*Solution*

Week 19 (1/20/03)

**Block and bouncing ball**

(a) Consider one of the collisions. Let it occur at a distance $\ell$ from the wall, and let $v$ and $V$ be the speeds of the ball and block, respectively, *after* the collision. We claim that the quantity $\ell(v-V)$ is invariant. That is, it is the same for each collision. This can be seen as follows.

The time to the next collision is given by $Vt + vt = 2\ell$ (because the sum of the distances traveled by the two objects is $2\ell$). Therefore, the next collision occurs at a distance $\ell'$ from the wall, where

$$\ell' = \ell - Vt = \ell - \frac{2\ell V}{V+v} = \frac{\ell(v-V)}{v+V}\,. \tag{1}$$

Therefore,

$$\ell'(v+V) = \ell(v-V). \tag{2}$$

We now make use of the fact that in an elastic collision, the relative speed before the collision equals the relative speed after the collision. (This is most easily seen by working in the center of mass frame, where this scenario clearly satisfies conservation of $E$ and $p$.) Therefore, if $v'$ and $V'$ are the speeds after the next collision, then

$$v + V = v' - V'. \tag{3}$$

Using this in eq. (2) gives

$$\ell'(v'-V') = \ell(v-V), \tag{4}$$

as we wanted to show.

What is the value of this invariant? After the first collision, the block continues to move at speed $V_0$, up to corrections of order $m/M$. And the ball acquires a speed of $2V_0$, up to corrections of order $m/M$. (This can be seen by working in the frame of the heavy block, or equivalently by using eq. (3) with $V' \approx V = V_0$ and $v = 0$.) Therefore, the invariant $\ell(v-V)$ is essentially equal to $L(2V_0 - V_0) = LV_0$.

Let $L_{\text{min}}$ be the closest distance to the wall. When the block reaches this closest point, its speed is (essentially) zero. Hence, all of the initial kinetic energy of the block now belongs to the ball. Therefore, $v = V_0\sqrt{M/m}$, and our invariant tells us that $LV_0 = L_{\text{min}}(V_0\sqrt{M/m} - 0)$. Thus,

$$L_{\text{min}} = L\sqrt{\frac{m}{M}}\,. \tag{5}$$

(b) (This solution is due to Slava Zhukov)

With the same notation as in part (a), conservation of momentum in a given collision gives

$$MV - mv = MV' + mv'. \tag{6}$$

This equation, along with eq. (3),[1] allows us to solve for $V'$ and $v'$ in terms of $V$ and $v$. The result, in matrix form, is

$$\begin{pmatrix} V' \\ v' \end{pmatrix} = \begin{pmatrix} \frac{M-m}{M+m} & \frac{-2m}{M+m} \\ \frac{2M}{M+m} & \frac{M-m}{M+m} \end{pmatrix} \begin{pmatrix} V \\ v \end{pmatrix}. \tag{7}$$

The eigenvectors and eigenvalues of this transformation are

$$\begin{aligned} A_1 = \begin{pmatrix} 1 \\ -i\sqrt{\frac{M}{m}} \end{pmatrix}, \qquad \lambda_1 &= \frac{(M-m)+2i\sqrt{Mm}}{M+m} \equiv e^{i\theta}, \\ A_2 = \begin{pmatrix} 1 \\ i\sqrt{\frac{M}{m}} \end{pmatrix}, \qquad \lambda_2 &= \frac{(M-m)-2i\sqrt{Mm}}{M+m} \equiv e^{-i\theta}, \end{aligned} \tag{8}$$

where

$$\theta \equiv \arctan\left(\frac{2\sqrt{Mm}}{M-m}\right) \approx 2\sqrt{\frac{m}{M}}\,. \tag{9}$$

The initial conditions are

$$\begin{pmatrix} V \\ v \end{pmatrix} = \begin{pmatrix} V_0 \\ 0 \end{pmatrix} = \frac{V_0}{2}(A_1 + A_2). \tag{10}$$

Therefore, the speeds after the $n$th bounce are given by

$$\begin{aligned} \begin{pmatrix} V_n \\ v_n \end{pmatrix} &= \frac{V_0}{2}(\lambda_1^n A_1 + \lambda_2^n A_2) \\ &= \frac{V_0}{2}\left(e^{in\theta}\begin{pmatrix} 1 \\ -i\sqrt{\frac{M}{m}} \end{pmatrix} + e^{-in\theta}\begin{pmatrix} 1 \\ i\sqrt{\frac{M}{m}} \end{pmatrix}\right) \\ &= V_0 \begin{pmatrix} \cos n\theta \\ \sqrt{\frac{M}{m}}\sin n\theta \end{pmatrix}. \end{aligned} \tag{11}$$

Let the block reach its closest approach to the wall at the $N$th bounce. Then $V_N = 0$, and so eq. (11) tells us that $N\theta = \pi/2$. Using the definition of $\theta$ from eq. (9), we have

$$\begin{aligned} N &= \frac{\pi/2}{\arctan \frac{2\sqrt{Mm}}{M-m}} \\ &\approx \frac{\pi}{4}\sqrt{\frac{M}{m}}\,. \end{aligned} \tag{12}$$

Remark: This solution is exact, up to the second line in eq. (12), where we finally use $M \gg m$. We can use the first line of eq. (12) to determine the relation between $m$ and $M$ for which the $N$th bounce leaves the block exactly at rest at its closest

[1] Alternatively, you could use conservation of energy, but this is a quadratic statement in the velocities, which makes things messy. Conservation of energy is built into the linear eq. (3).

approach to the wall. For this to happen, we need the $N$ in eq. (12) to be an integer. Letting $m/M \equiv r$, we can rewrite eq. (12) as

$$\frac{2\sqrt{r}}{1-r} = \tan\frac{\pi}{2N} \equiv \sqrt{\frac{1-\cos\beta}{1+\cos\beta}}\,, \tag{13}$$

where we have used the tan half-angle formula, with $\beta \equiv \pi/N$. Squaring both sides and solving the resulting quadratic equation for $r$ gives

$$r = \frac{3+\cos\beta - 2\sqrt{2+2\cos\beta}}{1-\cos\beta}\,. \tag{14}$$

If we want the block to come to rest after $N = 1$ bounce, then $\beta = \pi$ gives $r = 1$, which is correct. If we want $N = 2$, then $\beta = \pi/2$ gives $r = 3 - 2\sqrt{2} \approx 0.172$. If we want $N = 3$, then $\beta = \pi/3$ gives $r = 7 - 4\sqrt{3} \approx 0.072$. For general $N$, eq. (14) must be computed numerically. For large $N$, the second line in eq. (12) shows that $r$ goes like $1/N^2$. More precisely, $r \approx \pi^2/(16N^2)$.