Solution

Week 51  (9/1/03)

Accelerating spaceship

We will solve this problem by considering two nearby times and using the velocity-addition formula,

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}. \quad (1)$$

Using the definition of the proper acceleration, $a$, we have (with $v_1 \equiv v(t)$ and $v_2 \equiv a \, dt$)

$$v(t + dt) = \frac{v(t) + a \, dt}{1 + v(t)a \, dt/c^2}. \quad (2)$$

Expanding both sides to first order in $dt$ yields

$$\frac{dv}{dt} = a \left( 1 - \frac{v^2}{c^2} \right). \quad (3)$$

Separating variables and integrating gives, using $1/(1-z^2) = 1/2(1-z) + 1/2(1+z)$,

$$\int_0^v \left( \frac{1}{1 - v/c} + \frac{1}{1 + v/c} \right) dv = 2a \int_0^t dt. \quad (4)$$

This yields $\ln ((1+v/c)/(1-v/c)) = 2at/c$. Exponentiating, and solving for $v$, gives

$$v(t) = c \left( \frac{e^{2at/c} - 1}{e^{2at/c} + 1} \right) = c \tanh(at/c). \quad (5)$$

Note that for small $a$ or small $t$ (more precisely, for $at/c \ll 1$), we obtain $v(t) \approx at$, as we should. And for $at/c \gg 1$, we obtain $v(t) \approx c$, as we should.

Remarks: If $a$ happens to be a function of time, $a(t)$, then we can’t move the $a$ outside the integral in eq. (4), so we instead end up with the general formula,

$$v(t) = c \tanh \left( \frac{1}{c} \int_0^t a(t) \, dt \right). \quad (6)$$

If we define the rapidity, $\phi$, by

$$\phi(t) = \frac{1}{c} \int_0^t a(t) \, dt, \quad (7)$$

then we have

$$v = c \tanh \phi \quad \iff \quad \tanh \phi = \frac{v}{c}. \quad (8)$$

Note that whereas $v$ has $c$ as a limiting value, $\phi$ can become arbitrarily large. The $\phi$ associated with a given $v$ is simply $1/mc$ times the time integral of the force (felt by the astronaut) needed to bring the astronaut up to speed $v$. By applying a force for an arbitrarily long time, we can make $\phi$ arbitrarily large.

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1Equivalently, just take the derivative of $(v + w)/(1 + vw/c^2)$ with respect to $w$, and then set $w = 0.$
The quantity $\phi$ is very useful because many expressions in relativity (which we’ll just invoke here) take on a particularly nice form when written in terms of $\phi$. Consider, for example, the velocity-addition formula. Let $\beta_1 = \tanh \phi_1$ and $\beta_2 = \tanh \phi_2$. Then if we add $\beta_1$ and $\beta_2$ using the velocity-addition formula, eq. (1), we obtain

$$\frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} = \frac{\tanh \phi_1 + \tanh \phi_2}{1 + \tanh \phi_1 \tanh \phi_2} = \tanh(\phi_1 + \phi_2),$$

(9)

where we have used the addition formula for $\tanh \phi$ (which can be proved by writing things in terms of the exponentials $e^{\pm \phi}$). Therefore, while the velocities add in the strange manner of eq. (1), the rapidities add by standard addition.

The Lorentz transformation, 

$$\left( \begin{array}{c} x \\ c t \end{array} \right) = \left( \begin{array}{cc} \gamma & \gamma \beta \\ \gamma \beta & \gamma \end{array} \right) \left( \begin{array}{c} x' \\ c t' \end{array} \right),$$

(10)

also takes a nice form when written in terms of the rapidity. Note that $\gamma$ can be written as

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}} = \frac{1}{\sqrt{1 - \tanh^2 \phi}} = \cosh \phi,$$

(11)

and so

$$\gamma \beta \equiv \frac{\beta}{\sqrt{1 - \beta^2}} = \frac{\tanh \phi}{\sqrt{1 - \tanh^2 \phi}} = \sinh \phi.$$  

(12)

Therefore, the Lorentz transformation becomes

$$\left( \begin{array}{c} x \\ x' \end{array} \right) = \left( \begin{array}{cc} \cosh \phi & \sinh \phi \\ -\sinh \phi & \cosh \phi \end{array} \right) \left( \begin{array}{c} x' \\ c t' \end{array} \right),$$

(13)

This looks similar to a rotation in a plane, which is given by

$$\left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right) \left( \begin{array}{c} x' \\ y' \end{array} \right),$$

(14)

except that we now have hyperbolic trig functions instead of the usual trig functions. The fact that the invariant interval, $s^2 = c^2 t^2 - x^2$, does not depend on the frame is clear from eq. (13), because the cross terms in the squares cancel, and $\cosh^2 \phi - \sinh^2 \phi = 1$. (Compare with the invariance of $r^2 \equiv x^2 + y^2$ for rotations in a plane.)

Quantities associated with a Minkowski diagram also take a nice form when written in terms of the rapidity. In particular, the angle between the axes of the two relevant frames happens to be $\tan \theta = \beta$, where $\beta c$ is the relative speed between the frames. But $\beta = \tanh \phi$, so the angle between the axes is given by

$$\tan \theta = \tanh \phi.$$  

(15)

The integral $\int a(t) \, dt$ (which is $c$ times the rapidity) may be described as the naive, incorrect speed. That is, it is the speed the astronaut might think he has, if he has his eyes closed and knows nothing about the theory of relativity. (And indeed, his thinking would be essentially correct for small speeds.) The quantity $\int a(t) \, dt$ seems like a reasonably physical thing, so if there is any justice in the world, $\int a(t) \, dt = \int F(t) \, dt / m$ should have some meaning. And indeed, although it doesn’t equal $v$, all you have to do to get $v$ is take a tanh and throw in some factors of $c$.

The fact that rapidities add via simple addition when using the velocity-addition formula, as we saw in eq. (9), is evident from eq. (6). There is really nothing more going on here than the fact that

$$\int_{t_0}^{t_2} a(t) \, dt = \int_{t_0}^{t_1} a(t) \, dt + \int_{t_1}^{t_2} a(t) \, dt.$$  

(16)
To be explicit, let a force be applied from $t_0$ to $t_1$ that brings a mass up to speed $\beta_1 = \tanh \phi_1 = \tanh(\int_{t_0}^{t_1} a \, dt)$, and then let an additional force be applied from $t_1$ to $t_2$ that adds on an additional speed of $\beta_2 = \tanh \phi_2 = \tanh(\int_{t_1}^{t_2} a \, dt)$ (relative to the speed at $t_1$). Then the resulting speed may be looked at in two ways: (1) it is the result of relativistically adding the speeds $\beta_1 = \tanh \phi_1$ and $\beta_2 = \tanh \phi_2$, and (2) it is the result of applying the force from $t_0$ to $t_2$ (you get the same final speed, of course, whether or not you bother to record the speed along the way at $t_1$), which is $\beta = \tanh(\int_{t_0}^{t_2} a \, dt) = \tanh(\phi_1 + \phi_2)$, where the last equality comes from the obvious statement, eq. (16). Therefore, the relativistic addition of $\tanh \phi_1$ and $\tanh \phi_2$ gives $\tanh(\phi_1 + \phi_2)$, as we wanted to show.