Solution

Week 26 (3/10/03)

Drunken walk

(a) **First Solution:** Imagine a large number of copies of the given setup proceeding simultaneously. After each drunk takes his first step in all of the copies, the average position of all of them remains the same (namely, n steps from the river), because each one had a 50-50 chance of moving either way. Likewise, the average position remains unchanged after each successive step. This is true because the drunks who are still moving won't change the average position (because of their random motion), and the drunks who have stopped at an end of the street certainly won't change the average position (because they aren't moving). Therefore, the average position is always n steps from the river.

Let the drunks keep moving until all of them have stopped at either end. Let $P_r(n)$ and $P_p(n)$ be the probabilities of ending up at the river and police station, respectively, having started n steps from the river. Then after all the drunks have stopped, their average distance from the river is $0 \cdot P_r(n) + N \cdot P_p(n)$. But this must equal n. Hence, $P_p(n) = n/N$, and so $P_r(N) = 1 - (n/N)$.

Second Solution: Let the river and police station be located at positions 0 and N, respectively. Let $P_r(n)$ be the probability of ending up at the river, given a present position of n. Since after one step the drunk is equally likely to be at $n-1$ and $n+1$, we must have

$$P_r(n) = \frac{1}{2}P_r(n-1) + \frac{1}{2}P_r(n+1). \tag{1}$$

Therefore, $P_r(n)$ is a linear function of n. Invoking the requirements that $P_r(0) = 1$ and $P_r(N) = 0$, we find $P_r(n) = 1 - n/N$. The probability of ending up at the police station, $P_p(n)$, is then $P_p(n) = 1 - P_r(n) = n/N$.

(b) Let $g(k)$ be the expected number of steps it takes to reach an end of the street, having started k steps from the river. After one step from position k, there is a 1/2 chance of being at position $k-1$, and a 1/2 chance of being at position $k+1$. Therefore, the $g(k)$ are related by

$$g(k) = \frac{1}{2}g(k-1) + \frac{1}{2}g(k+1) + 1, \tag{2}$$

where $g(0) = g(N) = 0$. We must now solve this recursion relation.

Multiplying through by 2, and then summing all the eqs. (2) for values of k from 1 to m gives

$$\begin{aligned} g(1) + g(m) &= g(m+1) + 2m \\ \Longrightarrow \quad g(m+1) &= g(1) + g(m) - 2m. \end{aligned} \tag{3}$$

Note that if we set $m = N-1$, we obtain $0 = g(1) + g(N-1) - 2(N-1)$. Since $g(1) = g(N-1)$ by symmetry, we find $g(1) = g(N-1) = N-1$.

Summing all the eqs. (3) for values of m from 1 to $n-1$ gives

$$\begin{aligned} g(n) &= n \cdot g(1) - 2\sum_{1}^{n-1} m \\ &= n \cdot g(1) - n(n-1). \end{aligned} \tag{4}$$

Using $g(1) = N-1$, we find

$$\begin{aligned} g(n) &= n(N-1) - n(n-1) \\ &= n(N-n). \end{aligned} \tag{5}$$

This can be written as

$$g(n) = \left(\frac{N}{2}\right)^2 - \left(\frac{N}{2} - n\right)^2, \tag{6}$$

which is just an inverted parabola.