Solution

Week 42  (6/30/03)

How much change?

If the item costs between $N/2$ and $N$ dollars, then you can buy only one item. These extremes will produce remainders of $N/2$ and 0, respectively. The average amount of money left over in this region, which has length $N(1 - 1/2) = N/2$, is therefore $N/4$.

Likewise, if the item costs between $N/3$ and $N/2$ dollars, then you can buy only two items. These extremes will produce remainders of $N/3$ and 0, respectively. The average amount of money left over in this region, which has length $N(1/2 - 1/3) = N/6$, is therefore $N/6$.

Continuing in this manner, we see that if the item costs between $N/(n+1)$ and $N/n$, then you can buy only $n$ items. These extremes will produce remainders of $N/(n+1)$ and 0, respectively. The average amount of money left over in this region, which has length $N(1/n - 1/(n+1)) = N/n(n+1)$, is therefore $N/2(n+1)$.

The expected amount, $M$, of money left over is therefore

$$M = \sum_{n=1}^{\infty} \left( \frac{1}{n} \right) \frac{N}{2(n+1)}$$

$$= \frac{N}{2} \sum_{n=1}^{\infty} \left( \frac{1}{n(n+1)} - \frac{1}{(n+1)^2} \right)$$

$$= \frac{N}{2} \sum_{n=1}^{\infty} \left( \left( \frac{1}{n} - \frac{1}{n+1} \right) - \frac{1}{(n+1)^2} \right)$$

$$= \frac{N}{2} \left( 1 - \left( \frac{\pi^2}{6} - 1 \right) \right)$$

$$= N \left( 1 - \frac{\pi^2}{12} \right).$$

In the third line, we have used the fact that the sum in brackets telescopes to 1, and also that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$. Since $\pi^2/12 \approx 0.82$, the amount of money left over is roughly $(0.18)N$ dollars. Note that what we have essentially done in this problem is find the area under the graph in the following figure.