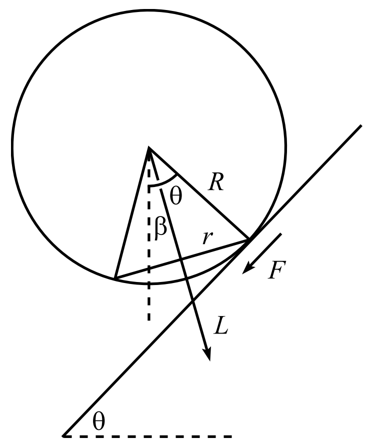
Solution

Week 33 (4/28/03)

Ball rolling in a cone

It turns out that the ball can move arbitrarily fast around the cone. As we will see, the plane of the contact circle (represented by the chord in the figure below) will need to be tilted downward from the contact point, so that the angular momentum has a rightward horizontal component, as shown.



Let's first look at $F = ma$ along the plane. Let Ω be the angular frequency of the ball's motion around the cone. Then the ball's horizontal acceleration is $m\ell\Omega^2$ to the left. So $F = ma$ along the plane gives (where F_f is the friction force)

$$mg\sin\theta + F_f = m\ell\Omega^2\cos\theta. \tag{1}$$

Now let's look at $\boldsymbol{\tau} = d\mathbf{L}/dt$. To get a handle on how fast the ball is spinning, consider what the setup looks like in the rotating frame in which the center of the ball is stationary (so the ball just spins in place as the cone spins around). Since there is no slipping, the contact points on the ball and the cone must have the same speed. That is,

$$\omega r = \Omega\ell \qquad \Longrightarrow \qquad \omega = \frac{\Omega\ell}{r}\,, \tag{2}$$

where ω is the angular speed of the ball in the rotating frame, and r is the radius of the contact circle on the ball.[1] The angular momentum of the ball in the lab frame equals $L = I\omega$ (at least for the purposes here[2]), and it points in the direction shown above.

[1]If the center of the ball travels in a circle of radius ℓ, then the ℓ here should actually be replaced with $\ell + R\sin\theta$, which is the radius of the contact circle on the cone. But since we're assuming that $R \ll \ell$, we can ignore the $R\sin\theta$ part.

[2]This $L = I\omega$ result isn't quite correct, because the angular velocity of the ball in the lab frame equals the angular velocity in the rotating frame (which tilts downwards with the ω magnitude we just found) plus the angular velocity of the rotating frame with respect to the lab frame (which points straight up with magnitude Ω). This second part of the angular velocity simply yields an additional vertical component of the angular momentum. But the vertical component of $\mathbf{L}$ doesn't change with time as the ball moves around the cone. It is therefore irrelevant, since we will be concerned only with $d\mathbf{L}/dt$ in what follows.

The $\mathbf{L}$ vector precesses around a cone in $\mathbf{L}$-space with the same frequency, Ω, as the ball moves around the cone. Only the horizontal component of $\mathbf{L}$ changes, and it traces out a circle of radius $L_{\text{hor}} = L \sin\beta$, at frequency Ω. Therefore,

$$\left|\frac{d\mathbf{L}}{dt}\right| = L_{\text{hor}}\Omega = (I\omega \sin\beta)\Omega = \frac{I\Omega^2 \ell \sin\beta}{r}\,, \tag{3}$$

and the direction of $d\mathbf{L}/dt$ is into the page.

The torque on the ball (relative to its center) is due to the friction force, F_f. Hence, $|\boldsymbol{\tau}| = F_f R$, and its direction is into the page. Therefore, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives (with $I = \eta m R^2$, where $\eta = 2/5$ in this problem)

$$\begin{aligned} F_f R &= \frac{I\Omega^2 \ell \sin\beta}{r} \\ \Longrightarrow \qquad F_f &= \frac{\eta m R \Omega^2 \ell \sin\beta}{r}\,. \end{aligned} \tag{4}$$

Using this F_f in eq. (1) gives

$$mg\sin\theta + \frac{\eta m R\Omega^2 \ell \sin\beta}{r} = m\ell\Omega^2 \cos\theta. \tag{5}$$

Solving for Ω gives

$$\Omega^2 = \frac{g\sin\theta}{\ell\left(\cos\theta - \frac{\eta R \sin\beta}{r}\right)}\,. \tag{6}$$

