Solution

Week 51 (9/1/03)

Accelerating spaceship

We will solve this problem by considering two nearby times and using the velocity-addition formula,

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2} . \tag{1}$$

Using the definition of the proper acceleration, a, we have (with $v_1 \equiv v(t)$ and $v_2 \equiv a\,dt$)

$$v(t + dt) = \frac{v(t) + a\,dt}{1 + v(t)a\,dt/c^2} . \tag{2}$$

Expanding both sides to first order in dt yields[1]

$$\frac{dv}{dt} = a\left(1 - \frac{v^2}{c^2}\right) . \tag{3}$$

Separating variables and integrating gives, using $1/(1-z^2) = 1/2(1-z) + 1/2(1+z)$,

$$\int_0^v \left(\frac{1}{1 - v/c} + \frac{1}{1 + v/c}\right) dv = 2a \int_0^t dt. \tag{4}$$

This yields $\ln\left((1+v/c)/(1-v/c)\right) = 2at/c$. Exponentiating, and solving for v, gives

$$v(t) = c\left(\frac{e^{2at/c} - 1}{e^{2at/c} + 1}\right) = c\tanh(at/c). \tag{5}$$

Note that for small a or small t (more precisely, for $at/c \ll 1$), we obtain $v(t) \approx at$, as we should. And for $at/c \gg 1$, we obtain $v(t) \approx c$, as we should.

Remarks: If a happens to be a function of time, $a(t)$, then we can't move the a outside the integral in eq. (4), so we instead end up with the general formula,

$$v(t) = c\tanh\left(\frac{1}{c}\int_0^t a(t)\,dt\right) . \tag{6}$$

If we define the *rapidity*, ϕ, by

$$\phi(t) \equiv \frac{1}{c}\int_0^t a(t)\,dt, \tag{7}$$

then we have

$$v = c\tanh\phi \qquad \Longleftrightarrow \qquad \tanh\phi = \frac{v}{c} . \tag{8}$$

Note that whereas v has c as a limiting value, ϕ can become arbitrarily large. The ϕ associated with a given v is simply $1/mc$ times the time integral of the force (felt by the astronaut) needed to bring the astronaut up to speed v. By applying a force for an arbitrarily long time, we can make ϕ arbitrarily large.

[1] Equivalently, just take the derivative of $(v + w)/(1 + vw/c^2)$ with respect to w, and then set $w = 0$.

The quantity ϕ is very useful because many expressions in relativity (which we'll just invoke here) take on a particularly nice form when written in terms of ϕ. Consider, for example, the velocity-addition formula. Let $\beta_1 = \tanh\phi_1$ and $\beta_2 = \tanh\phi_2$. Then if we add β_1 and β_2 using the velocity-addition formula, eq. (1), we obtain

$$\frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} = \frac{\tanh\phi_1 + \tanh\phi_2}{1 + \tanh\phi_1 \tanh\phi_2} = \tanh(\phi_1 + \phi_2), \tag{9}$$

where we have used the addition formula for $\tanh\phi$ (which can be proved by writing things in terms of the exponentials $e^{\pm\phi}$). Therefore, while the velocities add in the strange manner of eq. (1), the rapidities add by standard addition.

The Lorentz transformation,

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}, \tag{10}$$

also takes a nice form when written in terms of the rapidity. Note that γ can be written as

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\tanh^2\phi}} = \cosh\phi, \tag{11}$$

and so

$$\gamma\beta \equiv \frac{\beta}{\sqrt{1-\beta^2}} = \frac{\tanh\phi}{\sqrt{1-\tanh^2\phi}} = \sinh\phi. \tag{12}$$

Therefore, the Lorentz transformation becomes

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}. \tag{13}$$

This looks similar to a rotation in a plane, which is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}, \tag{14}$$

except that we now have hyperbolic trig functions instead of the usual trig functions. The fact that the *invariant interval*, $s^2 \equiv c^2t^2 - x^2$, does not depend on the frame is clear from eq. (13), because the cross terms in the squares cancel, and $\cosh^2\phi - \sinh^2\phi = 1$. (Compare with the invariance of $r^2 \equiv x^2 + y^2$ for rotations in a plane.)

Quantities associated with a Minkowski diagram also take a nice form when written in terms of the rapidity. In particular, the angle between the axes of the two relevant frames happens to be $\tan\theta = \beta$, where βc is the relative speed between the frames. But $\beta = \tanh\phi$, so the angle between the axes is given by

$$\tan\theta = \tanh\phi. \tag{15}$$

The integral $\int a(t)\,dt$ (which is c times the rapidity) may be described as the naive, incorrect speed. That is, it is the speed the astronaut might *think* he has, if he has his eyes closed and knows nothing about the theory of relativity. (And indeed, his thinking would be essentially correct for small speeds.) The quantity $\int a(t)\,dt$ seems like a reasonably physical thing, so if there is any justice in the world, $\int a(t)\,dt = \int F(t)\,dt/m$ should have *some* meaning. And indeed, although it doesn't equal v, all you have to do to get v is take a tanh and throw in some factors of c.

The fact that rapidities add via simple addition when using the velocity-addition formula, as we saw in eq. (9), is evident from eq. (6). There is really nothing more going on here than the fact that

$$\int_{t_0}^{t_2} a(t)\,dt = \int_{t_0}^{t_1} a(t)\,dt + \int_{t_1}^{t_2} a(t)\,dt. \tag{16}$$

To be explicit, let a force be applied from t_0 to t_1 that brings a mass up to speed $\beta_1 = \tanh\phi_1 = \tanh(\int_{t_0}^{t_1} a\,dt)$, and then let an additional force be applied from t_1 to t_2 that adds on an additional speed of $\beta_2 = \tanh\phi_2 = \tanh(\int_{t_1}^{t_2} a\,dt)$ (relative to the speed at t_1). Then the resulting speed may be looked at in two ways: (1) it is the result of relativistically adding the speeds $\beta_1 = \tanh\phi_1$ and $\beta_2 = \tanh\phi_2$, and (2) it is the result of applying the force from t_0 to t_2 (you get the same final speed, of course, whether or not you bother to record the speed along the way at t_1), which is $\beta = \tanh(\int_{t_0}^{t_2} a dt) = \tanh(\phi_1 + \phi_2)$, where the last equality comes from the obvious statement, eq. (16). Therefore, the relativistic addition of $\tanh\phi_1$ and $\tanh\phi_2$ gives $\tanh(\phi_1 + \phi_2)$, as we wanted to show.