

Solution

Week 21 (2/3/03)

Ball on turntable

Let the angular velocity of the turntable be $\Omega\hat{\mathbf{z}}$, and let the angular velocity of the ball be $\boldsymbol{\omega}$. If the ball is at position \mathbf{r} (with respect to the lab frame), then its velocity (with respect to the lab frame) may be broken up into the velocity of the turntable (at position \mathbf{r}) plus the ball's velocity with respect to the turntable. The non-slipping condition says that this latter velocity is given by $\boldsymbol{\omega} \times (a\hat{\mathbf{z}})$, where a is the radius of the ball. The ball's velocity with respect to the lab frame is therefore

$$\mathbf{v} = (\Omega\hat{\mathbf{z}}) \times \mathbf{r} + \boldsymbol{\omega} \times (a\hat{\mathbf{z}}). \quad (1)$$

The important point to realize in this problem is that the friction force from the turntable is responsible for changing both the ball's linear momentum and its angular momentum. In particular, $\mathbf{F} = d\mathbf{p}/dt$ gives

$$\mathbf{F} = m \frac{d\mathbf{v}}{dt}. \quad (2)$$

And the angular momentum of the ball is $\mathbf{L} = I\boldsymbol{\omega}$, so $\boldsymbol{\tau} = d\mathbf{L}/dt$ (relative to the center of the ball) gives

$$(-a\hat{\mathbf{z}}) \times \mathbf{F} = I \frac{d\boldsymbol{\omega}}{dt}, \quad (3)$$

because the force is applied at position $-a\hat{\mathbf{z}}$ relative to the center.

We will now use the previous three equations to demonstrate that the ball undergoes circular motion. Our goal will be to produce an equation of the form,

$$\frac{d\mathbf{v}}{dt} = \Omega'\hat{\mathbf{z}} \times \mathbf{v}, \quad (4)$$

since this describes circular motion, with frequency Ω' (to be determined). Plugging the expression for \mathbf{F} from eq. (2) into eq. (3) gives

$$\begin{aligned} (-a\hat{\mathbf{z}}) \times \left(m \frac{d\mathbf{v}}{dt} \right) &= I \frac{d\boldsymbol{\omega}}{dt} \\ \implies \frac{d\boldsymbol{\omega}}{dt} &= - \left(\frac{am}{I} \right) \hat{\mathbf{z}} \times \frac{d\mathbf{v}}{dt}. \end{aligned} \quad (5)$$

Taking the derivative of eq. (1) gives

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \Omega\hat{\mathbf{z}} \times \frac{d\mathbf{r}}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times (a\hat{\mathbf{z}}) \\ &= \Omega\hat{\mathbf{z}} \times \mathbf{v} - \left(\left(\frac{am}{I} \right) \hat{\mathbf{z}} \times \frac{d\mathbf{v}}{dt} \right) \times (a\hat{\mathbf{z}}). \end{aligned} \quad (6)$$

Since the vector $d\mathbf{v}/dt$ lies in the horizontal plane, it is easy to work out the cross-product in the right term (or just use the identity $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}$) to obtain

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \Omega\hat{\mathbf{z}} \times \mathbf{v} - \left(\frac{ma^2}{I} \right) \frac{d\mathbf{v}}{dt} \\ \implies \frac{d\mathbf{v}}{dt} &= \left(\frac{\Omega}{1 + (ma^2/I)} \right) \hat{\mathbf{z}} \times \mathbf{v}. \end{aligned} \quad (7)$$

For a uniform sphere, $I = (2/5)ma^2$, so we obtain

$$\frac{d\mathbf{v}}{dt} = \left(\frac{2}{7}\Omega\right) \hat{\mathbf{z}} \times \mathbf{v}. \quad (8)$$

Therefore, in view of eq. (4), we see that the ball undergoes circular motion, with a frequency equal to $2/7$ times the frequency of the turntable. This result for the frequency does not depend on initial conditions.

REMARKS:

1. Integrating eq. (8) from the initial time to some later time gives

$$\mathbf{v} - \mathbf{v}_0 = \left(\frac{2}{7}\Omega\right) \hat{\mathbf{z}} \times (\mathbf{r} - \mathbf{r}_0). \quad (9)$$

This may be written (as you can show) in the more suggestive form,

$$\mathbf{v} = \left(\frac{2}{7}\Omega\right) \hat{\mathbf{z}} \times \left(\mathbf{r} - \left(\mathbf{r}_0 + \frac{7}{2\Omega}(\hat{\mathbf{z}} \times \mathbf{v}_0)\right)\right). \quad (10)$$

This equation describes circular motion, with the center located at the point,

$$\mathbf{r}_c = \mathbf{r}_0 + \frac{7}{2\Omega}(\hat{\mathbf{z}} \times \mathbf{v}_0), \quad (11)$$

and with radius,

$$R = |\mathbf{r}_0 - \mathbf{r}_c| = \frac{7}{2\Omega}|\hat{\mathbf{z}} \times \mathbf{v}_0| = \frac{7v_0}{2\Omega}. \quad (12)$$

2. There are a few special cases to consider:

- If $v_0 = 0$ (that is, if the spinning motion of the ball exactly cancels the rotational motion of the turntable), then $R = 0$ and the ball remains in the same place (of course).
- If the ball is initially not spinning, and just moving along with the turntable, then $v_0 = \Omega r_0$. The radius of the circle is therefore $R = (7/2)r_0$, and its center is located at (from eq. (11))

$$\mathbf{r}_c = \mathbf{r}_0 + \frac{7}{2\Omega}(-\Omega\mathbf{r}_0) = -\frac{5\mathbf{r}_0}{2}. \quad (13)$$

The point on the circle diametrically opposite to the initial point is therefore at a radius $r_c + R = (5/2)r_0 + (7/2)r_0 = 6r_0$.

- If we want the center of the circle be the center of the turntable, then eq. (11) says that we need $(7/2\Omega)\hat{\mathbf{z}} \times \mathbf{v}_0 = -\mathbf{r}_0$. This implies that \mathbf{v}_0 has magnitude $v_0 = (2/7)\Omega r_0$ and points tangentially in the same direction as the turntable moves. (That is, the ball moves at $2/7$ times the velocity of the turntable beneath it.)
3. The fact that the frequency $(2/7)\Omega$ is a rational multiple of Ω means that the ball will eventually return to the same point on the turntable. In the lab frame, the ball will trace out two circles in the time it takes the turntable to undergo seven revolutions. From the point of view of someone on the turntable, the ball will “spiral” around five times before returning to the original position.
 4. If we look at a ball with moment of inertia $I = \eta ma^2$ (so a uniform sphere has $\eta = 2/5$), then you can show that the “ $2/7$ ” in eq. (8) gets replaced by “ $\eta/(1 + \eta)$ ”. If a ball has most of its mass concentrated at its center (so that $\eta \rightarrow 0$), then the frequency of the circular motion goes to 0, and the radius goes to ∞ (as long as $v_0 \neq 0$).