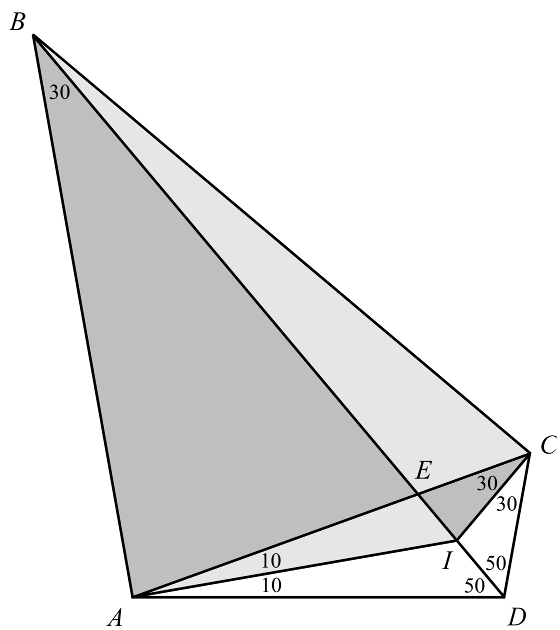
Solution

Week 14 (12/16/02)

Find the angles

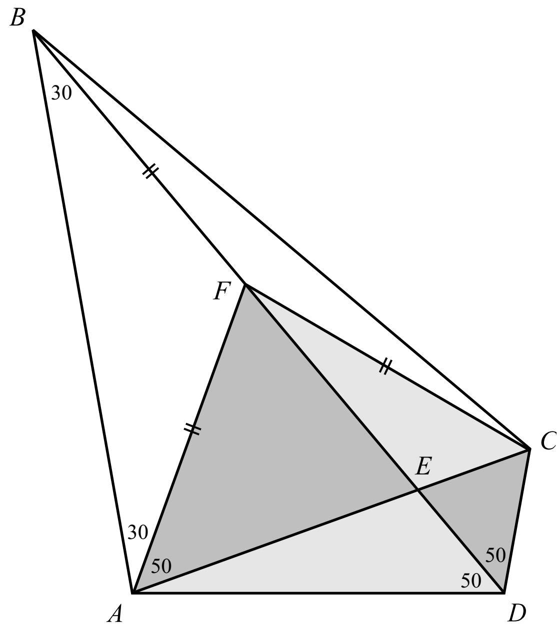
Although this problem seems simple at first glance, angle chasing won't provide the answer. Something a bit more sneaky is required. At the risk of going overboard, we'll give four solutions. You can check that all of the solutions rely on the equality of the two given 50° angles, and on the fact that $2(80^\circ) + 20^\circ = 180^\circ$.

First Solution: In the figure below, note that $\angle ACD = 60^\circ$ and $\angle ABD = 30^\circ$. Let AC and BD intersect at E. Draw the angle bisectors of triangle ACD. They meet at the incenter, I, located along segment ED. Since $\angle ECI = 30^\circ = \angle EBA$, triangles ECI and EBA are similar. Therefore, triangles EBC and EAI are also similar. Thus, $\angle EBC = \angle EAI = 10^\circ$. We then easily find $\angle ECB = 60^\circ$.



Second solution: In the figure below, note that $\angle ABD = 30^\circ$. Let AC and BD intersect at E. Draw segment AF, with F on BE, such that $\angle EAF = 50^\circ$. We then have $\angle FAB = 30^\circ$. So triangle FAB is isosceles, with $FA = FB$.

Since $\angle EDC = \angle EAF$, triangles EDC and EAF are similar. Therefore, triangles EAD and EFC are also similar. Hence, $\angle ECF = 50^\circ$, and triangle FCA is isosceles with $FC = FA$. Thus, $FC = FA = FB$, and so triangle FBC is also isosceles, with $\angle FBC = \angle FCB$. Since it is easy to show that these two angles must sum to 20°, they must each be 10°. Therefore, $\angle FBC = 10^\circ$ and $\angle ECB = 60^\circ$.



