Solution

Week 29 (3/31/03)

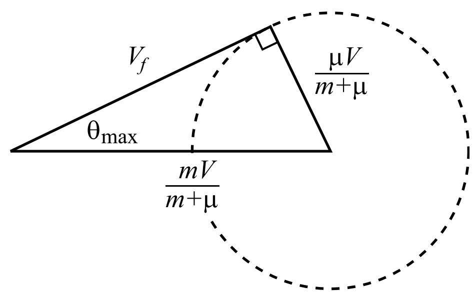
Balls in a semicircle

(a) Let $\mu \equiv M/N$ be the mass of each ball in the semicircle. We need the deflection angle in each collision to be $\theta = \pi/N$. However, if the ratio μ/m is too small, then this angle of deflection is not possible.

From Problem of the Week #25, the maximal angle of deflection in each collision is given by $\sin\theta = \mu/m$. (We'll just invoke this result here.) Since we want $\theta = \pi/N$ here, this $\sin\theta \le \mu/m$ condition becomes (using $\sin\theta \approx \theta$, for the small angle θ)

$$\theta \le \frac{\mu}{m} \qquad \Longrightarrow \qquad \frac{\pi}{N} \le \frac{M/N}{m} \qquad \Longrightarrow \qquad \pi \le \frac{M}{m}\,. \tag{1}$$

(b) Referring back to the solution to Problem #25, we see that m's speed after the first bounce is obtained from the following figure.



Looking at the right triangle, we see that the speed after the bounce is

$$V_f = V\,\frac{\sqrt{m^2-\mu^2}}{m+\mu}\,. \tag{2}$$

To first order in the small quantity μ/m, this equals

$$V_f \approx \frac{mV}{m+\mu} \approx V\left(1-\frac{\mu}{m}\right). \tag{3}$$

The same reasoning holds for each successive bounce, so the speed decreases by a factor of $(1-\mu/m)$ after each bounce. In the minimum M/m case found in part (a), we have

$$\frac{\mu}{m} = \frac{M/N}{m} = \frac{M/m}{N} = \frac{\pi}{N}\,. \tag{4}$$

Therefore, the ratio of m's final speed to initial speed equals

$$\frac{V_{\text{final}}}{V_{\text{initial}}} \approx \left(1-\frac{\pi}{N}\right)^N \approx e^{-\pi}. \tag{5}$$

That's a nice result, if there ever was one! Since $e^{-\pi}$ is roughly equal to 1/23, only about 4% of the initial speed remains.