Solution

Week 14  (12/16/02)

Find the angles

Although this problem seems simple at first glance, angle chasing won’t provide the answer. Something a bit more sneaky is required. At the risk of going overboard, we’ll give four solutions. You can check that all of the solutions rely on the equality of the two given 50° angles, and on the fact that 2(80°) + 20° = 180°.

First Solution: In the figure below, note that ∠ACD = 60° and ∠ABD = 30°. Let AC and BD intersect at E. Draw the angle bisectors of triangle ACD. They meet at the incenter, I, located along segment ED. Since ∠ECI = 30° = ∠EBA, triangles ECI and EBA are similar. Therefore, triangles EBC and EAI are also similar. Thus, ∠EBC = ∠EAI = 10°. We then easily find ∠ECB = 60°.

Second solution: In the figure below, note that ∠ABD = 30°. Let AC and BD intersect at E. Draw segment AF, with F on BE, such that ∠EAF = 50°. We then have ∠FAB = 30°. So triangle FAB is isosceles, with FA = FB.

Since ∠EDC = ∠EAF, triangles EDC and EAF are similar. Therefore, triangles EAD and EFC are also similar. Hence, ∠ECF = 50°, and triangle FCA is isosceles with FC = FA. Thus, FC = FA = FB, and so triangle FBC is also isosceles, with ∠FBC = ∠FCB. Since it is easy to show that these two angles must sum to 20°, they must each be 10°. Therefore, ∠FBC = 10° and ∠ECB = 60°.
Third Solution: In the figure below, note that $\angle ACD = 60^\circ$. Reflect triangle $ABC$ across $AB$ to yield triangle $ABG$. Note that $D$, $A$, and $G$ are collinear. From the law of sines in triangle $DBC$, we have

$$\frac{\sin 50^\circ}{BC} = \frac{\sin (60^\circ + \alpha)}{BD}. \quad (1)$$

From the law of sines in triangle $DBG$, we have

$$\frac{\sin 50^\circ}{BG} = \frac{\sin \alpha}{BD}. \quad (2)$$

But $BC = BG$, so we have $\sin (60^\circ + \alpha) = \sin \alpha$. Therefore, $60^\circ + \alpha$ and $\alpha$ must be supplementary angles, which gives $\alpha = 60^\circ$. We then easily obtain $\angle DBC = 10^\circ$. 

\[ \quad \]
Fourth Solution: We now present the brute-force method using the law of sines, just to show that it can be done.

In the figure below, let $AC$ and $BD$ intersect at $E$. Let the length of $AB$ be 1 unit. Then the law of sines in triangle $AED$ gives

$$a = \frac{\sin 50^\circ}{\sin 110^\circ}, \quad \text{and} \quad d = \frac{\sin 20^\circ}{\sin 110^\circ}. \quad (3)$$

The law of sines in triangles $AEB$ and $DEC$ then gives

$$b = \left(\frac{\sin 80^\circ}{\sin 30^\circ}\right) \left(\frac{\sin 50^\circ}{\sin 110^\circ}\right), \quad \text{and} \quad c = \left(\frac{\sin 50^\circ}{\sin 60^\circ}\right) \left(\frac{\sin 20^\circ}{\sin 110^\circ}\right). \quad (4)$$

The law of sines in triangle $BEC$ finally gives

$$\left(\frac{\sin 80^\circ \sin 50^\circ}{\sin 30^\circ \sin 110^\circ}\right)/\sin \alpha = \left(\frac{\sin 50^\circ \sin 20^\circ}{\sin 60^\circ \sin 110^\circ}\right)/\sin \beta. \quad (5)$$

Substituting $70^\circ - \alpha$ for $\beta$ yields (after some algebra)

$$\tan \alpha = \frac{\sin 60^\circ \sin 80^\circ \sin 70^\circ}{\sin 60^\circ \sin 80^\circ \cos 70^\circ + \sin 30^\circ \sin 20^\circ}. \quad (6)$$

Using $\sin 20^\circ = 2 \sin 10^\circ \cos 10^\circ = 2 \sin 10^\circ \sin 80^\circ$, along with $\sin 30^\circ = 1/2$, gives

$$\tan \alpha = \frac{\sin 60^\circ \sin 70^\circ}{\sin 60^\circ \cos 70^\circ + \sin 10^\circ}. \quad (7)$$

Finally, expanding $\sin 10^\circ = \sin(70^\circ - 60^\circ)$ gives the result

$$\tan \alpha = \tan 60^\circ. \quad (8)$$

Hence $\alpha = 60^\circ$, and so $\beta = 10^\circ$. 

![Diagram](image_url)