We see that it is possible for the ball to move around the cone infinitely fast if

$$\cos\theta = \frac{\eta \sin\beta}{x}\,, \tag{7}$$

where $x \equiv r/R$. But from the above figure, we see that β is given by

$$\beta = \theta - \sin^{-1}(r/R). \tag{8}$$

Therefore, eq. (7) gives

$$\begin{aligned} \cos\theta &= \frac{\eta}{x}\sin(\theta - \sin^{-1} x) \\ \Longrightarrow \qquad x\cos\theta &= \eta\sin\theta\cos(\sin^{-1} x) - \eta\cos\theta\sin(\sin^{-1} x) \\ \Longrightarrow \qquad x\cos\theta &= \eta\sin\theta\sqrt{1-x^2} - \eta\cos\theta x \\ \Longrightarrow \qquad x(1+\eta)\cos\theta &= \eta\sin\theta\sqrt{1-x^2}. \end{aligned} \tag{9}$$

Squaring and solving for x^2 gives

$$x^2 = \frac{\eta^2 \sin^2\theta}{(1+\eta)^2\cos^2\theta + \eta^2\sin^2\theta}\,. \tag{10}$$

In the problem at hand, we have $\eta = 2/5$, so

$$\frac{r}{R} \equiv x = \sqrt{\frac{4\sin^2\theta}{49\cos^2\theta + 4\sin^2\theta}}\,. \tag{11}$$

Remarks:

1. What value of θ allows largest the tilt angle of the contact circle (that is, the largest β)? From eq. (7), we see that maximizing β is equivalent to maximizing $x\cos\theta$, or equivalently $x^2\cos^2\theta$. Using the value of x^2 in eq. (10), we see that we want to maximize
$$x^2\cos^2\theta = \frac{\eta^2\sin^2\theta\cos^2\theta}{(1+\eta)^2\cos^2\theta + \eta^2\sin^2\theta}\,. \tag{12}$$
Taking the derivative with respect to θ and going through a bit of algebra, we find that the maximum is achieved when
$$\sin\theta = \sqrt{\frac{1+\eta}{1+2\eta}} = \sqrt{\frac{7}{9}} \qquad \Longrightarrow \qquad \theta = 61.9^\circ. \tag{13}$$
You can then show that
$$\sin\beta_{\text{max}} = \frac{1}{1+2\eta} = \frac{5}{9} \qquad \Longrightarrow \qquad \beta_{\text{max}} = 33.7^\circ. \tag{14}$$

2. Let's consider three special cases for the contact circle, namely, when it is a horizontal circle, a great circle, or a vertical circle.

   (a) *Horizontal circle*: In this case, we have $\beta = 0$, so eq. (6) gives
   $$\Omega^2 = \frac{g\tan\theta}{\ell}\,. \tag{15}$$
   In this case, $\mathbf{L}$ points vertically, which means that $d\mathbf{L}/dt$ is zero, which means that the torque is zero, which means that the friction force is zero. Therefore, the ball moves around the cone with the same speed as a particle sliding without friction. (You can show that such a particle does indeed have $\Omega^2 = g\tan\theta/\ell$.) The horizontal contact-point circle ($\beta = 0$) is the cutoff case between the sphere moving faster or slower than a frictionless particle.

   (b) *Great circle*: In this case, we have $r = R$ and $\beta = -(90^\circ - \theta)$. Hence, $\sin\beta = -\cos\theta$, and eq. (6) gives
   $$\Omega^2 = \frac{g\tan\theta}{\ell(1+\eta)}\,. \tag{16}$$
   This reduces to the frictionless-particle case when $\eta = 0$, as it should.

   (c) *Vertical circle*: In this case, we have $r = R\cos\theta$ and $\beta = -90^\circ$, so eq. (6) gives
   $$\Omega^2 = \frac{g\tan\theta}{\ell\left(1+\frac{\eta}{\cos^2\theta}\right)}\,. \tag{17}$$
   Again, this reduces to the frictionless-particle case when $\eta = 0$, as it should. But for $\theta \to 90^\circ$, Ω goes to zero, whereas in the other two cases above, Ω goes to ∞.