

Solution

Week 5 (10/14/02)

The raindrop

Let ρ be the mass density of the raindrop, and let λ be the average mass density in space of the water droplets. Let $r(t)$, $M(t)$, and $v(t)$ be the radius, mass, and speed of the raindrop, respectively.

We need three equations to solve for the above three unknowns. The equations we will use are two different expressions for dM/dt , and the $F = dp/dt$ expression for the raindrop.

The first expression for \dot{M} is obtained by simply taking the derivative of $M = (4/3)\pi r^3 \rho$, which gives

$$\dot{M} = 4\pi r^2 \dot{r} \rho \quad (1)$$

$$= 3M \frac{\dot{r}}{r}. \quad (2)$$

The second expression for \dot{M} is obtained by noting that the change in M is due to the acquisition of water droplets. The raindrop sweeps out volume at a rate given by its cross-sectional area times its velocity. Therefore,

$$\dot{M} = \pi r^2 v \lambda. \quad (3)$$

The force on the droplet is Mg , and the momentum is Mv . Therefore, $F = dp/dt$ gives

$$Mg = \dot{M}v + M\dot{v}. \quad (4)$$

We now have three equations involving the three unknowns, r , M , and v .¹

Our goal is to find \dot{v} . We will do this by first finding \ddot{r} . Eqs. (1) and (3) give

$$v = \frac{4\rho}{\lambda} \dot{r} \quad (5)$$

$$\implies \dot{v} = \frac{4\rho}{\lambda} \ddot{r}. \quad (6)$$

Plugging eqs. (2, 5, 6) into eq. (4) gives

$$Mg = \left(3M \frac{\dot{r}}{r}\right) \left(\frac{4\rho}{\lambda} \dot{r}\right) + M \left(\frac{4\rho}{\lambda} \ddot{r}\right). \quad (7)$$

Therefore,

$$\tilde{g}r = 12\dot{r}^2 + 4r\ddot{r}, \quad (8)$$

where we have defined $\tilde{g} \equiv g\lambda/\rho$, for convenience. The only parameter in eq. (8) is \tilde{g} . Therefore, $r(t)$ can depend only on \tilde{g} and t . Hence, by dimensional analysis, r must take the form

$$r = A\tilde{g}t^2, \quad (9)$$

¹Note that we *cannot* write down the naive conservation-of-energy equation (which would say that the decrease in the water's potential energy equals the increase in its kinetic energy), because mechanical energy is *not* conserved. The collisions between the raindrop and the droplets are completely inelastic. The raindrop will, in fact, heat up. See the remark at the end of the solution.

where A is a numerical constant, to be determined. Plugging this expression for r into eq. (8) gives

$$\begin{aligned} \tilde{g}(A\tilde{g}t^2) &= 12(2A\tilde{g}t)^2 + 4(A\tilde{g}t^2)(2A\tilde{g}) \\ \implies A &= 48A^2 + 8A^2. \end{aligned} \quad (10)$$

Therefore, $A = 1/56$, and so $\ddot{r} = 2A\tilde{g} = \tilde{g}/28 = g\lambda/28\rho$. Eq. (6) then gives the acceleration of the raindrop as

$$\dot{v} = \frac{g}{7}, \quad (11)$$

independent of ρ and λ .

REMARK: A common invalid solution to this problem is the following, which (incorrectly) uses conservation of energy.

The fact that v is proportional to \dot{r} (shown in eq. (5)) means that the volume swept out by the raindrop is a cone. The center-of-mass of a cone is $1/4$ of the way from the base to the apex (as you can show by integrating over horizontal circular slices). Therefore, if M is the mass of the raindrop after it has fallen a height h , then an (incorrect) application of conservation of energy gives

$$\frac{1}{2}Mv^2 = Mg\frac{h}{4} \quad \implies \quad v^2 = \frac{gh}{2}. \quad (12)$$

Taking the derivative of this (or equivalently, using $v^2 = 2ad$), we obtain

$$\dot{v} = \frac{g}{4}. \quad (\text{Incorrect}) \quad (13)$$

The reason why this solution is invalid is that the collisions between the raindrop and the droplets are completely inelastic. Heat is generated, and the overall kinetic energy of the raindrop is smaller than you would otherwise expect.

Let's calculate how much mechanical energy is lost (and therefore how much the raindrop heats up) as a function of the height fallen. The loss in mechanical energy is

$$E_{\text{lost}} = Mg\frac{h}{4} - \frac{1}{2}Mv^2. \quad (14)$$

Using $v^2 = 2(g/7)h$, this becomes

$$\Delta E_{\text{int}} = E_{\text{lost}} = \frac{3}{28}Mgh, \quad (15)$$

where ΔE_{int} is the gain in internal thermal energy.

The energy required to heat 1g of water by $1\text{ }^\circ\text{C}$ is 1 calorie (= 4.18 Joules). Therefore, the energy required to heat 1 kg of water by $1\text{ }^\circ\text{C}$ is $\approx 4200\text{ J}$. In other words,

$$\Delta E_{\text{int}} = 4200 M \Delta T, \quad (16)$$

where M is measured in kilograms, and T is measured in Celsius. Eqs. (15) and (16) give the increase in temperature as a function of h ,

$$4200 \Delta T = \frac{3}{28}gh. \quad (17)$$

How far must the raindrop fall before it starts to boil? If we assume that the water droplets' temperature is near freezing, then the height through which the raindrop must fall to have $\Delta T = 100\text{ }^\circ\text{C}$ is found to be

$$h \approx 400\text{ km}. \quad (18)$$

We have, of course, idealized the problem. But needless to say, there is no need to worry about getting burned by the rain.

A typical value for h more like a few kilometers, which would raise the temperature only by about one degree. This effect, of course, is washed out by many other factors.