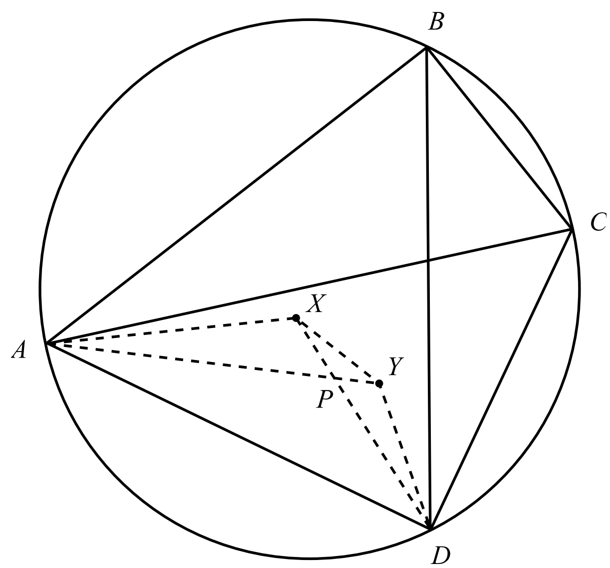
Solution

Week 28   (3/24/03)

Rectangle in a circle

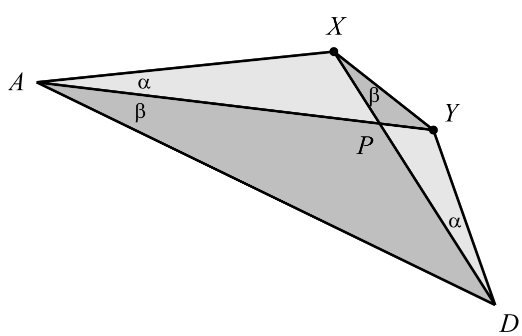
In the figure below, let the incenters of triangles ADB and ADC be X and Y, respectively.



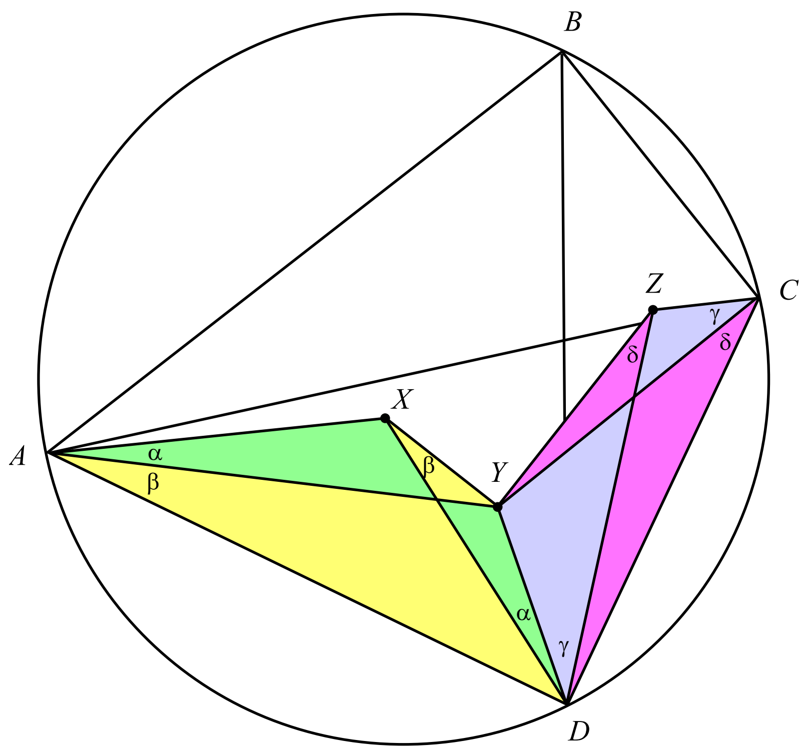
Angle $\angle XAY$ can be written as

$$\begin{aligned}\angle XAY &= \angle XAD - \angle YAD \\ &= \frac{1}{2}\angle BAD - \frac{1}{2}\angle CAD \\ &= \frac{1}{2}\angle BAC \\ &= \frac{1}{4}(\widehat{BC}).\end{aligned}$$

A similar argument (with A, B, X interchanged with D, C, Y) shows that angle $\angle YDX$ also equals $(1/4)(\widehat{BC})$. This equality of angles $\angle XAY$ and $\angle YDX$ implies that triangles XAP and YDP are similar. This in turn implies that triangles PXY and PAD are similar. Therefore, $\angle PXY = \angle PAD$. These results may be summarized in the following figure.



We may now repeat the above procedure with the incenters (Y and Z) of triangles DCA and DCB. The result is two more pairs of equal angles, as shown.



The four angles shown have the values,

$$\begin{aligned}
\alpha &= (1/4)(\widehat{BC}),\\
\beta &= (1/2)\angle CAD = (1/4)(\widehat{CD}),\\
\gamma &= (1/4)(\widehat{AB}),\\
\delta &= (1/2)\angle ACD = (1/4)(\widehat{AD}).
\end{aligned}$$

Therefore,

$$\alpha + \beta + \gamma + \delta = \frac{1}{4}(\widehat{BC} + \widehat{CD} + \widehat{AB} + \widehat{AD}) = \frac{1}{4}(360^\circ) = 90^\circ. \quad (1)$$

We now note that angle $\angle XYZ$ is given by

$$\begin{aligned}
\angle XYZ &= 360^\circ - \angle XYD - \angle ZYD\\
&= 360^\circ - (180^\circ - \alpha - \beta) - (180^\circ - \gamma - \delta)\\
&= \alpha + \beta + \gamma + \delta\\
&= 90^\circ.
\end{aligned}$$

The same reasoning holds for the other three vertices of the incenter quadrilateral. Therefore, this quadrilateral is a rectangle, as we wanted to show.