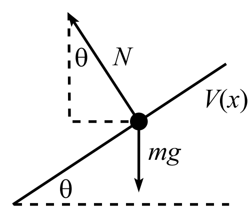
Solution

Week 23 (2/17/03)

$V(x)$ **versus a hill**

Quick solution: Consider the normal force, N, acting on the bead at a given point. Let θ be the angle that the tangent to $V(x)$ makes with the horizontal, as shown.



The horizontal $F = ma$ equation is

$$-N\sin\theta = m\ddot{x}. \tag{1}$$

The vertical $F = ma$ equation is

$$N\cos\theta - mg = m\ddot{y} \qquad \Longrightarrow \qquad N\cos\theta = mg + m\ddot{y}. \tag{2}$$

Dividing eq. (1) by eq. (2) gives

$$-\tan\theta = \frac{\ddot{x}}{g+\ddot{y}}\,. \tag{3}$$

But $\tan\theta = V'(x)$. Therefore,

$$\ddot{x} = -(g+\ddot{y})V'. \tag{4}$$

We see that this is not equal to $-gV'$. In fact, there is in general no way to construct a curve with height $y(x)$ which gives the same horizontal motion that a 1-D potential $V(x)$ gives, for all initial conditions. We would need $(g+\ddot{y})y' = V'$, for all x. But at a given x, the quantities V' and y' are fixed, whereas $\ddot{y}$ depends on the initial conditions. For example, if there is a bend in the wire, then $\ddot{y}$ will be large if $\dot{y}$ is large. And $\dot{y}$ depends (in general) on how far the bead has fallen.

Eq. (4) holds the key to constructing a situation that does give the $\ddot{x} = -gV'$ result for a 1-D potential $V(x)$. All we have to do is get rid of the $\ddot{y}$ term. So here's what we do. We grab our $y = V(x)$ wire and then move it up (and/or down) in precisely the manner that makes the bead stay at the same height with respect to the ground. (Actually, constant vertical speed would be good enough.) This will make the $\ddot{y}$ term vanish, as desired. (Note that the vertical movement of the curve doesn't change the slope, V', at a given value of x.)

REMARK: There is one case where $\ddot{x}$ is (approximately) equal to $-gV'$, even when the wire remains stationary. In the case of small oscillations of the bead near a minimum of $V(x)$, $\ddot{y}$ is small compared to g. Hence, eq. (4) shows that $\ddot{x}$ is approximately equal to

$-gV'$. Therefore, for small oscillations, it is reasonable to model a particle in a 1-D potential $mgV(x)$ as a particle sliding in a valley whose height is given by $y = V(x)$.

Long solution: The component of gravity along the wire is what causes the change in speed of the bead. That is,

$$-g\sin\theta = \frac{dv}{dt}\,, \tag{5}$$

where θ is given by

$$\tan\theta = V'(x) \qquad \Longrightarrow \qquad \sin\theta = \frac{V'}{\sqrt{1+V'^2}}\,, \qquad \cos\theta = \frac{1}{\sqrt{1+V'^2}}\,. \tag{6}$$

We are, however, not concerned with the rate of change of v, but rather with the rate of change of $\dot{x}$. In view of this, let us write v in terms of $\dot{x}$. Since $\dot{x} = v\cos\theta$, we have $v = \dot{x}/\cos\theta = \dot{x}\sqrt{1+V'^2}$. (Dots denote d/dt. Primes denote d/dx.) Therefore, eq. (5) becomes

$$\begin{aligned} \frac{-gV'}{\sqrt{1+V'^2}} &= \frac{d}{dt}\Big(\dot{x}\sqrt{1+V'^2}\Big) \\ &= \ddot{x}\sqrt{1+V'^2} + \frac{\dot{x}V'(dV'/dt)}{\sqrt{1+V'^2}}\,. \end{aligned} \tag{7}$$

Hence, $\ddot{x}$ is given by

$$\ddot{x} = \frac{-gV'}{1+V'^2} - \frac{\dot{x}V'(dV'/dt)}{1+V'^2}\,. \tag{8}$$

We'll simplify this in a moment, but first a remark.

Remark: A common incorrect solution to this problem is the following. The acceleration along the curve is $g\sin\theta = -g(V'/\sqrt{1+V'^2})$. Calculating the horizontal component of this acceleration brings in a factor of $\cos\theta = 1/\sqrt{1+V'^2}$. Therefore, we might think that

$$\ddot{x} = \frac{-gV'}{1+V'^2}\,. \tag{9}$$

But we have missed the second term in eq. (8). Where is the mistake? The error is that we forgot to take into account the possible change in the curve's slope. (Eq. (9) is true for straight lines.) We addressed only the acceleration due to a change in *speed*. We forgot about the acceleration due to a change in the *direction* of motion. (The term we missed is the one with dV'/dt.) Intuitively, if we have sharp enough bend in the wire, then $\dot{x}$ can change at an arbitrarily large rate. In view of this fact, eq. (9) is definitely incorrect, because it is bounded (by $g/2$, in fact).

To simplify eq. (8), note that $V' \equiv dV/dx = (dV/dt)/(dx/dt) \equiv \dot{V}/\dot{x}$. Therefore,

$$\begin{aligned} \dot{x}V'\frac{dV'}{dt} &= \dot{x}V'\frac{d}{dt}\left(\frac{\dot{V}}{\dot{x}}\right) \\ &= \dot{x}V'\left(\frac{\dot{x}\ddot{V} - \dot{V}\ddot{x}}{\dot{x}^2}\right) \\ &= V'\ddot{V} - V'\ddot{x}\left(\frac{\dot{V}}{\dot{x}}\right) \\ &= V'\ddot{V} - V'^2\ddot{x}. \end{aligned} \tag{10}$$

Substituting this into eq. (8), we obtain

$$\ddot{x} = -(g + \ddot{V})V', \tag{11}$$

in agreement with eq. (4), since $y(x) = V(x)$.

Eq. (11) is valid for a curve $V(x)$ that remains fixed. If we grab the wire and start moving it up and down, then the above solution is invalid, because the starting point, eq. (5), rests on the assumption that gravity is the only force that does work on the bead. But if we move the wire, then the normal force also does work.

It turns out that for a moving wire, we simply need to replace the $\ddot{V}$ in eq. (11) by $\ddot{y}$. This can be seen by looking at things in the (instantaneously inertial) vertically-moving frame in which the wire is at rest. In this new frame, the normal force does no work, so the above solution is valid. And in this new frame, $\ddot{y} = \ddot{V}$. Eq. (11) therefore becomes $\ddot{x} = -(g + \ddot{y})V'$. Shifting back to the lab frame (which moves at constant speed with respect to the instantaneous inertial frame of the wire) doesn't change $\ddot{y}$. We thus arrive at eq. (4), valid for a stationary or vertically moving wire.