(This clever solution and generalization comes from Mike Robinson.)

Put the circle in the complex plane, with its center at the origin. Let the given vertex of the $N$-gon be located at the point $(1,0)$. Let $a \equiv e^{2\pi i/N}$, so that $a^N = 1$. Then the other vertices are located at the points $a^n$, where $n = 1, \ldots, N - 1$.

Let the distance between the vertex at $(1,0)$ and the vertex at $a^n$ be $\ell_n$. Then the desired product (call it $P_N$) of the $N - 1$ segments from the given vertex to the other vertices is

$$P_N = |1 - a| |1 - a^2| \cdots |1 - a^{N-1}| = (1 - a)(1 - a^2) \cdots (1 - a^{N-1}), \quad (1)$$

where the third line comes from the fact that the product is real, because $(1 - a^k)$ is the complex conjugate of $(1 - a^{N-k})$, so the phases in the product cancel in pairs.

Consider the function,

$$F(z) \equiv z^N - 1. \quad (2)$$

One factorization of $F(z)$ is

$$F(z) = (z - 1)(z^{N-1} + z^{N-2} + \cdots + 1). \quad (3)$$

Another factorization is

$$F(z) = (z - 1)(z - a)(z - a^2) \cdots (z - a^{N-1}), \quad (4)$$

because the righthand side is simply the factorization that yields the zeros of $z^N - 1$ (namely, all the numbers of the form $a^n$). These two factorizations give

$$(z - a)(z - a^2) \cdots (z - a^{N-1}) = z^{N-1} + z^{N-2} + \cdots + 1. \quad (5)$$

This equality holds for any value of $z$. In particular, if we set $z = 1$ we obtain $P_N = N$, as desired.

**Remark:** Consider the product of the $N$ lengths from an arbitrary point $z$ in the complex plane, to all $N$ vertices of the $N$-gon.
This product equals the absolute value of the righthand side of eq. (4). Hence, it equals $|F(z)| = |z^N - 1|$. Note what this gives in the $N \to \infty$ limit. If $z$ equals any of the $N$th roots of 1, we obtain zero, of course. But if $z$ is any point inside the unit circle, we obtain $|0 - 1| = 1$, independent of both $z$ and $N$.

The product of all the lengths except the length to the point $(1,0)$, as in the original statement of the problem, equals

$$\left| \frac{z^N - 1}{z - 1} \right| = |z^{N-1} + z^{N-2} + \cdots + 1|.$$  \hspace{1cm} (6)

Note that as $N \to \infty$, this goes to infinity for $z = 1$ (it’s just our original result of $N$, by looking at the righthand side). But it goes to the finite number $1/|z - 1|$ for any $z$ inside the unit circle (by looking at the lefthand side).