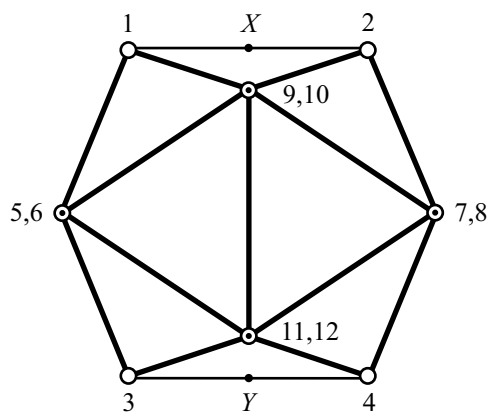


Solution

Week 17 (1/6/03)

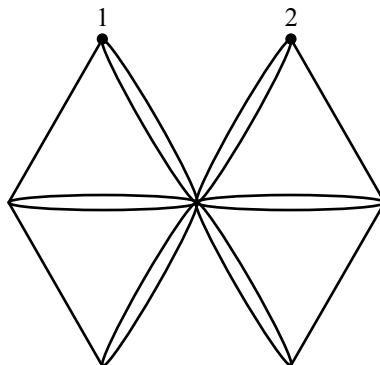
Icosahedron of resistors

First Solution: We will calculate the effective resistance between vertices 1 and 2 in the figure below. When the icosahedron is viewed from the angle shown, four vertices lie directly behind four other vertices; each of these pairs is represented by a dot inside a circle. And thirteen edges lie directly behind thirteen other edges; each of these pairs is represented by a bold line. The remaining two (of the 30 total) edges not represented in the figure are the ones connecting vertices 5 and 6, and 7 and 8. For future reference, points X and Y are defined to be the midpoints of the edges shown.

