Equal segments

The construction will proceed inductively. Given a segment divided into \( N \) equal segments, we will show how to divide it into \( N + 1 \) equal segments. For purposes of concreteness and having manageable figures, we will just do the case \( N = 3 \). Generalization to arbitrary \( N \) will be clear.

In the figure below, let the segment \( AB \) be divided into three equal segments by \( D_1 \) and \( D_2 \). From an arbitrary point \( P \) (assume that \( P \) is on the side of \( AB \) opposite to the infinite line \( L \), although it need not be), draw lines through \( A \) and \( B \), which meet \( L \) at points \( M \) and \( N \), respectively. Draw segments \( MD_1, MD_2, MB, NA, ND_1 \), and \( ND_2 \). Let the resulting intersections (the ones closest to segment \( AB \)) be \( Q_1, Q_2 \), and \( Q_3 \), as shown.

Claim: The lines \( PQ_1, PQ_2, \) and \( PQ_3 \) divide \( AB \) into four equal segments.

Proof: First, note that \( Q_1, Q_2, \) and \( Q_3 \) are collinear on a line parallel to \( AB \) and \( L \). This is true because the ratio of similar triangles \( AQ_1D_1 \) and \( NQ_1M \) is the same as the ratio of similar triangles \( D_1Q_2D_2 \) and \( NQ_2M \) (because \( AD_1 = D_1D_2 \)). Therefore, the altitude from \( Q_1 \) to \( AD_1 \) equals the altitude from \( Q_2 \) to \( D_1D_2 \). The same reasoning applies to \( Q_3 \), so all the \( Q_i \) are equal distances from \( AB \). Let the line determined by the \( Q_i \) intersect \( PM \) and \( PN \) at \( Q_0 \) and \( Q_4 \), respectively.

We now claim that the distances \( Q_iQ_{i+1}, i = 0, \ldots, 3 \) are equal. They are equal because the ratio of similar triangles \( Q_0AQ_1 \) and \( MAN \) is the same as the ratio of similar triangles \( Q_1D_1Q_2 \) and \( MD_1N \) (because the ratio of the altitudes from \( A \) in the first pair is the same as the ratio of the altitudes from \( D_1 \) in the second pair). Hence, \( Q_0Q_1 = Q_1Q_2 \). Likewise for the other \( Q_iQ_{i+1} \).

Therefore, since \( Q_0Q_4 \) is parallel to \( AB \), the intersections of the lines \( PQ_i \) with \( AB \) divide \( AB \) into four equal segments. \( \blacksquare \)
**Remark:** To divide $AB$ into five equal segments, we can use the same figure, with most of the work having already been done. The only new lines we need to draw are $NQ_0$ and $MQ_4$, to give a total of four intersections on a horizontal line one “level” below the $Q_i$. If we continue with this process, we obtain figures looking like the one below. The horizontal lines in this figure are divided into equal parts by the intersections of the diagonal lines. The initial undivided segment is the top one.\(^1\)

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\(^1\)This segment must be used again for the $N = 2$ segment, because the above procedure yields only one point $Q_1$, and this single point doesn’t determine a line parallel to $L$. 

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We leave for you the following exercise: Given a segment of length $\ell$, and a line parallel to it, construct a segment with a length equal to an arbitrary multiple of $\ell$, using only a ruler.