Solution

Week 22 (2/10/03)

Trading envelopes

(a) Let your envelope contain N dollars. Then the other envelope contains either $2N$ or $N/2$ dollars. If you switch, the expected value of your assets is $\frac{1}{2}(2N)+\frac{1}{2}(N/2)=5N/4$. This is greater than N. Therefore, you should switch.

(b) There are (at least) two possible modes of reasoning, yielding different results:

- It seems that we should be able to use the same reasoning as in part (a). If you have N dollars in your envelope, then the other one has either $2N$ or $N/2$. Since you had a 50-50 chance of picking either envelope, the other envelope should have a 50-50 chance of containing $2N$ or $N/2$ dollars. If you switch, there is a 1/2 chance you win N dollars, and a 1/2 chance you lose $N/2$ dollars. Therefore, the expectation value for your gain is $N/4$ dollars. So you should switch.
- If the correct strategy is to switch (that is, if there is an average gain from trading), then if person A picks one envelope and person B picks the other, then they are both better off if they switch. This cannot be true. Likewise, it cannot be true that they are both better off if they do not switch. Therefore, it doesn't matter whether or not they switch.

The second reasoning is correct. The flaw in the first reasoning is that the other envelope does *not* have a 50-50 chance of containing $2N$ or $N/2$ dollars. Such a 50-50 distribution would yield a zero probability of the envelopes containing a finite and nonzero quantity (as we'll explain below). In a nutshell, it is incorrect to assume that because you have a 50-50 chance of picking each envelope, the envelope you don't pick has a 50-50 chance of having twice or half the amount in your envelope.

(c) As we have stated, the fundamental difference between the scenarios in parts (a) and (b) is that the second envelope in scenario (b) does not have a 50-50 chance of containing half or twice the amount in your envelope. Let's look at this further.

Consider the following slightly modified game, which has all the essentials of the original one. Consider a game where powers of 2 (positive, negative, or zero) are the only numbers of dollars allowed in the envelopes.

The fact that in scenario (b) there is not a 50-50 chance that the other envelope has $2N$ or $N/2$ dollars is most easily seen by looking at the simplest distribution of money in the envelopes, the case where only two quantities are used. Let's say that I always put \$4 in one envelope and \$8 in the other. (And assume that you have a bad memory and can't remember anything from one game to the next.) If your strategy is to switch, and if you initially have \$4, then you will definitely win \$4 on the switch. And if you initially have \$8, then you will definitely lose \$4 on the switch. Since you have a 50-50 chance of starting with the \$4 or \$8 envelope, you will on average neither win nor lose any money. In this example, it is clear that if you have, for example, \$4, there is *not* a 50-50 chance that the other envelope contains \$2 or \$8. Rather, there is a 100% chance that it contains \$8.

You can try make a situation in scenario (b) that comes "close" to always having a 50-50 chance that the other envelope has twice or half the amount in your envelope. For example, let there be a $1/n$ chance that the envelopes contain 2^k and 2^{k+1} dollars, for all k from 1 to n. Then indeed if there are 2^m dollars in your envelope, for $m = 2, \ldots, n-1$, then there is a 50-50 chance that the other envelope has twice or half that amount. In all these $n-2$ cases, you will win money, on average, if you switch. And you will certainly win money if you switch in the case where you have the minimum amount, 2^1 dollars. You will, however, lose a great deal of money if you happen to start out with 2^{n+1} dollars. This only happens $1/(2n)$ of the time, but it in fact precisely cancels, on average, the winnings from all the other $n-1$ cases (as you can show). Therefore, it doesn't matter if you switch.

If you want to produce a 50-50 chance that the other envelope has twice or half the amount in your envelope for all m, then you have to assign equal probabilities to all of the $(2^k, 2^{k+1})$ pairs, for $-\infty < k < \infty$. But the assignment of equal probabilities to an infinite set requires that all of these probabilities are zero, which means that there is a zero chance of putting a finite amount of money in the envelopes. Since it's stated that there *is* some amount of money in the envelopes, we conclude that all the probabilities of the $(2^k, 2^{k+1})$ pairs are not equal. The setup in part (b) is therefore not the same as in part (a), and there is no paradox.