Solution

Week 12 (12/2/02)

Decreasing numbers

First Solution: Let $E(x)$ be the expected number of numbers you have *yet to pick*, given that you have just picked the number x. Then, for example, $E(0) = 1$, because the next number you pick is guaranteed to be greater than $x = 0$, whereupon the game stops. Let's calculate $E(x)$.

Imagine picking the next number, having just picked x. There is a $(1-x)$ chance that this next number is greater than x, in which case the game stops. So in this case it takes you just one pick after the number x. If, on the other hand, you pick a number, y, which is less than x, then you can expect to pick $E(y)$ numbers after that. So in this case it takes you an average of $E(y)+1$ total picks after the number x. These two scenarios my be combined to give the equation,

$$\begin{aligned} E(x) &= 1 \cdot (1-x) + \int_0^x (E(y)+1)\, dy \\ &= 1 + \int_0^x E(y)\, dy \qquad (1) \end{aligned}$$

Differentiating this with respect to x gives $E'(x) = E(x)$. Therefore, $E(x) = Ae^x$, where A is some constant. The condition $E(0) = 1$ gives $A = 1$. Hence

$$E(x) = e^x. \qquad (2)$$

The total number of picks, T, is simply $T = E(1)$, because the first pick is automatically less than 1, so the number of picks *after* starting a game with the number 1 is equal to the total number of picks in a game starting with a random number. Since $E(1) = e$, we have

$$T = e. \qquad (3)$$

Second Solution: Let the first number you pick be x_1, the second x_2, the third x_3, and so on. There is a $p_2 = 1/2$ chance that $x_2 < x_1$. There is a $p_3 = 1/3!$ chance that $x_3 < x_2 < x_1$. There is a $p_4 = 1/4!$ chance that $x_4 < x_3 < x_2 < x_1$, and so on.

You must make at least two picks in this game. The probability that you make exactly two picks is equal to the probability that $x_2 > x_1$, which is $1 - p_2 = 1/2$.

The probability that you make exactly three picks is equal to the probability that $x_2 < x_1$ and $x_3 > x_2$. This equals the probability that $x_2 < x_1$ minus the probability that $x_3 < x_2 < x_2$, that is, $p_2 - p_3$.

The probability that you make exactly four picks is equal to the probability that $x_3 < x_2 < x_1$ and $x_4 > x_3$. This equals the probability that $x_3 < x_2 < x_1$ minus the probability that $x_4 < x_3 < x_2 < x_2$, that is, $p_3 - p_4$.

Continuing in this manner, we find that the expected total number of picks, T, is

$$T = 2(1-p_2) + 3(p_2 - p_3) + 4(p_3 - p_4) + \cdots$$

$$
\begin{aligned}
&= 2\left(1-\frac{1}{2!}\right)+3\left(\frac{1}{2!}-\frac{1}{3!}\right)+4\left(\frac{1}{3!}-\frac{1}{4!}\right)+\cdots \\
&= 2+\frac{(3-2)}{2!}+\frac{(4-3)}{3!}+\frac{(5-4)}{4!}+\cdots \\
&= 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots \\
&= e. \qquad (4)
\end{aligned}
$$

Third Solution: Let $p(x)\,dx$ be the probability that a number between x and $x+dx$ is picked as part of the decreasing sequence. Then we may find $p(x)$ by adding up the probabilities, $p_j(x)\,dx$, that a number between x and $x+dx$ is picked on the jth pick.

The probability that such a number is picked first is dx. The probability that it is picked second is $(1-x)dx$, because $1-x$ is the probability that the first number is greater than x. The probability that it is picked third is $(1/2)(1-x)^2dx$, because $(1-x)^2$ is the probability that the first two numbers are greater than x, and $1/2$ is the probability that these numbers are picked in decreasing order. Likewise, the probability that it is picked fourth is $(1/3!)(1-x)^3dx$. Continuing in this manner, we see that the probability that it is picked sooner or later in the decreasing sequence is

$$
\begin{aligned}
p(x)\,dx &= \left(1+(1-x)+\frac{(1-x)^2}{2!}+\frac{(1-x)^3}{3!}+\cdots\right)dx \\
&= e^{1-x}\,dx. \qquad (5)
\end{aligned}
$$

The expected number of numbers picked in the decreasing sequence is therefore $\int_0^1 e^{1-x}dx = e-1$. Adding on the last number picked (which is not in the decreasing sequence) gives a total of e numbers picked, as above.

REMARKS:

1. What is the average value of the smallest number you pick? The probability that the smallest number is between x and $x+dx$ equals $e^{1-x}(1-x)\,dx$. This is true because $p(x)\,dx = e^{1-x}\,dx$ is the probability that you pick a number between x and $x+dx$ as part of the decreasing sequence (from the third solution above), and then $(1-x)$ is the probability that the next number you pick is larger. The average value, s, of the smallest number you pick is therefore $s=\int_0^1 e^{1-x}(1-x)x\,dx$. Letting $y \equiv 1-x$ for convenience, and integrating (say, by parts), we have

$$
\begin{aligned}
s &= \int_0^1 e^y y(1-y)\,dy \\
&= \left(-y^2e^y+3ye^y-3e^y\right)\Big|_0^1 \\
&= 3-e \\
&\approx 0.282. \qquad (6)
\end{aligned}
$$

Likewise, the average value of the final number you pick is $\int_0^1 e^{1-x}(1-x)(1+x)/2\,dx$, which you can show equals $2-e/2 \approx 0.64$. The $(1+x)/2$ in this integral arises from the fact that if you do pick a number greater than x, its average value will be $(1+x)/2$.

2. We can also ask questions such as: Continue the game as long as $x_2 < x_1$, and $x_3 > x_2$, and $x_4 < x_3$, and $x_5 > x_4$, and so on, with the numbers alternating in size. What is the expected number of numbers you pick?

   We can apply the method of the first solution here. Let $A(x)$ be the expected number of numbers you have yet to pick, for $x = x_1, x_3, x_5, \ldots$. And let $B(x)$ be the expected number of numbers you have yet to pick, for $x = x_2, x_4, x_6, \ldots$. From the reasoning in the first solution, we have

$$
\begin{aligned}
A(x) &= 1 \cdot (1-x) + \int_0^x (B(y)+1)\,dy &= 1 + \int_0^x B(y)\,dy, \\
B(x) &= 1 \cdot x + \int_x^1 (A(y)+1)\,dy &= 1 + \int_x^1 A(y)\,dy. \qquad (7)
\end{aligned}
$$

   Differentiating these two equations yields $A'(x) = B(x)$ and $B'(x) = -A(x)$. If we then differentiate the first of these and substitute the result into the second, we obtain $A''(x) = -A(x)$. Likewise, $B''(x) = -B(x)$. The solutions to these equations may be written as

$$
A(x) = \alpha \sin x + \beta \cos x \qquad \text{and} \qquad B(x) = \alpha \cos x - \beta \sin x. \qquad (8)
$$

   The condition $A(0) = 1$ yields $\beta = 1$. The condition $B(1) = 1$ then gives $\alpha = (1 + \sin 1)/\cos 1$. The desired answer to the problem equals $B(0)$, since we could imagine starting the game with someone picking a number greater than 0, which is guaranteed. (Similarly, the desired answer also equals $A(1)$.) So the expected total number of picks is $B(0) = (1 + \sin 1)/\cos 1$. This has a value of about 3.41, which is greater than the $e \approx 2.72$ answer to our original problem.