Solution

Week 10 (11/18/02)

Product of lengths

(This clever solution and generalization comes from Mike Robinson.)

Put the circle in the complex plane, with its center at the origin. Let the given vertex of the N-gon be located at the point $(1, 0)$. Let $a \equiv e^{2\pi i/N}$, so that $a^N = 1$. Then the other vertices are located at the points a^n, where $n = 1, \dots, N-1$.

Let the distance between the vertex at $(1, 0)$ and the vertex at a^n be ℓ_n. Then the desired product (call it P_N) of the $N-1$ segments from the given vertex to the other vertices is

$$\begin{aligned} P_N &= \ell_1 \ell_2 \dots \ell_{N-1} \\ &= |1-a||1-a^2| \cdots |1-a^{N-1}| \\ &= (1-a)(1-a^2)\cdots(1-a^{N-1}), \end{aligned} \tag{1}$$

where the third line comes from the fact that the product is real, because $(1-a^k)$ is the complex conjugate of $(1-a^{N-k})$, so the phases in the product cancel in pairs.

Consider the function,

$$F(z) \equiv z^N - 1. \tag{2}$$

One factorization of $F(z)$ is

$$F(z) = (z-1)(z^{N-1} + z^{N-2} + \cdots + 1). \tag{3}$$

Another factorization is

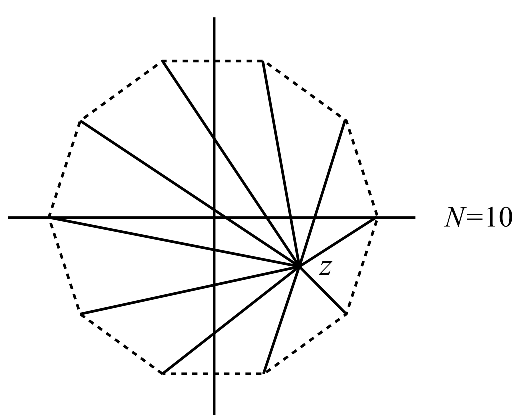
$$F(z) = (z-1)(z-a)(z-a^2)\cdots(z-a^{N-1}), \tag{4}$$

because the righthand side is simply the factorization that yields the zeros of $z^N - 1$ (namely, all the numbers of the form a^n). These two factorizations give

$$(z-a)(z-a^2)\cdots(z-a^{N-1}) = z^{N-1} + z^{N-2} + \cdots + 1. \tag{5}$$

This equality holds for any value of z. In particular, if we set $z = 1$ we obtain $P_N = N$, as desired.

Remark: Consider the product of the N lengths from an arbitrary point z in the complex plane, to *all* N vertices of the N-gon.



This product equals the absolute value of the righthand side of eq. (4). Hence, it equals $|F(z)| = |z^N - 1|$. Note what this gives in the $N \to \infty$ limit. If z equals any of the Nth roots of 1, we obtain zero, of course. But if z is any point inside the unit circle, we obtain $|0 - 1| = 1$, independent of both z and N.

The product of all the lengths *except* the length to the point (1,0), as in the original statement of the problem, equals

$$\left| \frac{z^N - 1}{z - 1} \right| = |z^{N-1} + z^{N-2} + \cdots + 1|. \tag{6}$$

Note that as $N \to \infty$, this goes to infinity for $z = 1$ (it's just our original result of N, by looking at the righthand side). But it goes to the finite number $1/|z - 1|$ for any z inside the unit circle (by looking at the lefthand side).