Solution

Week 27 (3/17/03)

Relativistic cookies

Let the diameter of the cookie cutter be L, and consider the two following reasonings.

- In the lab frame, the dough is length-contracted, so the diameter L corresponds to a distance larger than L (namely γL) in the dough's frame. Therefore, when you buy a cookie, it is stretched out by a factor γ in the direction of the belt.[1]

- In the frame of the dough, the cookie cutter is length-contracted in the direction of motion. It has length L/γ. So in the frame of the dough, the cookies have a length of only L/γ. Therefore, when you buy a cookie, it is squashed by a factor γ in the direction of the belt.

Which reasoning is correct? The first one is. The cookies are stretched out. The fallacy in the second reasoning is that the various parts of the cookie cutter do *not* strike the dough simultaneously in the dough frame. What the dough sees is this: The cutter moves to, say, the left. The right side of the cutter stamps the dough, then nearby parts of the cutter stamp it, and so on, until finally the left side of the cutter stamps the dough. But by this time the front (that is, the left) of the cutter has moved farther to the left. So the cookie turns out to be longer than L. It takes a little work to demonstrate that the length is actually γL, but let's do that now (by working in the dough frame).

Consider the moment when the the rightmost point of the cutter strikes the dough. In the dough frame, a clock at the rear (the right side) of the cutter reads Lv/c^2 more than a clock at the front (the left side). The front clock must therefore advance by Lv/c^2 by the time it strikes the dough. (This is true because all points on the cutter strike the dough simultaneously in the cutter frame. Hence, all cutter clocks read the same when they strike.) But due to time dilation, this takes a time $\gamma(Lv/c^2)$ in the dough frame. During this time, the cutter travels a distance $v(\gamma Lv/c^2)$. Since the front of the cutter was initially a distance L/γ (due to length contraction) ahead of the back, the total length of the cookie in the dough frame equals

$$\begin{aligned} \ell &= \frac{L}{\gamma} + v\left(\frac{\gamma L v}{c^2}\right) \\ &= \gamma L\left(\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right) \\ &= \gamma L\left(\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\right) \\ &= \gamma L, \end{aligned}$$

as we wanted to show.

[1]The shape is an ellipse, since that's what a stretched-out circle is. The eccentricity of an ellipse is the focal distance divided by the semi-major axis length. As an exercise, you can show that this equals $\beta \equiv v/c$ here.