Solution

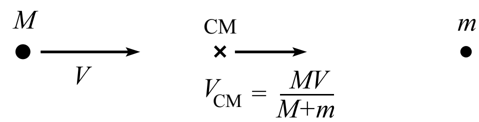
Week 25 (3/3/03)

Maximum deflection angle

First Solution: Although it is possible to solve this problem by working in the lab frame (see the second solution below), it is much easier to make use of the center-of-mass frame. Let M have initial speed V in the lab frame. Then the CM moves with speed

$$V_{\text{CM}} = \frac{MV}{M+m}\,, \tag{1}$$

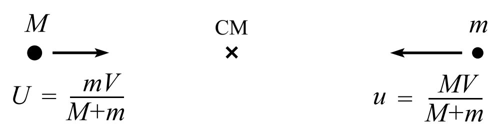
as shown.



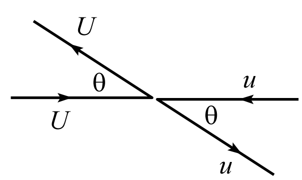
The speeds of the masses in the CM frame are therefore equal to

$$U = V - V_{\text{CM}} = \frac{mV}{M+m}\,, \qquad \text{and} \qquad u = |-V_{\text{CM}}| = \frac{MV}{M+m}\,, \tag{2}$$

as shown.

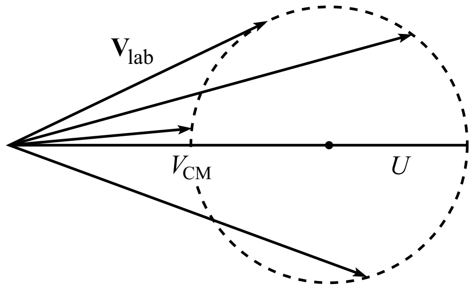


In the CM frame, the collision is simple. The particles keep the same speeds, but simply change their directions (while still moving in opposite directions), as shown.

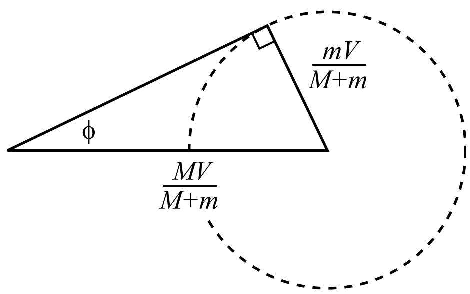


The angle θ is free to have any value. This scenario clearly satisfies conservation of energy and momentum; therefore, it is what happens.

The important point to note is that since θ can have any value, the tip of the $\mathbf{U}$ velocity vector can be located anywhere on a circle of radius U. If we then shift back to the lab frame, we see that the final velocity of M with respect to the lab frame, $\mathbf{V}_{\text{lab}}$, is obtained by adding $\mathbf{V}_{\text{CM}}$ to the vector $\mathbf{U}$ (which can point anywhere on the dotted circle below). A few possibilities for $\mathbf{V}_{\text{lab}}$ are shown.



The largest angle of deflection is obtained when $\mathbf{V}_{\rm lab}$ is tangent to the dotted circle, in which case we have the following situation.



The maximum angle of deflection, ϕ, is therefore given by

$$\sin\phi = \frac{U}{V_{\rm CM}} = \frac{\frac{mV}{M+m}}{\frac{MV}{M+m}} = \frac{m}{M}\,. \tag{3}$$

Second Solution: Let V' and v' be the final speeds, and let ϕ and γ be the scattering angles of M and m, respectively, in the lab frame. Then conservation of p_x, p_y, and E give

$$\begin{aligned} MV &= MV'\cos\phi + mv'\cos\gamma, && (4)\\ 0 &= MV'\sin\phi - mv'\sin\gamma, && (5)\\ \frac{1}{2}MV^2 &= \frac{1}{2}MV'^2 + \frac{1}{2}mv'^2. && (6) \end{aligned}$$

Putting the ϕ terms on the left-hand sides of eqs. (4) and (5), and then squaring and adding these equations, gives

$$M^2(V^2 + V'^2 - 2VV'\cos\phi) = m^2v'^2. \tag{7}$$

Equating this expression for $m^2v'^2$ with the one obtained by multiplying eq. (6) through by m gives

$$\begin{aligned} & M(V^2 + V'^2 - 2VV'\cos\phi) &=&\; m(V^2 - V'^2) \\ \Longrightarrow\quad & (M+m)V'^2 - (2MV\cos\phi)V' + (M-m)V^2 &=&\; 0. \qquad (8) \end{aligned}$$

A solution to this quadratic equation in V' exists if and only if the discriminant is non-negative. Therefore, we must have

$$
\begin{aligned}
(2MV\cos\phi)^2 - 4(M+m)(M-m)V^2 &\geq 0 \\
\Longrightarrow \qquad m^2 &\geq M^2(1-\cos^2\phi) \\
\Longrightarrow \qquad m^2 &\geq M^2\sin^2\phi \\
\Longrightarrow \qquad \frac{m}{M} &\geq \sin\phi. \qquad (9)
\end{aligned}
$$

REMARKS: If $M < m$, then eq. (9) says that any value of ϕ is possible. In particular, it it possible for M to bounce directly backwards. In the language of the first solution above, if $M < m$ then $V_{\rm CM} < U$, so the dotted circle passes to the left of the left vertex of the triangle. This means that ϕ can take on any value.

The method of the first solution provides an easy way to demonstrate the result that if the two masses are equal, then they always scatter at a relative angle of 90° (a familiar result in billiards). If $M = m$, then

$$u = U = V_{\rm CM} = \frac{MV}{M+M} = \frac{V}{2}\,. \qquad (10)$$

Therefore, the **u** and **U** vectors in the figure below form a diameter of the dotted circle, which means that the final velocities of M and m in the lab frame are perpendicular.