*Solution*

Week 34 (5/5/03)

**Counterfeit coin**

(a) Note that there are three possible outcomes to each weighing: left side heavier, right side heavier, or both sides equal. In order to do the given task in as few weighings as possible, we will need as much information from each weighing as possible. Hence, all three possibilities should be realizable for each weighing (except for the final weighing in some scenarios, as we will see below). So, for example, an initial weighing of six coins against six coins is probably not a good idea, because it is not possible for the scale to balance. We should expect to have to switch coins from one side of the scale to the other, from one weighing to the next, in order to make the three possibilities realizable for a given weighing. Having said that, here is one scheme that does the task in three weighings (there are other variations that also work):

Weigh four coins (labelled $A_1$, $A_2$, $A_3$, $A_4$) against four others ($B_1$, $B_2$, $B_3$, $B_4$). The remaining four will be labelled $C_1$, $C_2$, $C_3$, $C_4$. There are three possible outcomes to this weighing:

(1) The $A$ group is heavier than the $B$ group. We know in this case that the $C$ coins are "good", and the "bad" coin is either an $A$ or a $B$. If the bad coin is an $A$, it is heavy. If the bad coin is a $B$, it is light.

Now weigh $(A_1, A_2, B_1)$ against $(A_3, A_4, B_2)$. There are three possible outcomes:

- If the $(A_1, A_2, B_1)$ side is heavier, the bad coin must be $A_1$, $A_2$, or $B_2$. Weigh $A_1$ against $A_2$. If $A_1$ is heavier, it is the bad (heavy) coin; if $A_2$ is heavier, it is the bad (heavy) coin; if they are equal, $B_1$ is the bad (light) coin.
- If the $(A_3, A_4, B_2)$ side is heavier, the bad coin must be $A_3$, $A_4$, or $B_1$. Use the same strategy as in the previous case.
- If they are equal, the bad coin must be $B_3$ or $B_4$. Simply weigh $B_3$ against a good coin.

(2) The $B$ group is heavier than the $A$ group. This case is the same as the previous one, but with "heavy" switched with "light".

(3) The $A$ and $B$ groups balance. So the bad coin is a $C$.

Weigh $(C_1, C_2)$ against ($C_3$, good-coin). There are three possible outcomes:

- If the $(C_1, C_2)$ side is heavier, weigh $C_1$ against $C_2$. If $C_1$ is heavier, it is the bad (heavy) coin; if $C_2$ is heavier, it is the bad (heavy) coin; if they are equal, $C_3$ is the bad (light) coin.
- If the ($C_3$, good-coin) side is heavier, this is equivalent to the previous case, with "heavy" switched with "light".
- If they are equal, the bad coin is $C_4$. Weigh $C_4$ against a good coin to determine if it is heavy or light.

(b) **Lemma:** *Let there be $N$ coins, about which our information is the following: The $N$ coins may be divided into two sets, $\{H\}$ and $\{L\}$, such that i) if a coin is in $\{H\}$ and it turns out to be the bad coin, it is heavy; and ii) if a coin is in $\{L\}$ and it turns out to be the bad coin, it is light. Then, given $n$ weighings, the maximum value of $N$ for which we can identify the bad coin, and also determine whether it is heavy or light, is $N = 3^n$.*

**Proof:** For the case $n = 0$, the lemma is obviously true, because by assumption we know which of the two sets, $\{H\}$ and $\{L\}$, the one coin is in. We will show by induction that the lemma is true for all $n$.

Assume the lemma true for $n$ weighings. Let us show that it is then true for $n + 1$ weighings. We will do this by first showing that $N = 3^{n+1}$ is solvable, and then showing that $N = 3^{n+1} + 1$ is not always solvable.

By assumption, the $N = 3^{n+1}$ coins are divided into $\{H\}$ and $\{L\}$ sets. On both sides of the scale, put $h$ coins from $\{H\}$ and $l$ coins from $\{L\}$, with $h + l = 3^n$. (Either $h$ or $l$ may be zero, if necessary.) There are then $3^n$ coins left over.

There are three possible outcomes to this weighing:

- If the left side is heavier, the bad coin must be one of the $h$ $H$-coins from the left or one of the $l$ $L$-coins from the right.
- If the right side is heavier, the bad coin must be one of the $h$ $H$-coins from the right or one of the $l$ $L$-coins from the left.
- If the scale balances, the bad coin must be one of the $3^n$ leftover coins.

In each of these cases, the problem is reduced to a setup with $3^n$ coins which are divided into $\{H\}$ and $\{L\}$ sets. But this is assumed to be solvable with $n$ weighings, by induction. Therefore, since $N = 3^0 = 1$ is solvable for $n = 0$, we conclude that $N = 3^n$ is solvable for all $n$.

