Solution

Week 15 (12/23/02)

Maximal gravity

Assume that the material has been shaped and positioned so that the field at P is maximum. Let this field point in the x-direction. The key to this problem is to realize that all the small elements of mass dm on the surface of the material must give equal contributions to the x-component of the field at P. If this were not the case, then we could simply move a tiny piece of the material from one point on the surface to another, thereby increasing the field at P, in contradiction to our assumption that the field at P is maximum.

Label the points on the surface by their distance r from P, and by the angle θ that the line of this distance subtends with the x-axis. Then a small mass dm on the surface provides an x-component of the gravitational field equal to

$$F_x = \cos\theta \frac{G\,dm}{r^2}\,. \tag{1}$$

Since this contribution cannot depend on the location of the mass dm on the surface, we must have $r^2 \propto \cos\theta$. The surface may therefore be described by the equation,

$$r^2 = a^2 \cos\theta, \tag{2}$$

where the constant a^2 depends on the volume of the material. The desired shape clearly exhibits cylindrical symmetry around the x-axis, so let us consider a cross section in the x-y plane. In terms of x and y (with $x^2 + y^2 = r^2$ and $\cos\theta = x/r$), eq. (2) becomes

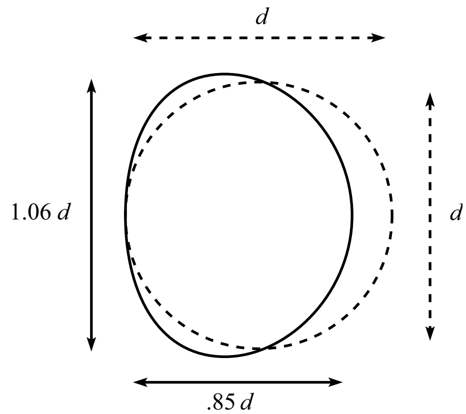
$$r^3 = a^2 x \qquad \Longrightarrow \qquad r^2 = a^{4/3}x^{2/3} \qquad \Longrightarrow \qquad y^2 = a^{4/3}x^{2/3} - x^2. \tag{3}$$

To get a sense of what this surface looks like, note that $dy/dx = \infty$ at both $x = 0$ and $x = a$ (the point on the surface furthest from P). So the surface is smooth and has no cusps. We can easily calculate the volume in terms of a, and we find

$$V = \int_0^a \pi y^2\,dx = \int_0^a \pi(a^{4/3}x^{2/3} - x^2)\,dx = \frac{4\pi}{15}a^3. \tag{4}$$

Since the diameter of a sphere of volume V is $(6V/\pi)^{1/3}$, we see that a sphere with the same volume would have a diameter of $(8/5)^{1/3}a \approx 1.17a$. Hence, our shape is squashed by a factor of $(5/8)^{1/3} \approx 0.85$ along the x-direction, compared to a sphere of the same volume.

We may also calculate the maximum height in the y-direction. You can show that is occurs at $x = 3^{-3/4}a \approx 0.44a$ and has a value of $2(4/27)^{1/3}a \approx 1.24a$. Hence, our shape is stretched by a factor of $2(4/27)^{1/4}(5/8)^{1/3} \approx 1.24/1.17 \approx 1.06$ in the y-direction, compared to a sphere of the same volume. Cross sections of our shape and a sphere with the same volume are shown below.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