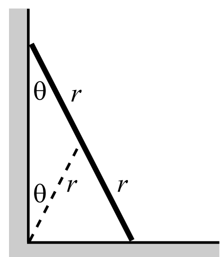
*Solution*

Week 47 (8/4/03)

**Sliding ladder**

The important point to realize in this problem is that the ladder loses contact with the wall before it hits the ground. Let's find where this loss of contact occurs.

Let $r = \ell/2$, for convenience. While the ladder is in contact with the wall, its CM moves in a circle of radius $r$. (This follows from the fact that the median to the hypotenuse of a right triangle has half the length of the hypotenuse). Let $\theta$ be the angle between the wall and the radius from the corner to the CM. (This is also the angle between the ladder and the wall.)




We'll solve this problem by assuming that the CM always moves in a circle, and then determining the position at which the horizontal CM speed starts to decrease, that is, the point at which the normal force from the wall would have to become negative. Since the normal force of course can't be negative, this is the point where the ladder loses contact with the wall.

By conservation of energy, the kinetic energy of the ladder equals the loss in potential energy, which is $mgr(1-\cos\theta)$. This kinetic energy may be broken up into the CM translational energy plus the rotation energy. The CM translational energy is simply $mr^2\dot{\theta}^2/2$ (because the CM travels in a circle of radius $r$). The rotational energy is $I\dot{\theta}^2/2$. (The same $\dot{\theta}$ applies here as in the CM translational motion, because $\theta$ is the angle between the ladder and the vertical, and thus is the angle of rotation of the ladder.) Letting $I \equiv \eta m r^2$ to be general ($\eta = 1/3$ for our ladder), the conservation of energy statement is $(1+\eta)mr^2\dot{\theta}^2/2 = mgr(1-\cos\theta)$. Therefore, the speed of the CM, which is $v = r\dot{\theta}$, equals

$$v = \sqrt{\frac{2gr}{1+\eta}}\sqrt{(1-\cos\theta)}\,. \tag{1}$$

The horizontal component of this is

$$v_x = \sqrt{\frac{2gr}{1+\eta}}\sqrt{(1-\cos\theta)}\cos\theta. \tag{2}$$

Taking the derivative of $\sqrt{(1-\cos\theta)}\cos\theta$, we see that the horizontal speed is maximum when $\cos\theta = 2/3$. Therefore the ladder loses contact with the wall when

$$\cos\theta = \frac{2}{3} \qquad \Longrightarrow \qquad \theta \approx 48.2^\circ. \tag{3}$$

Note that this is independent of $\eta$. This means that, for example, a dumbbell (two masses at the ends of a massless rod, with $\eta = 1$) will lose contact with the wall at the same angle.

Plugging this value of $\theta$ into eq. (2), and using $\eta = 1/3$, we obtain a final horizontal speed of

$$v_x = \frac{\sqrt{2gr}}{3} \equiv \frac{\sqrt{g\ell}}{3}\,. \tag{4}$$

Note that this is 1/3 of the $\sqrt{2gr}$ horizontal speed that the ladder would have if it were arranged (perhaps by having the top end slide down a curve) to eventually slide horizontally along the ground.

REMARK: The normal force from the wall is zero at the start and finish, so it must reach a maximum at some intermediate value of $\theta$. Let's find this $\theta$. Taking the derivative of $v_x$ in eq. (2) to find $a_x$, and then using $\dot{\theta} \propto \sqrt{1-\cos\theta}$ from eq. (1), we see that the force from the wall is proportional to

$$a_x \;\propto\; \frac{\sin\theta(3\cos\theta - 2)}{\sqrt{1-\cos\theta}}\,\dot{\theta} \;\propto\; \sin\theta(3\cos\theta - 2). \tag{5}$$

Taking the derivative of this, we find that the force from the wall is maximum when

$$\cos\theta = \frac{1+\sqrt{19}}{6} \qquad \Longrightarrow \qquad \theta \approx 26.7^\circ. \tag{6}$$