Let us now show that $N = 3^{n+1}+1$ is not always solvable with $n+1$ weighings. Assume inductively that $N = 3^n + 1$ is not always solvable with $n$ weighings. ($N = 3^0 + 1 = 2$ is certainly not solvable for $n = 0$.) For the first weighing, the leftover pile can have at most $3^n$ coins in it, since the bad coin may end up being there. There must therefore be at least $2 \cdot 3^n + 1$ total coins on the scale (which then implies that there must be at least $2 \cdot 3^n + 2$ total coins on the scale, since the number must be even). Depending on how the $\{H\}$ and $\{L\}$ coins are distributed on the scale, the first weighing will (assuming the scale doesn't balance) tell us that the bad coin is either in a subset containing $s$ coins, or in the complementary subset containing $(2 \cdot 3^n + 2) - s$ coins. One of these sets will necessarily have at least $3^n + 1$ coins in it, which by assumption is not necessarily solvable. ∎

Returning to the original problem, let us first consider a modified setup where we have an additional known good coin at our disposal.

**Claim:** *Given $N$ coins and $W$ weighings, and given an additional known good coin, the maximum value for $N$ for which we can identify the bad coin,* *and also determine whether it is heavy or light, is $N_W^{\mathrm{g}} = (3^W - 1)/2$, where the superscript "g" signifies that we have a known good coin available.*

**Proof:** The claim is true for $W = 1$. Assume inductively that it is true for $W$ weighings. We will show that it is then true for $W + 1$ weighings.

In the first of our $W+1$ weighings, we may have (by the inductive assumption) at most $(3^W - 1)/2$ leftover coins not involved in the weighing, since the bad coin may end up being there (in which case we have many good coins from the scale at out disposal).

From the above lemma, we may have at most $3^W$ suspect coins on the scale. We can indeed have this many, if we bring in a known good coin to make the number of weighed coins, $3^W + 1$, even (so that we can have an equal number on each side). If the scale doesn't balance, the $3^W$ suspect coins satisfy the hypotheses of the lemma (they can be divided into $\{H\}$ and $\{L\}$ sets), so if the bad coin is among these $3^W$ coins, it can be determined in $W$ weighings.

Therefore,

$$N_{W+1}^{\mathrm{g}} = N_W^{\mathrm{g}} + 3^W = \frac{3^W - 1}{2} + 3^W = \frac{3^{W+1} - 1}{2}\,, \tag{1}$$

as we wanted to show. ■

**Corollary:** *Given $N$ coins and $W$ weighings (and not having an additional known good coin available), the maximum value of $N$ for which we can identify the bad coin, and also determine whether it is heavy or light, is*

$$N_w^{\mathrm{ng}} = \frac{3^W - 1}{2} - 1, \tag{2}$$

*where the superscript "ng" signifies that we do not have a known good coin available.*

**Proof:** If we are not given a known good coin, the only modification to the reasoning in the above claim is that we can't put a total of $3^W$ suspect coins on the scale, because $3^W$ is odd. We are limited to a total of $3^W - 1$ coins on the scale, so we now obtain

$$N_{W+1}^{\mathrm{ng}} = N_W^{\mathrm{g}} + (3^W - 1) = \frac{3^W - 1}{2} + (3^W - 1) = \frac{3^{W+1} - 1}{2} - 1. \tag{3}$$

Note that if the scale balances, so that we know the bad coin is a leftover coin, then from that point on, we do indeed have a known good coin at our disposal (any coin on the scale), so $N_W^{\mathrm{g}} = (3^W - 1)/2$ is indeed what appears in the above equation. ■

Therefore, $N_W$ is decreased by one if we don't have a known good coin at the start.

Remark: It is possible to write down an upper bound for $N_W^{\mathrm{g}}$ and $N_W^{\mathrm{ng}}$, without going through all of the above work. We may do this by considering the number of possible outcomes of the $W$ weighings. There are 3 possibilities for each weighing (left side heavier, right side heavier, or both sides equal), so there are at most $3^W$ possible outcomes. Each of these outcomes may be labelled by a string of $W$ letters, for example, $LLRER$ if $W = 5$ (with "$L$" for "left", etc.).

However, the $EEEEE$ string (where the scale always balances) does not give enough information to determine whether the bad coin is heavy or light. Also, the "mirror image" outcome (namely $RRLEL$, for the $LLRER$ case above) corresponds to equivalent information, with "left" and "right" simply reversed. Therefore, there are $(3^W - 1)/2$ effectively different strings. Hence, we can start with no more than $(3^W - 1)/2$ coins (because we can imagine initially labelling each coin with a string, so that if that particular string is the one obtained, then the corresponding coin is the bad coin).

As we saw above, this upper bound of $(3^W - 1)/2$ is obtainable if we have an additional known good coin at our disposal. But we fall short of the bound by one, if we initially do not have an additional good coin.

Note that although there may be different possible strategies for coin placement at various points in the weighing process (so that there are actually far more than $3^W$ possible outcomes, taking into consideration different placements of coins), only $3^W$ of the possible outcomes are realizable in a given scheme. Whatever weighing strategy you pick, you can write down an "if, then" tree before you start the process. Once you pick a scheme, there are $3^W$ possible outcomes (and thus $(3^W - 1)/2$ effectively different outcomes).