Solution

Week 36 (5/19/03)

Monochromatic Triangle

(a) Let us try to avoid forming a monochromatic triangle, a task that we will show is impossible. Consider one point and the sixteen lines drawn from it to the other sixteen points. From the pigeonhole principle (if you have n pigeons and $n-1$ pigeonholes, then at least two pigeons must go in one pigeonhole), we see that at least six of these lines must be of the same color. Let this color be red.

Now consider the six points at the ends of these red lines. Look at the lines going from one of these points to the other five. In order to not form a red triangle, each of these five lines must be either green or blue. Hence (by the pigeonhole principle) at least three of them must be of the same color. Let this color be green.

Finally, consider the three points at the ends of the three green lines. If any one of the three lines connecting them is red, a red triangle is formed. And if any one of the three lines connecting them is green, a green triangle is formed. Therefore, they must all be blue, and a blue triangle is formed.

(b) Consider the problem for the case of $n = 4$, in order to get an idea of how the solution generalizes. We claim that 66 points will necessitate a monochromatic triangle. As in the case of $n = 3$, isolate one point and paint all the lines from it to the other points. Since we have 65 other points and 4 colors, the pigeonhole principle requires that at least 17 of these lines be of the same color. In order to not have a monochromatic triangle, the lines joining the endpoints of these 17 lines must use only the remaining three colors, and the problem is reduced to the case of $n = 3$.

Generalizing this reasoning yields the following result:

Claim: *If n colors and P_n points necessitate a monochromatic triangle, then $n+1$ colors and*

$$P_{n+1} = (n+1)(P_n - 1) + 2 \tag{1}$$

points also necessitate a monochromatic triangle.

Proof: Isolate one point, and paint each of the $(n+1)(P_n - 1) + 1$ lines to the other points one of $n+1$ colors. From the pigeonhole principle, at least P_n of these lines must be the same color. In order to not have a monochromatic triangle, the points at the ends of these P_n lines must be joined by the n other colors. But by hypothesis, there must then be a monochromatic triangle. ■

If we use the above recursion relation in itself (that is, if we write the P_n in eq. (1) in terms of P_{n-1}, and then write P_{n-1} in terms of P_{n-2}, and so on), the pattern becomes clear. Using the initial condition $P_1 = 3$, we arrive at the following expression for P_n:

$$P_n = n!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) + 1, \tag{2}$$

as you can easily verify. The sum in the parentheses is smaller than e by a margin that is less than $1/n!$. Therefore, P_n does indeed equal the smallest integer greater than $n!e$.

REMARK: For $n = 1, 2, 3$, the numbers $\lceil n!e \rceil$ (which equal 3,6,17, respectively) are the smallest numbers which necessitate a monochromatic triangle. (For $n = 1$, two points don't even form a triangle. And for $n = 2$, you can easily construct a diagram that doesn't contain a monochromatic triangle. For $n = 3$, things are much more difficult, but in 1968 Kalbfleisch and Stanton showed that 16 points do *not* necessitate a monochromatic triangle.) For $n \geq 4$, the problem of finding the smallest number of points that necessitate a monochromatic triangle is unsolved, I believe.