Solution

Week 37 (5/26/03)

Bouncing down a plane

Let us ignore the tilt of the plane for a moment and determine how the ω_f and v_f after a bounce are related to the ω_i and v_i before the bounce (where v denotes the velocity component parallel to the plane). Let the positive directions of velocity and force be to the right along the plane, and let the positive direction of angular velocity be counterclockwise. If we integrate the force and torque over the small time of a bounce, we obtain

$$\begin{aligned} F &= \frac{dP}{dt} \qquad \Longrightarrow \qquad \int F\,dt = \Delta P, \\ \tau &= \frac{dL}{dt} \qquad \Longrightarrow \qquad \int \tau\,dt = \Delta L. \end{aligned} \tag{1}$$

But $\tau = RF$. And since R is constant, we have

$$\Delta L = \int RF\,dt = R\int F\,dt = R\Delta P. \tag{2}$$

Therefore,

$$I(\omega_f - \omega_i) = Rm(v_f - v_i). \tag{3}$$

But conservation of energy gives

$$\begin{aligned} \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2 &= \frac{1}{2}mv_i^2 + \frac{1}{2}I\omega_i^2 \\ \Longrightarrow \qquad I(\omega_f^2 - \omega_i^2) &= m(v_i^2 - v_f^2). \end{aligned} \tag{4}$$

Dividing this equation by eq. (3) gives[1]

$$R(\omega_f + \omega_i) = -(v_f + v_i). \tag{5}$$

We can now combine this equation with eq. (3), which can be rewritten (using $I = (2/5)mR^2$) as

$$\frac{2}{5}R(\omega_f - \omega_i) = v_f - v_i. \tag{6}$$

Given v_i and ω_i, the previous two equations are two linear equations in the two unknowns, v_f and ω_f. Solving for v_f and ω_f, and then writing the result in matrix notation, gives

$$\begin{pmatrix} v_f \\ R\omega_f \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & -4 \\ -10 & -3 \end{pmatrix}\begin{pmatrix} v_i \\ R\omega_i \end{pmatrix} \equiv \mathcal{A}\begin{pmatrix} v_i \\ R\omega_i \end{pmatrix}. \tag{7}$$

[1] We have divided out the trivial $\omega_f = \omega_i$ and $v_f = v_i$ solution, which corresponds to slipping motion on a frictionless plane. The nontrivial solution we will find shortly is the non-slipping one. Basically, to conserve energy, there must be no work done by friction. But since work is force times distance, this means that either the plane is frictionless, or that there is no relative motion between ball's contact point and the plane. Since we are given that the plane has friction, the latter (non-slipping) case must be the one we are concerned with.

Note that

$$\mathcal{A}^2 = \frac{1}{49}\begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix} = \mathcal{I}. \tag{8}$$

Now let us consider the effects of the tilted plane. Since the ball's speed perpendicular to the plane is unchanged by each bounce, the ball spends the same amount of time in the air between any two successive bounces. This time equals $T = 2V/g\cos\theta$, because the component of gravity perpendicular to the plane is $g\cos\theta$. During this time, the speed along the plane increases by $(g\sin\theta)T = 2V\tan\theta \equiv V_0$.

Let $\mathbf{Q}$ denote the $(v, R\omega)$ vector at a given time (where v denotes the velocity component parallel to the plane). The ball is initially projected with $\mathbf{Q} = \mathbf{0}$. Therefore, right before the first bounce, we have $\mathbf{Q}_1^{\mathbf{before}} = (V_0, 0) \equiv \mathbf{V}_0$. (We have used the fact that ω doesn't change while the ball is in the air.) Right after the first bounce, we have $\mathbf{Q}_1^{\mathbf{after}} = \mathcal{A}\mathbf{V}_0$. We then have $\mathbf{Q}_2^{\mathbf{before}} = \mathcal{A}\mathbf{V}_0 + \mathbf{V}_0$, and so $\mathbf{Q}_2^{\mathbf{after}} = \mathcal{A}(\mathcal{A}\mathbf{V}_0 + \mathbf{V}_0)$. Continuing in this manner, we see that

$$\begin{aligned} \mathbf{Q}_n^{\mathbf{before}} &= (\mathcal{A}^{n-1} + \cdots + \mathcal{A} + \mathcal{I})\mathbf{V}_0, \quad \text{and} \\ \mathbf{Q}_n^{\mathbf{after}} &= (\mathcal{A}^{n} + \cdots + \mathcal{A}^2 + \mathcal{A})\mathbf{V}_0. \end{aligned} \tag{9}$$

However, $\mathcal{A}^2 = \mathcal{I}$, so all the even powers of $\mathcal{A}$ equal $\mathcal{I}$. The value of $\mathbf{Q}$ after the nth bounce is therefore given by

$$\begin{aligned} n \text{ even} \quad &\Longrightarrow \quad \mathbf{Q}_n^{\mathbf{after}} = \frac{n}{2}(\mathcal{A} + \mathcal{I})\mathbf{V}_0. \\ n \text{ odd} \quad &\Longrightarrow \quad \mathbf{Q}_n^{\mathbf{after}} = \frac{1}{2}\Big((n+1)\mathcal{A} + (n-1)\mathcal{I}\Big)\mathbf{V}_0. \end{aligned} \tag{10}$$

Using the value of $\mathcal{A}$ defined in eq. (7), we find

$$\begin{aligned} n \text{ even} \quad &\Longrightarrow \quad \begin{pmatrix} v_n \\ R\omega_n \end{pmatrix} = \frac{n}{7}\begin{pmatrix} 5 & -2 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} V_0 \\ 0 \end{pmatrix}. \\ n \text{ odd} \quad &\Longrightarrow \quad \begin{pmatrix} v_n \\ R\omega_n \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 5n-2 & -2n-2 \\ -5n-5 & 2n-5 \end{pmatrix}\begin{pmatrix} V_0 \\ 0 \end{pmatrix}. \end{aligned} \tag{11}$$

Therefore, the speed along the plane after the nth bounce equals (using $V_0 \equiv 2V\tan\theta$)

$$v_n = \frac{10nV\tan\theta}{7} \quad (n \text{ even}), \qquad \text{and} \qquad v_n = \frac{(10n-4)V\tan\theta}{7} \quad (n \text{ odd}). \tag{12}$$

Remark: Note that after an even number of bounces, eq. (11) gives $v = -R\omega$. This is the "rolling" condition. That is, the angular speed exactly matches up with the translation speed, so v and ω are unaffected by the bounce. (The vector $(1, -1)$ is an eigenvector of $\mathcal{A}$.) At the instant that an even-n bounce occurs, the v and ω are the same as they would be for a ball that simply rolls down the plane. At the instant after an odd-n bounce, the v is smaller than it would be for the rolling ball, but the ω is larger. (Right before an odd-n bounce, the v is larger but the ω is smaller.)