v$. By conservation of energy, it will therefore rise to a height of $H = d + (3v)^2/(2g)$. But $v^2 = 2gh$, so we have

$$H = d + 9h. \tag{2}$$

(b) Just before B_1 hits the ground, all of the balls are moving downward with speed $v = \sqrt{2gh}$.

We will inductively determine the speed of each ball after it bounces off the one below it. If B_i achieves a speed of v_i after bouncing off B_{i-1}, then what is the speed of B_{i+1} after it bounces off B_i? The relative speed of B_{i+1} and B_i (right before they bounce) is $v + v_i$. This is also the relative speed after they bounce. Since B_i is still moving upwards at essentially speed v_i, the final upward speed of B_{i+1} is therefore $(v + v_i) + v_i$. Thus,

$$v_{i+1} = 2v_i + v. \tag{3}$$

Since $v_1 = v$, we obtain $v_2 = 3v$ (in agreement with part (a)), $v_3 = 7v$, $v_4 = 15v$, etc. In general,

$$v_n = (2^n - 1)v, \tag{4}$$

which is easily seen to satisfy eq. (3), with the initial value $v_1 = v$.

From conservation of energy, B_n will bounce to a height of

$$H = \ell + \frac{\left((2^n - 1)v\right)^2}{2g} = \ell + (2^n - 1)^2 h. \tag{5}$$

[1]It turns out that the relative speed is the same before and after any elastic collision, independent of what the masses are. This is easily seen by working in the center-of-mass frame, where the masses simply reverse their velocities.

If h is 1 meter, and we want this height to equal 1000 meters, then (assuming ℓ is not very large) we need $2^n - 1 > \sqrt{1000}$. Five balls won't quite do the trick, but six will, and in this case the height is almost four kilometers.

Escape velocity from the earth (which is $v_{\text{esc}} = \sqrt{2gR} \approx 11,200\,\text{m/s}$) is reached when

$$v_n \geq v_{\text{esc}} \quad \Longrightarrow \quad (2^n - 1)\sqrt{2gh} \geq \sqrt{2gR} \quad \Longrightarrow \quad n \geq \ln_2\left(\sqrt{\frac{R}{h}} + 1\right). \tag{6}$$

With $R = 6.4 \cdot 10^6\,\text{m}$ and $h = 1\,\text{m}$, we find $n \geq 12$. Of course, the elasticity assumption is absurd in this case, as is the notion that one can find 12 balls with the property that $m_1 \gg m_2 \gg \cdots \gg m_{12}$.