Solution

Week 4 (10/7/02)

Passing the spaghetti

(a) For the case of $n = 3$, it is obvious that the two people not at the head of the table have equal 1/2 chances of being the last served (BTLS).

For the case of $n = 4$, label the diners as A,B,C,D (with A being the head), and consider D's probability of BTLS. The various paths of spaghetti that allow D to be the last served are:

$$\text{ABC...},\ \text{ABABC...},\ \text{ABABABC...},\ \text{etc.} \tag{1}$$

The sum of the probabilities of these is

$$\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots = \frac{1/4}{1 - 1/4} = \frac{1}{3}\,. \tag{2}$$

By symmetry, B also has a 1/3 chance of BLTS, and then that leaves a 1/3 chance for C. Hence, B, C, and D all have equal 1/3 chances of BLTS.

The probabilities for $n = 5$ are a bit tedious to calculate in this same manner, so at this point we will (for lack of a better option) make the following guess:

Claim: *For arbitrary n, all diners not at the head of the table have equal $1/(n-1)$ chances of being the last served (BTLS).*

This seems a bit counterintuitive (because you might think that the diners further from the head have a greater chance of BTLS), but it is in fact correct.

Proof: Two things must happen to a given diner for BTLS:

(1) The plate must approach the given diner from the right or left and reach the person next to that diner.

(2) The plate must then reverse its direction and make its way (in whatever manner) all the way around the table until it reaches the person on the other side of the given diner.

For any of the (non-head) diners, the probability that the first of these conditions will be satisfied is 1. This condition will therefore not differentiate the probabilities of BTLS.

Given that (1) has happened, there is some definite probability of (2) happening, independent of where the diner is located. This is true because the probability of traveling all the way around the table does not depend on where this traveling starts. Hence, (2) also does not differentiate between the $n - 1$ (non-head) probabilities of BTLS.

Thus, all the $n-1$ (non-head) probabilities of BTLS are equal, and are therefore equal to $1/(n-1)$. ■

(b) This problem is equivalent to asking how many steps it takes, on average, for a random walk in one dimension to hit n sites.

Let f_n be this expected number of steps. And let g_n be defined as follows. Assume that n sites have been visited, and that the present position is at one of the ends of this string of n sites. Then g_n is the expected number of steps it takes to reach a new site.

We then have

$$f_n = f_{n-1} + g_{n-1}. \tag{3}$$

This is true because in order to reach n sites, you must first reach $n-1$ sites (which takes f_{n-1} steps, on average). And then you must reach one more site, starting at the end of the string of $n-1$ sites (which takes g_{n-1} steps, on average).

Claim: $g_n = n$.

Proof: Let the sites which have been visited be labeled $1, 2, \ldots, n$. Let the present position be site 1.

There is a 1/2 chance that the next step will be to site number 0, in which case it only takes one step to reach a new site.

There is a 1/2 chance that the next step will be to site number 2. By considering this site to be an end-site of the string $2, 3, \ldots, n-1$ (which has size $n-2$), we see that it takes g_{n-2} steps (on average) to reach sites 1 or n. And then from each of these, it takes of g_n steps (on average) to reach a new site.

Putting this together gives $g_n = \frac{1}{2}(1) + \frac{1}{2}(1 + g_{n-2} + g_n)$, or

$$g_n = g_{n-2} + 2. \tag{4}$$

Since we obviously have $g_1 = 1$, and since it is easy to see from the above reasoning that $g_2 = 2$ (or equivalently, that $g_0 = 0$), we inductively obtain $g_n = n$. ∎

Therefore, $f_n = f_{n-1} + (n-1)$. Using $f_1 = 0$, we see by induction that

$$f_n = \frac{n(n-1)}{2}\,. \tag{5}$$