Solution

Week 43 (7/7/03)

Infinite Atwood's machine

First Solution: If the strength of gravity on the earth were multiplied by a factor η, then the tension in all of the strings in the Atwood's machine would likewise be multiplied by η. This is true because the only way to produce a quantity with the units of tension (that is, force) is to multiply a mass by g. Conversely, if we put the Atwood's machine on another planet and discover that all of the tensions are multiplied by η, then we know that the gravity there must be ηg.

Let the tension in the string above the first pulley be T. Then the tension in the string above the second pulley is $T/2$ (because the pulley is massless). Let the downward acceleration of the second pulley be a_2. Then the second pulley effectively lives in a world where gravity has strength $g - a_2$.

Consider the subsystem of all the pulleys except the top one. This infinite subsystem is identical to the original infinite system of all the pulleys. Therefore, by the arguments in the first paragraph above, we must have

$$\frac{T}{g} = \frac{T/2}{g - a_2}\,, \tag{1}$$

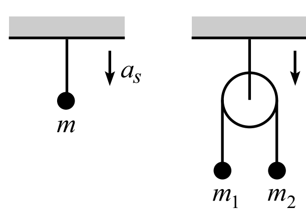
which gives $a_2 = g/2$. But a_2 is also the acceleration of the top mass, so our answer is $g/2$.

Remarks: You can show that the relative acceleration of the second and third pulleys is $g/4$, and that of the third and fourth is $g/8$, etc. The acceleration of a mass far down in the system therefore equals $g(1/2 + 1/4 + 1/8 + \cdots) = g$, which makes intuitive sense.

Note that $T = 0$ also makes eq. (1) true. But this corresponds to putting a mass of zero at the end of a finite pulley system (see the following solution).

Second Solution: Consider the following auxiliary problem.

Problem: Two setups are shown below. The first contains a hanging mass m. The second contains a pulley, over which two masses, m_1 and m_2, hang. Let both supports have acceleration a_s downward. What should m be, in terms of m_1 and m_2, so that the tension in the top string is the same in both cases?



Answer: In the first case, we have

$$mg - T = ma_s. \tag{2}$$

In the second case, let a be the acceleration of m_2 relative to the support (with downward taken to be positive). Then we have

$$\begin{aligned} m_1 g - \frac{T}{2} &= m_1(a_s - a), \\ m_2 g - \frac{T}{2} &= m_2(a_s + a). \end{aligned} \tag{3}$$

Note that if we define $g' \equiv g - a_s$, then we may write the above three equations as

$$\begin{aligned} mg' &= T, \\ m_1 g' &= \frac{T}{2} - m_1 a, \\ m_2 g' &= \frac{T}{2} + m_2 a. \end{aligned} \tag{4}$$

Eliminating a from the last two of these equations gives $4m_1m_2g' = (m_1 + m_2)T$. Using this value of T in the first equation then gives

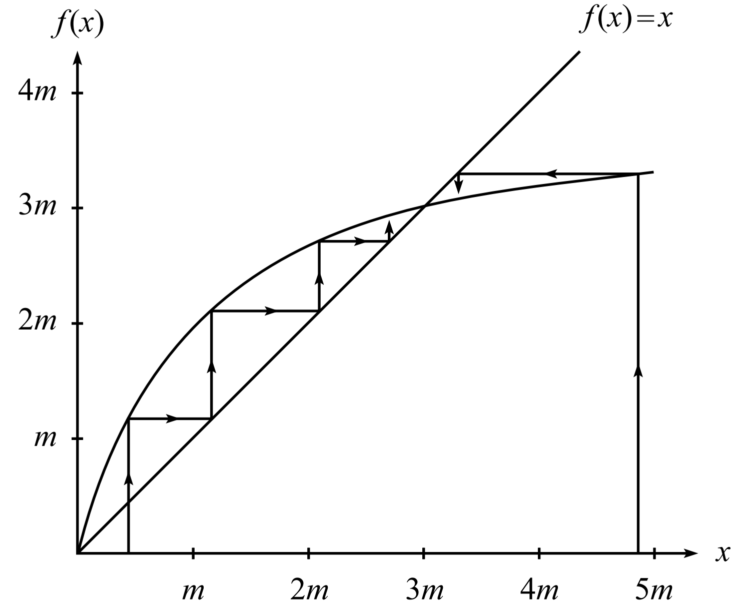
$$m = \frac{4m_1m_2}{m_1 + m_2}\,. \tag{5}$$

Note that the value of a_s is irrelevant. (We effectively have a fixed support in a world where the acceleration from gravity is g'.) This auxiliary problem shows that the two-mass system in the second case may be equivalently treated as a mass m, given by eq. (5), as far as the upper string is concerned. ■

Now let's look at our infinite Atwood's machine. Start at the bottom. (Assume that the system has N pulleys, where $N \to \infty$.) Let the bottom mass be x. Then the auxiliary problem shows that the bottom two masses, m and x, may be treated as an effective mass $f(x)$, where

$$\begin{aligned} f(x) &= \frac{4mx}{m + x} \\ &= \frac{4x}{1 + (x/m)}\,. \end{aligned} \tag{6}$$

We may then treat the combination of the mass $f(x)$ and the next m as an effective mass $f(f(x))$. These iterations may be repeated, until we finally have a mass m and a mass $f^{(N-1)}(x)$ hanging over the top pulley. So we must determine the behavior of $f^N(x)$, as $N \to \infty$. This behavior is clear if we look at the following plot of $f(x)$.



Note that $x = 3m$ is a fixed point of f. That is, $f(3m) = 3m$. This plot shows that no matter what x we start with, the iterations approach $3m$ (unless we start at $x = 0$, in which case we remain there). These iterations are shown graphically by the directed lines in the plot. After reaching the value $f(x)$ on the curve, the line moves horizontally to the x value of $f(x)$, and then vertically to the value $f(f(x))$ on the curve, and so on.

Therefore, since $f^N(x) \to 3m$ as $N \to \infty$, our infinite Atwood's machine is equivalent to (as far as the top mass is concerned) just two masses, m and $3m$. You can then quickly show that that the acceleration of the top mass is $g/2$.

Note that as far as the support is concerned, the whole apparatus is equivalent to a mass $3m$. So $3mg$ is the upward force exerted by the support.