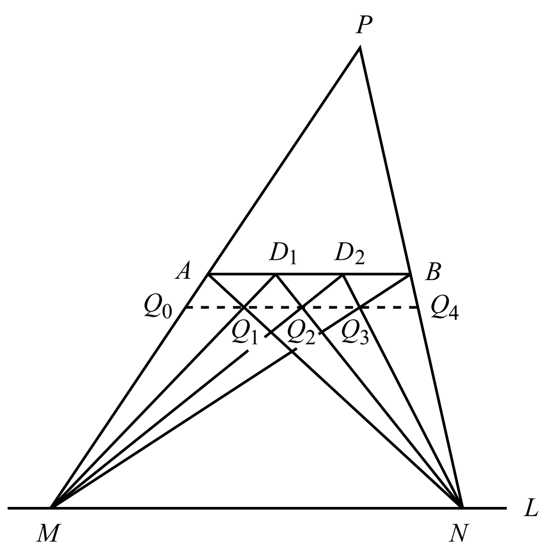
Solution

Week 50 (8/25/03)

Equal segments

The construction will proceed inductively. Given a segment divided into N equal segments, we will show how to divide it into $N+1$ equal segments. For purposes of concreteness and having manageable figures, we will just do the case $N = 3$. Generalization to arbitrary N will be clear.

In the figure below, let the segment AB be divided into three equal segments by D_1 and D_2. From an arbitrary point P (assume that P is on the side of AB opposite to the infinite line L, although it need not be), draw lines through A and B, which meet L at points M and N, respectively. Draw segments MD_1, MD_2, MB, NA, ND_1, and ND_2. Let the resulting intersections (the ones closest to segment AB) be Q_1, Q_2, and Q_3, as shown.



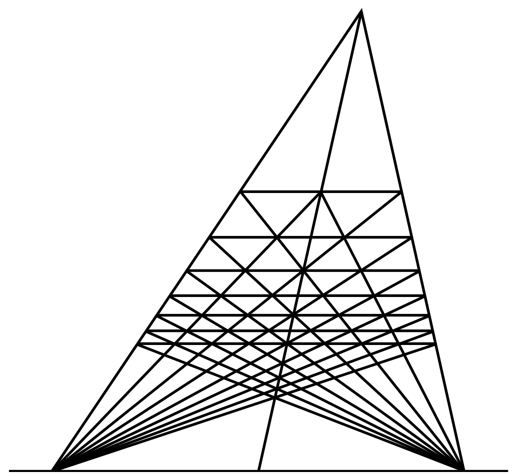
Claim: *The lines PQ_1, PQ_2, and PQ_3 divide AB into four equal segments.*

Proof: First, note that Q_1, Q_2, and Q_3 are collinear on a line parallel to AB and L. This is true because the ratio of similar triangles AQ_1D_1 and NQ_1M is the same as the ratio of similar triangles $D_1Q_2D_2$ and NQ_2M (because $AD_1 = D_1D_2$). Therefore, the altitude from Q_1 to AD_1 equals the altitude from Q_2 to D_1D_2. The same reasoning applies to Q_3, so all the Q_i are equal distances from AB. Let the line determined by the Q_i intersect PM and PN at Q_0 and Q_4, respectively.

We now claim that the distances Q_iQ_{i+1}, $i = 0, \ldots, 3$ are equal. They are equal because the ratio of similar triangles Q_0AQ_1 and MAN is the same as the ratio of similar triangles $Q_1D_1Q_2$ and MD_1N (because the ratio of the altitudes from A in the first pair is the same as the ratio of the altitudes from D_1 in the second pair). Hence, $Q_0Q_1 = Q_1Q_2$. Likewise for the other Q_iQ_{i+1}.

Therefore, since Q_0Q_4 is parallel to AB, the intersections of the lines PQ_i with AB divide AB into four equal segments. ■

Remark: To divide AB into five equal segments, we can use the same figure, with most of the work having already been done. The only new lines we need to draw are NQ_0 and MQ_4, to give a total of four intersections on a horizontal line one "level" below the Q_i. If we continue with this process, we obtain figures looking like the one below. The horizontal lines in this figure are divided into equal parts by the intersections of the diagonal lines. The initial undivided segment is the top one.[1]



We leave for you the following exercise: Given a segment of length ℓ, and a line parallel to it, construct a segment with a length equal to an arbitrary multiple of ℓ, using only a ruler.

[1]This segment must be used again for the $N = 2$ segment, because the above procedure yields only one point Q_1, and this single point doesn't determine a line parallel to L.