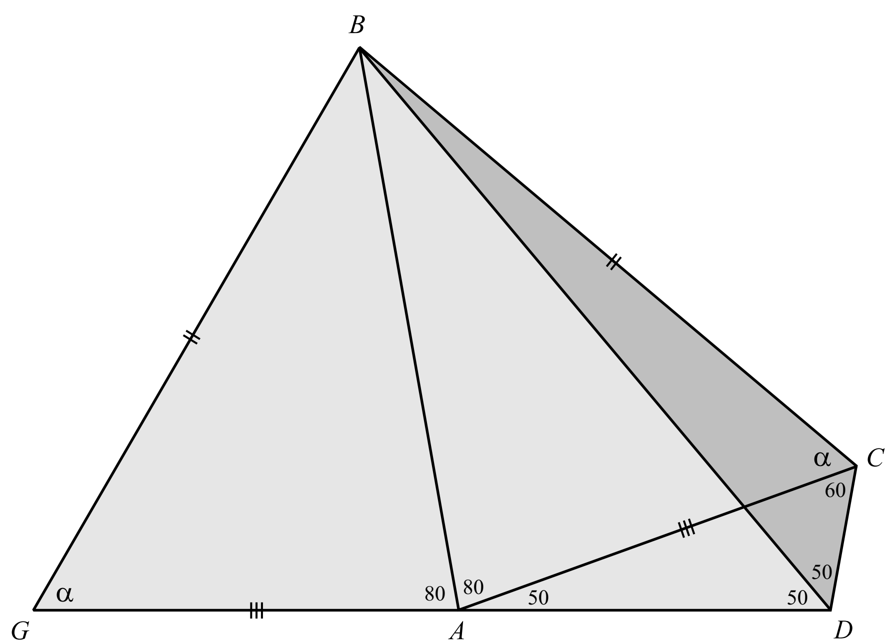
Third Solution: In the figure below, note that $\angle ACD = 60^\circ$. Reflect triangle ABC across AB to yield triangle ABG. Note that D, A, and G are collinear. From the law of sines in triangle DBC, we have

$$\frac{\sin 50^\circ}{BC} = \frac{\sin(60^\circ + \alpha)}{BD}. \tag{1}$$

From the law of sines in triangle DBG, we have

$$\frac{\sin 50^\circ}{BG} = \frac{\sin \alpha}{BD}. \tag{2}$$

But $BC = BG$, so we have $\sin(60^\circ + \alpha) = \sin \alpha$. Therefore, $60^\circ + \alpha$ and α must be supplementary angles, which gives $\alpha = 60^\circ$. We then easily obtain $\angle DBC = 10^\circ$.



Fourth Solution: We now present the brute-force method using the law of sines, just to show that it can be done.

In the figure below, let AC and BD intersect at E. Let the length of AB be 1 unit. Then the law of sines in triangle AED gives

$$a = \frac{\sin 50^\circ}{\sin 110^\circ}, \qquad \text{and} \qquad d = \frac{\sin 20^\circ}{\sin 110^\circ}. \tag{3}$$

The law of sines in triangles AEB and DEC then gives

$$b = \left(\frac{\sin 80^\circ}{\sin 30^\circ}\right)\left(\frac{\sin 50^\circ}{\sin 110^\circ}\right), \qquad \text{and} \qquad c = \left(\frac{\sin 50^\circ}{\sin 60^\circ}\right)\left(\frac{\sin 20^\circ}{\sin 110^\circ}\right). \tag{4}$$

The law of sines in triangle BEC finally gives

$$\left(\frac{\sin 80^\circ \sin 50^\circ}{\sin 30^\circ \sin 110^\circ}\right) \Big/ \sin\alpha = \left(\frac{\sin 50^\circ \sin 20^\circ}{\sin 60^\circ \sin 110^\circ}\right) \Big/ \sin\beta. \tag{5}$$

Substituting $70^\circ - \alpha$ for β yields (after some algebra)

$$\tan\alpha = \frac{\sin 60^\circ \sin 80^\circ \sin 70^\circ}{\sin 60^\circ \sin 80^\circ \cos 70^\circ + \sin 30^\circ \sin 20^\circ}. \tag{6}$$

Using $\sin 20^\circ = 2\sin 10^\circ \cos 10^\circ = 2\sin 10^\circ \sin 80^\circ$, along with $\sin 30^\circ = 1/2$, gives

$$\tan\alpha = \frac{\sin 60^\circ \sin 70^\circ}{\sin 60^\circ \cos 70^\circ + \sin 10^\circ}. \tag{7}$$

Finally, expanding $\sin 10^\circ = \sin(70^\circ - 60^\circ)$ gives the result

$$\tan\alpha = \tan 60^\circ. \tag{8}$$

Hence $\alpha = 60^\circ$, and so $\beta = 10^\circ$.