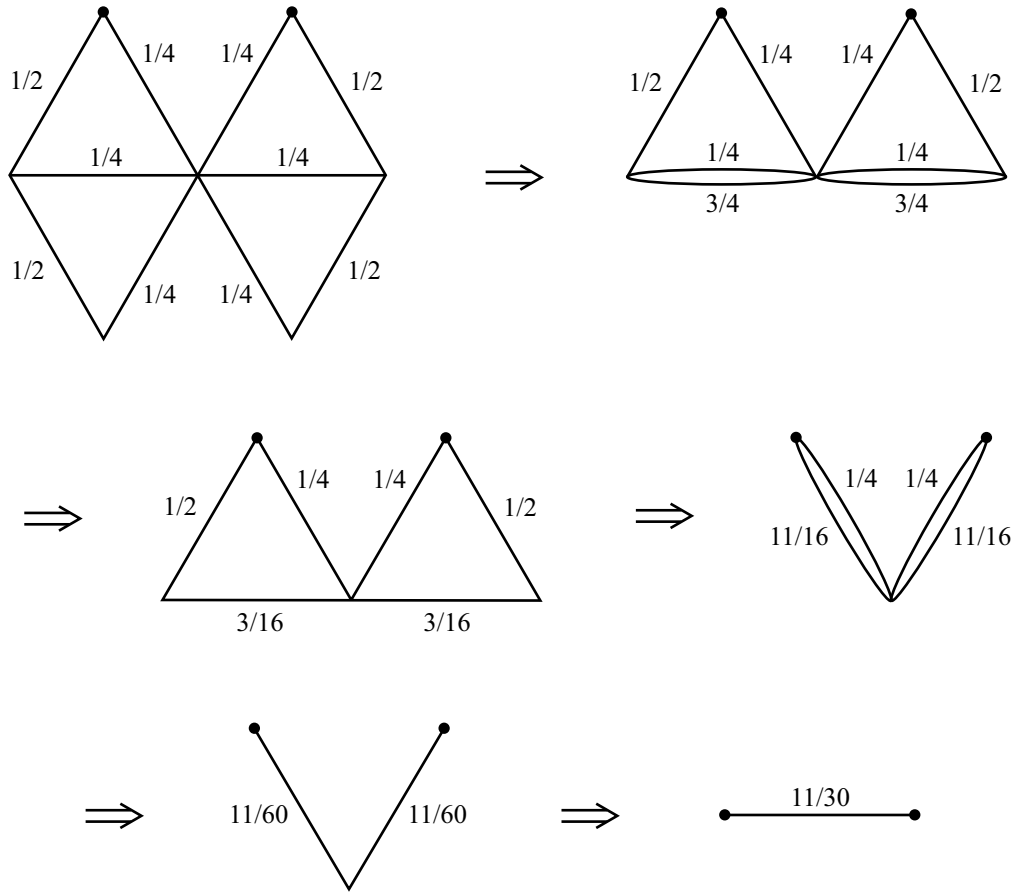


If a potential difference is created between vertices 1 and 2, then the potentials at vertices 5 and 6 are equal, and likewise for the other three pairs of vertices. We may therefore bring each of these four pairs of points together and identify each pair as one point. The resulting circuit is simply the figure above, now planar, where each bold line represents a $(1/2)\Omega$ resistor (because it arises from two 1Ω resistors in parallel).

We now note that all points on the vertical bisector of the circuit (that is, X , Y , the 9-10 pair, and the 11-12 pair) are at equal potentials, so we may bring them all together and identify them as one point. Since X and Y split the top and bottom resistors into two $(1/2)\Omega$ resistors, we arrive at the following circuit, where every line in the figure represents a $(1/2)\Omega$ resistor.



This circuit may be reduced as follows:



The effective resistance between two adjacent vertices is therefore $(11/30)\Omega$.

Second Solution: Let vertices 1 and 2 be adjacent. Consider the setup where a current $1A$ enters through vertex 1, and a current $(1/11)A$ leaves through the other 11 vertices. Note that, due to symmetry, a current $(1/5)A$ flows through each of the 5 edges leaving vertex 1. Hence, the voltage difference between vertices 1 and 2 is

$$V_1 - V_2 = \left(\frac{1}{5}A\right)(1\Omega) = \frac{1}{5}V. \quad (1)$$

Consider a second setup, where a current $1A$ leaves through vertex 2, and a current $(1/11)A$ enters through the other 11 vertices. Again, note that a current $(1/5)A$ flows through each of the 5 edges entering vertex 2. Hence, the voltage difference between vertices 1 and 2 is

$$V_1 - V_2 = \left(\frac{1}{5}A\right)(1\Omega) = \frac{1}{5}V. \quad (2)$$

If we superimpose these two setups on each other, then we arrive at the setup where:

- A current $(12/11)A$ enters through vertex 1,

- A current $(12/11)A$ leaves through vertex 2,
- No current enters or leaves through the other 10 vertices, and
- The potential difference between vertices 1 and 2 is

$$V_1 - V_2 = \frac{1}{5}V + \frac{1}{5}V = \frac{2}{5}V. \quad (3)$$

We have therefore constructed precisely the experimental setup that serves to determine the effective resistance between vertices 1 and 2. That is, we have put a current in at vertex 1, taken it out at vertex 2, and measured the voltage difference between the two points. The effective resistance between vertices 1 and 2 is therefore given by

$$V = IR \quad \Longrightarrow \quad \frac{2}{5}V = \left(\frac{12}{11}A\right) R_{\text{eff}} \quad \Longrightarrow \quad R_{\text{eff}} = \frac{11}{30}\Omega. \quad (4)$$

REMARK: Note that the sum of the effective resistances across all of the 30 edges in the icosahedron is $11\Omega = (N - 1)\Omega$, where N is the number of vertices. You can quickly use the method of the second solution above to calculate the sum of the effective resistances across all of the edges in any of the other regular polyhedra. You will find that the result is always $(N - 1)\Omega$. Does this result hold for more general figures? Stay tuned for a future problem of the week...