

Solution

Week 49 (8/18/03)

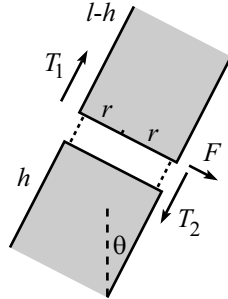
Falling chimney

Before we start dealing with the forces in the rods, let's first determine $\ddot{\theta}$ as a function of θ (the angle through which the chimney has fallen). Let ℓ be the height of the chimney. Then the moment of inertia around the pivot point on the ground is $m\ell^2/3$ (if we ignore the width), and the torque (around the pivot point) due to gravity is $\tau = mg(\ell/2) \sin \theta$. Therefore, $\tau = dL/dt$ gives $mg(\ell/2) \sin \theta = (1/3)m\ell^2\ddot{\theta}$, or

$$\ddot{\theta} = \frac{3g \sin \theta}{2\ell}. \quad (1)$$

Let's now determine the forces in the rods. Our strategy will be to imagine that the chimney consists of a chimney of height h , with another chimney of height $\ell - h$ placed on top of it. We'll find the forces in the rods connecting these two "sub-chimneys", and then we'll maximize one of these forces (T_2 , defined below) as a function of h .

The forces on the top piece are gravity and the forces from the two rods at each end of the bottom board. Let's break these latter forces up into transverse and longitudinal forces along the chimney. Let T_1 and T_2 be the two longitudinal components, and let F be the sum of the transverse components, as shown.



We have picked the positive directions for T_1 and T_2 such that positive T_1 corresponds to a compression in the left rod, and positive T_2 corresponds to a tension in the right rod (which is what the forces will turn out to be, as we'll see). It turns out that if the width (which we'll call $2r$) is much less than the height, then $T_2 \gg F$ (as we will see below), so the tension in the right rod is essentially equal to T_2 . We will therefore be concerned with maximizing T_2 .

In writing down the force and torque equations for the top piece, we have three equations (the radial and tangential $F = ma$ equations, and $\tau = dL/dt$ around the CM), and three unknowns (F , T_1 , and T_2). If we define the fraction $f \equiv h/\ell$, then the top piece has length $(1 - f)\ell$ and mass $(1 - f)m$, and its CM travels in a circle of radius $(1 + f)\ell/2$. Therefore, our three force and torque equations are, respectively,

$$\begin{aligned} T_2 - T_1 + (1 - f)mg \cos \theta &= (1 - f)m \left(\frac{(1 + f)\ell}{2} \right) \dot{\theta}^2, \\ F + (1 - f)mg \sin \theta &= (1 - f)m \left(\frac{(1 + f)\ell}{2} \right) \ddot{\theta}, \end{aligned}$$

$$(T_1 + T_2)r - F\frac{(1-f)\ell}{2} = (1-f)m\left(\frac{(1-f)^2\ell^2}{12}\right)\ddot{\theta}. \quad (2)$$

At this point, we could plow forward and solve this system of three equations in three unknowns. But things simplify greatly in the limit where $r \ll \ell$. The third equation says that $T_1 + T_2$ is of order $1/r$, and the first equation says that $T_2 - T_1$ is of order 1. These imply that $T_1 \approx T_2$, to leading order in $1/r$. Therefore, we may set $T_1 + T_2 \approx 2T_2$ in the third equation. Using this approximation, along with the value of $\ddot{\theta}$ from eq. (1), the second and third equations become

$$\begin{aligned} F + (1-f)mg \sin \theta &= \frac{3}{4}(1-f^2)mg \sin \theta, \\ 2rT_2 - F\frac{(1-f)\ell}{2} &= \frac{1}{8}(1-f)^3mg\ell \sin \theta. \end{aligned} \quad (3)$$

This first of these equations gives

$$F = \frac{mg \sin \theta}{4}(-1 + 4f - 3f^2), \quad (4)$$

and then the second gives

$$T_2 \approx \frac{mg\ell \sin \theta}{8r}f(1-f)^2. \quad (5)$$

As stated above, this is much greater than F (because $\ell/r \gg 1$), so the tension in the right rod is essentially equal to T_2 . Taking the derivative of T_2 with respect to f , we see that it is maximum at

$$f \equiv \frac{h}{\ell} = \frac{1}{3}. \quad (6)$$

Therefore, the chimney is most likely to break at a point one-third of the way up (assuming that the width is much less than the height). Interestingly, $f = 1/3$ makes the force F in eq. (4) exactly equal to zero.