Solution

Week 35 (5/12/03)

Rising hoop

Let θ be the angle through which the bead has fallen, and let N be the normal force from the hoop on the bead, with inward taken to be positive. Then the radial $F = ma$ equation for the bead is

$$N + mg\cos\theta = \frac{mv^2}{R}\,. \tag{1}$$

The height the bead has fallen is $R - R\cos\theta$, so conservation of energy gives

$$\frac{mv^2}{2} = mgR(1-\cos\theta) \qquad \Longrightarrow \qquad v^2 = 2gR(1-\cos\theta). \tag{2}$$

Therefore, the radial $F = ma$ equation becomes

$$\begin{aligned} N &= \frac{mv^2}{R} - mg\cos\theta \\ &= 2mg(1-\cos\theta) - mg\cos\theta \\ &= mg(2 - 3\cos\theta). \end{aligned} \tag{3}$$

By Newton's third law, this is the force from the bead on the hoop, with outward taken to be positive. Note that this force is positive (that is, the bead pulls outward on the hoop) if $\theta > \cos^{-1}(2/3) \approx 48.2^\circ$.

Since there are two beads, the total upward force on the hoop from the beads is

$$2N\cos\theta = 2mg(2 - 3\cos\theta)\cos\theta. \tag{4}$$

The θ that yields the maximum value of this upward force is obtained by taking the derivative, which gives

$$\begin{aligned} 0 &= \frac{d}{d\theta}(2\cos\theta - 3\cos^2\theta) \\ &= -2\sin\theta + 6\sin\theta\cos\theta. \end{aligned} \tag{5}$$

Therefore, the maximum value is achieved when $\cos\theta = 1/3$, in which case the upward force equals

$$2mg\left(2 - 3\left(\frac{1}{3}\right)\right)\left(\frac{1}{3}\right) = \frac{2mg}{3}\,. \tag{6}$$

The hoop will rise up off the ground if this maximum upward force is larger than the weight of the hoop. That is, if

$$\frac{2mg}{3} > Mg \qquad \Longrightarrow \qquad \frac{m}{M} > \frac{3}{2}\,. \tag{7}$$

Remark: Alternatively, we can solve for the minimum value of m/M by setting the upward force, $2mg(2 - 3\cos\theta)\cos\theta$, equal to the weight of the hoop, Mg, and then using the quadratic formula to solve for $\cos\theta$. A solution for $\cos\theta$ exists only if the discriminant is positive, which is the case only if $m/M > 3/2$.