Solution

Week 44  (7/14/03)

Relatively prime numbers

The probability that two random numbers both have a given prime $p$ as a factor is $1/p^2$. The probability that they do not have $p$ as a common factor is thus $1 - 1/p^2$. Therefore, the probability that two numbers have no common prime factors is

$$P = (1 - 1/2^2)(1 - 1/3^2)(1 - 1/5^2)(1 - 1/7^2)(1 - 1/11^2) \cdots.$$  \hspace{1cm} (1)

Using

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots,$$  \hspace{1cm} (2)

this can be rewritten as

$$P = \left(\left(1 + 1/2^2 + 1/3^2 + 1/4^2 + 1/5^2 + 1/6^2 + \cdots\right)^{-1}. \hspace{1cm} (3) \right.$$ By the Unique Factorization Theorem (every positive integer is expressible as the product of primes in exactly one way), we see that the previous expression is equivalent to

$$P = (1 + 1/2^2 + 1/3^2 + 1/4^2 + 1/5^2 + 1/6^2 + \cdots)^{-1}. \hspace{1cm} (4)$$

And since the sum of the squares of the reciprocals of all of the positive integers is known to be $\pi^2/6$, the desired probability is $P = 6/\pi^2 \approx 61\%$.

Remarks:

1. The probability that $n$ random numbers all have a given prime $p$ as a factor is $1/p^n$. So the probability that they do not all have $p$ as a common factor is $1 - 1/p^n$. In exactly the same manner as above, we find that the probability, $P_n$, that $n$ numbers have no common factor among all of them is

$$P_n = (1 + 1/2^n + 1/3^n + 1/4^n + 1/5^n + 1/6^n + \cdots)^{-1}. \hspace{1cm} (5)$$

This is, by definition, the Riemann zeta function, $\zeta(n)$. It can be calculated exactly for even values of $n$, but only numerically for odd values. For the case of $n = 4$, we can use the known value $\zeta(4) = \pi^4/90$ to see that the probability that four random numbers do not all have a common factor is $P_4 = 90/\pi^4 \approx 92\%$.

2. We can also perform the somewhat silly exercise of applying this result to the case of $n = 1$. The question then becomes: What is the probability, $P_1$, that a randomly chosen positive integer does not have a factor? Well, 1 is the only positive integer without any factors, so the probability is $1/\infty = 0$. And indeed,

$$P_1 = (1 - 1/2)(1 - 1/3)(1 - 1/5)(1 - 1/7)\cdots \hspace{1cm} (6)$$

$$= (1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + \cdots)^{-1}$$

$$= 1/\infty,$$

because the sum of the reciprocals of all of the positive integers diverges.
3. Let $\phi(n)$ equal the number of integers less than $n$ that are relatively prime to $n$. Then $\phi(n)/n$ equals the probability that a randomly chosen integer is relatively prime to $n$. (This is true because any integer is relatively prime to $n$ if and only if its remainder, when divided by $n$, is relatively prime to $n$.) The result of our original problem therefore tells us that the average value of $\phi(n)/n$ is $6/\pi^2$.

4. To be precise about what we mean by probabilities in this problem, we really should word the question as: Let $N$ be a very large integer. Pick two random integers less than or equal to $N$. What is the probability that these numbers are relatively prime, in the limit where $N$ goes to infinity?

The solution would then be slightly modified, in that the relevant primes $p$ would be cut off at $N$, and “edge effects” due to the finite size of $N$ would have to be considered. It is fairly easy to see that the answer obtained in this limit is the same as the answer obtained above.